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COMPLEX CONFORMAL CONNECTION ON THE LOCALLY CONFORMAL KÄHLER MANIFOLDS

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Abstract. The object of the paper is to investigate the curvature tensor of the complex conformal connection on locally conformal Kähler manifolds.

§ 1. Complex conformal connection.

Let (M, g, J), $\dim M = 2n$, be an almost Hermitian manifold with metric g and complex strusture J. This means that, with respect to the local coordinates (x^i), we have

$$J^i_t J^t_j = -\delta^i_j, \quad g_{pq} J^p_i J^q_j = g_{ij}.$$

The change of metric

$$\widetilde{g}_{ij} = e^{2p} g_{ij},$$

there p is a certain scalar function, is conformal change of metric. Here and in the sequel, we suppose that all functions are of the class C^3.

In [4], K. Yano proved the following.

The affine connection $\widetilde{\nabla}$ with components $\widetilde{\Gamma}^h_{ji}$ which satisfies

$$\text{(1.1)} \qquad \begin{aligned} &\text{(a)} \quad \widetilde{\nabla}_k \widetilde{g}_{ji} = 0, \\ &\text{(b)} \quad \widetilde{\nabla}_k J^h_j = \nabla_k J^h_j, \\ &\text{(c)} \quad \widetilde{\Gamma}^h_{ji} - \widetilde{\Gamma}^h_{ij} = -2F_{ji} q^h, \end{aligned}$$

where q^h is a vector field, $F_{ji} = J^t_j g_{ti}$ and ∇ is the operator of covariant differentiation with respect to the Christoffel symbols $\{^{\,h}_{ji}\}$ of the metric g, is given by

$$\text{(1.2)} \qquad \widetilde{\Gamma}^h_{ji} = \{^{\,h}_{ji}\} + p_j \delta^h_i + p_i \delta^h_j - g_{ji} p^h + q_j J^h_i + q_i J^h_j - F_{ji} q^h,$$

where p_i is the gradient of p and

$$(1.3)\qquad \begin{aligned} &\text{(a)}\quad p^h = p_t g^{th},\\ &\text{(b)}\quad q_i = -p_t J_i^t,\\ &\text{(c)}\quad q^h = q_t g^{th}. \end{aligned}$$

The relation (1.3 (b)) is the consequence of the condition (1.1(b)). Yano proved this for the Kähler manifolds, i.e. he used the condition $\widetilde{\nabla}_k J_j^h = 0$. In the general case (1.1(b)), the proof is the same, and so we omit it. We note that we have, as a consequence of (1.3(b)),

$$(1.4)\qquad q^h = p^t J_t^h, \quad p_i = q_t J_i^t, \quad p^h = -J_t^h q^t.$$

The connection (1.2) is called *complex conformal connection.* Yano also proved the following [4]:

If in an $2n$–dimensional Kähler manifold ($2n \geq 4$) there exist a scalar function p such that the complex conformal connection (1.2) is of zero curvature, then the Bochner curvature tensor of the manifold vanishes.

This theorem can be generalized in the following way [3]:

If the curvature tensor $\widetilde{R}$ of the connection (1.2) in an $2n$–dimensional Kähler manifold ($2n \geq 4$) is algebraic curvature tensor and satisfies the Kähler condition

$$R(X, Y, JZ, JW) = R(X, Y, Z, W),$$

then its Weyl component is the Bochner tensor of the considered Kähler manifold.

The object of this paper is to investigate complex conformal connection (1.2) on the locally conformal Kähler manifolds. After preliminary § 2, we find in § 3, the necessary and sufficient condition such that the curvature tensor of the connection (1.2) is an algebraic curvature tensor. In § 4, we give an example of locally conformal Kähler manifold satisfying this condition. In § 5, we prove the relation (5.7).

§ 2. Curvature tensor of a complex conformal connection.

Let

$$\widetilde{R}_{kji}{}^s = \partial_k \widetilde{\Gamma}^s_{ji} - \partial_j \widetilde{\Gamma}^s_{ki} + \widetilde{\Gamma}^s_{kt}\widetilde{\Gamma}^t_{ji} - \widetilde{\Gamma}^s_{jt}\widetilde{\Gamma}^t_{ki}$$

be the curvature tensor of the connection (1.2), and

$$\widetilde{R}_{kjih} = \widetilde{g}_{sh}\widetilde{R}_{kji}{}^s = e^{2p} g_{sh}\widetilde{R}_{kji}{}^s.$$

Then

$$
\begin{aligned}
e^{-2p}\widetilde{R}_{kjih} = R_{kjih}& \\
&+ g_{jh}(\nabla_k p_i - p_k p_i + q_k q_i + \frac{1}{2} g_{ki} p_t p^t) \\
&- g_{kh}(\nabla_j p_i - p_j p_i + q_j q_i + \frac{1}{2} g_{ji} p_t p^t) \\
&- g_{ji}(\nabla_k p_h - p_k p_h + q_k q_h + \frac{1}{2} g_{kh} p_t p^t) \\
&+ g_{ki}(\nabla_j p_h - p_j p_h + q_j q_h + \frac{1}{2} g_{jh} p_t p^t) \\
&+ F_{jh}(\nabla_k q_i - p_k q_i - q_k p_i + \frac{1}{2} F_{ki} p_t p^t) \\
&- F_{kh}(\nabla_j q_i - p_j q_i - q_j p_i + \frac{1}{2} F_{ji} p_t p^t) \\
&- F_{ji}(\nabla_k q_h - p_k q_h - q_k p_h + \frac{1}{2} F_{kh} p_t p^t) \\
&+ F_{ki}(\nabla_j q_h - p_j q_h - q_j p_h + \frac{1}{2} F_{jh} p_t p^t) \\
&+ 2F_{jk}(p_i q_h - q_i p_h) + F_{ih}(\nabla_k q_j - \nabla_j q_k) \\
&+ q_h(\nabla_j F_{ki} - \nabla_k F_{ji}) + q_i(\nabla_k F_{jh} - \nabla_j F_{kh}) \\
&+ q_j \nabla_k F_{ih} - q_k \nabla_j F_{ih},
\end{aligned}
\tag{2.1}
$$

where R_{kjih} is the Riemannian curvature tensor, i.e. the curvature tensor of the Levi–Civita connection $\{{h \atop ji}\}$.

Now, let us suppose that (M, g, J) is a locally conformal Kähler manifold, i.e. we suppose that

$$g_{ij} = e^{2\sigma} \mathring{g}_{ij}, \tag{2.2}$$

where $\mathring{g}$ is the metric of a Kähler manifold $(M, \mathring{g}, J)$ and σ is a scalar function. Then [1]

$$\nabla_i J_j^k = \delta_i^k \sigma_t J_j^t + g_{ij} \sigma^t J_t^k - J_i^k \sigma_j + F_{ij} \sigma^k, \tag{2.3}$$

where

$$\sigma_i = \frac{\partial \sigma}{\partial x^i}, \quad \sigma^h = g^{ht} \sigma_t.$$

From (2.3), we get

$$\nabla_i F_{jh} = g_{ih} \sigma_t J_j^t - g_{ij} \sigma_t J_h^t - F_{ih} \sigma_j + F_{ij} \sigma_h. \tag{2.4}$$

Substituting (2.4) into (2.1), we find

$$
\begin{aligned}
e^{-2p}\widetilde{R}_{kjih} = R_{kjih}& \\
&+ g_{jh} p_{ki} + g_{ki} p_{jh} - g_{kh} p_{ji} - g_{ji} p_{kh} \\
&+ F_{jh} q_{ki} + F_{ki} q_{jh} - F_{kh} q_{ji} - F_{ji} q_{kh} \\
&+ F_{kj} \beta_{ih} - F_{ih} \alpha_{kj},
\end{aligned}
\tag{2.5}
$$

where

$$
\begin{aligned}
p_{ji} &= \nabla_j p_i - p_j p_i + q_j q_i + \frac{1}{2} g_{ji} p_t p^t - q_i \sigma_t J_j^t - q_j \sigma_t J_i^t, \\
q_{ji} &= \nabla_j q_i - p_j q_i - q_j p_i + \frac{1}{2} F_{ji} p_t p^t + q_i \sigma_j + q_j \sigma_i, \\
\alpha_{kj} &= -(\nabla_k q_j - \nabla_j q_k), \\
\beta_{ih} &= 2(p_i q_h - q_i p_h + q_h \sigma_i - q_i \sigma_h).
\end{aligned}
\tag{2.6}
$$

§ 3. The condition such that $\widetilde{R}$ be the algebraic curvature tensor.

Let us suppose that $\widetilde{R}$ is algebraic curvature tensor, i.e. that it satisfies

$$
\begin{aligned}
&\text{(a)} \quad \widetilde{R}_{kjih} = -\widetilde{R}_{jkih} = -\widetilde{R}_{kjhi}, \\
&\text{(b)} \quad \widetilde{R}_{kjih} = \widetilde{R}_{ihkj}, \\
&\text{(c)} \quad \widetilde{R}_{kjih} + \widetilde{R}_{jikh} + \widetilde{R}_{ikjh} = 0.
\end{aligned}
\tag{3.1}
$$

The conditions (3.1(a)) are satisfied because, according (2.6), both α_{kj} and β_{ih} are skewsymmetric.

The condition (3.1(c)) is satisfied if and only if

$$
\begin{aligned}
F_{jh}(q_{ki} - q_{ik} - \alpha_{ik}) + F_{ih}(q_{jk} - q_{kj} - \alpha_{kj}) + F_{kh}(q_{ij} - q_{ji} - \alpha_{ji}) \\
+ F_{ki}(2q_{jh} + \beta_{jh}) + F_{jk}(2q_{ih} + \beta_{ih}) \\
+ F_{ij}(2q_{kh} + \beta_{kh}) = 0.
\end{aligned}
\tag{3.2}
$$

But, according (2.6), we have

$$
-q_{ji} + q_{ij} - \alpha_{ji} = F_{ij} p_t p^t.
$$

This means that (3.2) can be rewritten in the form

$$
\begin{aligned}
F_{ik}(2q_{jh} + \beta_{jh} + F_{jh} p_t p^t) + F_{kj}(2q_{ih} + \beta_{ih} + F_{ih} p_t p^t) \\
+ F_{ji}(2q_{kh} + \beta_{kh} + F_{kh} p_t p^t) = 0.
\end{aligned}
\tag{3.3}
$$

Transvecting (3.3) with $J_t^i g^{tj}$, we get

$$
2(n-1)(2q_{kh} + \beta_{kh} + F_{kh} p_t p^t) = 0.
$$

Thus, for $n > 1$, the condition

$$
2q_{kh} + \beta_{kh} + F_{kh} p_t p^t = 0
\tag{3.4}
$$

is necessary and sufficient for (3.1(c)) to be satisfied.

Taking into account (2.6), we can rewrite (3.4) in the form

$$\nabla_k q_h = 2q_k p_h - 2q_h\sigma_k - F_{kh}p_t p^t \tag{3.5}$$

Now,

$$\begin{aligned}\alpha_{jh} &= -\nabla_j q_h + \nabla_h q_j \\ &= -2q_j p_h + 2q_h p_j + 2q_h\sigma_j - 2q_j\sigma_h + 2F_{jh}p_t p^t\end{aligned}$$

and in view of

$$\beta_{jh} = 2(p_j q_h - q_j p_h + q_h\sigma_j - q_j\sigma_h),$$

we have

$$\alpha_{jh} - \beta_{jh} = 2F_{jh}p_t p^t. \tag{3.6}$$

Finally, from (3.4), we get

$$q_{kh} + q_{hk} = 0. \tag{3.7}$$

Now, we can discuss (3.1(b)). This relation is satisfied if and only if

$$\begin{aligned}F_{jh}(q_{ki} + q_{ik}) + F_{ki}(q_{jh} + q_{hj}) - F_{kh}(q_{ji} + q_{ij}) - F_{ji}(q_{kh} + q_{hk}) \\ + F_{kj}(\alpha_{ih} - \beta_{ih}) - F_{ih}(\alpha_{kj} - \beta_{kj}) = 0.\end{aligned} \tag{3.8}$$

But, in view of (3.6) and (3.7), (3.8) is satisfied identicaly.

From (3.5), in view of (1.3) and (1.4), we have

$$(\nabla_k q_t)J_j^t = -2q_k q_j - 2p_j\sigma_k - g_{ij}\sigma_t\sigma^t.$$

But

$$(\nabla_k q_t)J_j^t = \nabla_k(q_t J_j^t) - q_t\nabla_k J_j^t,$$

and using (2.3), we get

$$(\nabla_k q_t)J_j^t = \nabla_k p_j - q_k\sigma_t J_j^t - g_{kj}q_t\sigma^s J_s^t + p_k\sigma_j - F_{kj}q_t\sigma^t.$$

Thus,

$$\begin{aligned}\nabla_k p_j = &-2q_k q_j - 2p_j\sigma_k + g_{kj}(p_t\sigma^t - p_t p^t) \\ &- p_k\sigma_j + F_{kj}q_t\sigma^t,\end{aligned} \tag{3.9}$$

But $\nabla_k p_j$ is a symmetric tensor because p_j is a gradient. Therefore,

$$p_j\sigma_k - p_k\sigma_j + q_j\sigma_t J_k^t - q_k\sigma_t J_j^t = 2F_{kj}q_t\sigma^t.$$

Transvecting this with p^j, we find

$$p_t p^t\sigma_k - p_k\sigma_t p^t = -q_k q_t\sigma^t. \tag{3.10}$$

Transvecting (3.10) with q^k, we obtain

$$p_t p^t \sigma_k q^k = -p_k p^k q_t \sigma^t$$

that is, $\sigma_t q^t = 0$ because of $p_t p^t \neq 0$. Thus, (3.10) reduces to

$$p_t p^t \sigma_k = p_k \sigma_t p^t.$$

If $\sigma_t p^t = 0$, then $\sigma_k = 0$, i.e. $\sigma = \text{const.}$, which means that (M, g, J) ia a Kähler manifold. Thus, $\sigma_t p^t \neq 0$ and we have

$$p_k = f\sigma_k, \quad q_k = -f\sigma_t J_k^t, \quad p_t p^t = f^2 \sigma_t \sigma^t, \tag{3.11}$$

where

$$f = \frac{p_t p^t}{p_t \sigma^t},$$

is some scalar function. Now, (3.9) and (3.5) reduce, respectively to

$$\nabla_k p_j = -(f + 2f^2)\sigma_t \sigma_s J_k^t J_j^s - 3f\sigma_j \sigma_k + (f - f^2)\sigma_t \sigma^t g_{kj}, \tag{3.12}$$

and

$$\nabla_k q_j = -2f^2 \sigma_t J_k^t \sigma_j + 2f\sigma_t J_j^t \sigma_k - f^2 F_{kj} \sigma_t \sigma^t, \tag{3.13}$$

and we can state the theorem

Theorem 1. *Let (M, g, J) be a locally conformal Kähler manifold,* $\dim M = 2n$, $2n \geq 4$, *and let $\widetilde{R}$ be the curvature tensor of the complex conformal connection (1.2) on it. Then $\widetilde{R}$ is an algebraic curvature tensor if and only if (3.12), or equivalently (3.13), holds. The functions σ and f are determined by (2.2) and (3.11) respectively.*

In view of (2.6), (3.11), (3.12) and (3.13), we have

$$\begin{aligned} p_{ji} &= (f - f^2)\sigma_t \sigma_s J_j^t J_i^s - (3f + f^2)\sigma_j \sigma_i + (f - \frac{1}{2}f^2)\sigma_t \sigma^t g_{ji}, \\ q_{ji} &= -(f^2 + f)\sigma_t J_j^t \sigma_i + (f^2 + f)\sigma_t J_i^t \sigma_j - \frac{1}{2}\sigma_t \sigma^t F_{ji}, \\ \alpha_{kj} &= 2(f^2 + f)\sigma_t J_k^t \sigma_j - 2(f^2 + f)\sigma_t J_j^t \sigma_k + 2f^2 \sigma_t \sigma^t F_{kj}, \\ \beta_{ih} &= 2(f^2 + f)\sigma_t J_i^t \sigma_h - 2(f^2 + f)\sigma_t J_h^t \sigma_i. \end{aligned} \tag{3.14}$$

§ 4. Example.

In [2] it is proved the following.

Let a Riemannian space M, $dim M = 2n$, have a metric defined by

$$\begin{aligned} \overset{\circ}{g}_{ab} = \overset{\circ}{g}_{a+n\, b+n} &= \partial_{ab}G + \partial_{a+n\, b+n}G, \\ \overset{\circ}{g}_{a\, b+n} &= \partial_{a\, b+n}G - \partial_{a+n\, b}G, \end{aligned} \tag{4.1}$$

where

$$G = G(x^1 + S(x^2, \ldots, x^n, x^{2+n}, \ldots, x^{2n})),$$

$G'G'' \neq 0$, G, S are functions of given arguments, a, $b = 1, 2, \ldots, n$; $|\overset{\circ}{g}_{ij}| \neq 0$.

Then this space is the Kähler space which admits a scalar function σ such that the vector field $\sigma_i = \frac{\partial\sigma}{\partial x^i}$ satisfies

$$\overset{\circ}{\nabla}_j \sigma_i = \partial \overset{\circ}{g}_{ji} + c(\sigma_i\sigma_j + \sigma_p\sigma_q J_i^p J_j^q), \tag{4.2}$$

where $\overset{\circ}{\nabla}$ denotes the Levi–Civita connection with respect to the metric (4.1), and a and c are some functions.

We note that the latin indices i, j, k, p, q, t run over the range $1, 2, ..., 2n$.

In local coordinates in which the conditions (4.1) are valid, the complex structure is defined by

$$J_b^{a+n} = -J_{b+n}^a = \delta_b^a, \quad J_b^a = J_{b+n}^{a+n} = 0. \tag{4.3}$$

As for vector field σ_i, it has the components $\sigma_i = \frac{\partial\sigma}{\partial x^i} = \overset{\circ}{g}_{i1}$ and thus $\sigma^i = \overset{\circ}{g}{}^{ij}\sigma_j = \delta_1^i$. Therefore

$$\begin{aligned} \sigma_1 &= \frac{\partial\sigma}{\partial x^1} = \overset{\circ}{g}_{11} = G,'' \\ \sigma_{1+n} &= \frac{\partial\sigma}{\partial x^{1+n}} = \overset{\circ}{g}_{1+n\,1} = 0, \\ \sigma_\alpha &= \frac{\partial\sigma}{\partial x^\alpha} = \overset{\circ}{g}_{\alpha 1} = G''\frac{\partial S}{\partial x^\alpha}, \\ \sigma_{\alpha+n} &= \frac{\partial\sigma}{\partial x^{\alpha+n}} = \overset{\circ}{g}_{\alpha+n\,1} = G''\frac{\partial S}{\partial x^{\alpha+n}}, \quad \alpha = 2, ..., n. \end{aligned}$$

Thus, we see that

$$\sigma = G'. \tag{4.4}$$

Also,

$$\sigma_t\sigma^t = \overset{\circ}{g}_{ij}\sigma^i\sigma^j = \overset{\circ}{g}_{ij}\delta_1^i\delta_1^j = \overset{\circ}{g}_{11} = G''. \tag{4.5}$$

As for the functions a and c, we have [2]:

$$a = \frac{1}{2}\frac{G''}{G'}, \quad c = \frac{1}{2}\frac{G'''G' - (G'')^2}{G'(G'')^2}.$$

In view of (4.4) and (4.5), the function a can be expressed in the form

$$a = \frac{1}{2}\frac{\sigma_t\sigma^t}{\sigma}.$$

Now, let us suppose that G is the solution of the differential equation

$$\frac{1}{2}\frac{G'''G' - (G'')^2}{G'(G'')^2} = \frac{1}{G'} - 1. \tag{4.6}$$

Then $c = 1/G' - 1$ and therefore $2a/(\sigma_t\sigma^t) = 1 + c$. Let us put

$$f = -\frac{1}{2\sigma}. \tag{4.7}$$

Then

$$a = -f\sigma_t\sigma^t, \quad c = -2f - 1,$$

and (4.2) can be rewritten as follows

$$\overset{\circ}{\nabla}_j\sigma_i = (-f\sigma_t\sigma^t)\overset{\circ}{g}_{ij} - (1+2f)(\sigma_i\sigma_j + \sigma_p\sigma_q J_i^p J_j^q). \tag{4.8}$$

Now, let us consider the conformal change

$$g = e^{2\sigma}\overset{\circ}{g}.$$

Then (M, g, J) is the locally conformal Kähler manifold. Obviously, we have

$$\left\{{k \atop ij}\right\} = \overset{\circ}{\left\{{k \atop ij}\right\}} + \delta_i^k\sigma_j + \delta_j^k\sigma_i - \overset{\circ}{g}_{ij}\sigma^k$$

and

$$\nabla_j\sigma_i = \overset{\circ}{\nabla}_j\sigma_i - 2\sigma_i\sigma_j + \overset{\circ}{g}_{ij}\sigma_t\sigma^t,$$

where ∇ and $\overset{\circ}{\nabla}$ denote the Levi–Civita connections with respect to the metrics g and $\overset{\circ}{g}$ respectively. Substituting (4.8) into the last relation, we find

$$\nabla_j\sigma_i = -(3+2f)\sigma_j\sigma_i - (1+2f)\sigma_p\sigma_q J_j^p J_i^q + (1-f)\sigma_t\sigma^t g_{ji}. \tag{4.9}$$

Finally, let us put $p_i = f\sigma_i$ (see (3.11)). Then

$$\nabla_j p_i = \frac{\partial f}{\partial x^j}\sigma_i + f\nabla_j\sigma_i.$$

But, taking into account (4.7), we have

$$\frac{\partial f}{\partial x^j} = 2f^2\sigma_j.$$

This means that

$$\nabla_j\sigma_i = \frac{1}{f}\nabla_j p_i - 2f\sigma_i\sigma_j.$$

Substitutuing this into (4.9), we find

$$\nabla_j p_i = -3f\sigma_j\sigma_i - f(1+2f)\sigma_p\sigma_q J_j^p J_i^q + f(1-f)\sigma_t\sigma^t g_{ji}.$$

But this is just the condition (3.12). Thus we can state

Theorem 2. *Let us consider the Kähler manifold* $(M, \overset{\circ}{g}, J)$ *with metric (4.1) and complex structure (4.3). Let the function* G *be the solution of the differential equation (4.6). If we put*

$$\sigma = G', \quad g_{ij} = e^{2\sigma}\overset{\circ}{g}_{ij},$$

then (M, g, J) *is the locally conformal Kähler manifold satisfying the conditions of Theorem 1, where* $f = -1/(2\sigma) = -1/(2G')$.

§ 5. Some more results.

Transvecting (2.5) with $g^{kh} = e^{2p}\widetilde{g}^{kh}$ and denoting by $\widetilde{R}_{ji}$ and R_{ji} the corresponding Ricci tensors, we find

$$\begin{aligned}\widetilde{R}_{ji} = R_{ji} &- 2(n-1)p_{ji} - g_{ji}p_{kh}g^{kh} \\ &+ J_j^t(q_{ti} + \beta_{it}) - J_i^t(q_{jt} + \alpha_{tj}).\end{aligned} \tag{5.1}$$

But, according (3.14), we have

$$\begin{aligned}p_{kh}g^{kh} &= [(2n-2)f - (n+2)f^2]\sigma_t\sigma^t, \\ (q_{ti} + \beta_{it})J_j^t &= 3(f^2+f)\sigma_j\sigma_i + 3(f^2+f)\sigma_p\sigma_q J_i^p J_j^q + \frac{1}{2}f^2 g_{ij}\sigma_t\sigma^t, \\ (q_{jt} + \alpha_{tj})J_i^t &= -3(f^2+f)\sigma_i\sigma_j - 3(f^2+f)\sigma_p\sigma_q J_i^p J_j^q - \frac{\sigma}{2}f^2 g_{ij}\sigma_t\sigma^t,\end{aligned}$$

because of which, (5.1) becomes

$$\begin{aligned}\widetilde{R}_{ji} = R_{ji} &+ [6nf + 2(n+2)f^2]\sigma_i\sigma_j \\ &+ [-2(n-4)f + 2(n+2)f^2]\sigma_p\sigma_q J_i^p J_j^q \\ &+ [-4(n-1)f + 2(n+2)f^2]\sigma_t\sigma^t g_{ij}.\end{aligned}$$

Let us put

$$
(5.2)\qquad \begin{aligned}
&6nf + 2(n+2)f^2 = A,\\
&-2(n-4)f + 2(n+2)f^2 = B,\\
&[-4(n-1)f + 2(n+2)f^2]\sigma_t\sigma^t = C.
\end{aligned}
$$

Then the preceding relation can be rewritten in the form

$$
(5.3)\qquad \widetilde{R}_{ji} = R_{ji} + A\sigma_i\sigma_j + B\sigma_p\sigma_q J_i^p J_j^q + Cg_{ij},
$$

from which it follows

$$
(5.4)\qquad \widetilde{R}_{pq}J_i^pJ_j^q = R_{pq}J_i^pJ_j^q + B\sigma_i\sigma_j + A\sigma_p\sigma_q J_i^p J_j^q + Cg_{ij}.
$$

We obtain from (5.3) and (5.4)

$$
\begin{aligned}
\sigma_i\sigma_j =& \Big(\frac{A}{A^2-B^2}\widetilde{R}_{ij} - \frac{B}{A^2-B^2}\widetilde{R}_{pq}J_i^pJ_j^q\Big)\\
&- \Big(\frac{A}{A^2-B^2}R_{ij} - \frac{B}{A^2-B^2}R_{pq}J_i^pJ_j^q\Big) - \frac{C}{A+B}g_{ij},\\
\sigma_p\sigma_qJ_i^pJ_j^q =& -\Big(\frac{B}{A^2-B^2}\widetilde{R}_{ij} - \frac{A}{A^2-B^2}\widetilde{R}_{pq}J_i^pJ_j^q\Big)\\
&+ \Big(\frac{B}{A^2-B^2}R_{ij} - \frac{A}{A^2-B^2}R_{pq}J_i^pJ_j^q\Big) - \frac{C}{A+B}g_{ij},\\
\sigma_i\sigma_tJ_j^t =& \Big(\frac{A}{A^2-B^2}\widetilde{R}_{ti}J_j^t + \frac{B}{A^2-B^2}\widetilde{R}_{jt}J_i^t\Big)\\
&- \Big(\frac{A}{A^2-B^2}R_{ti}J_j^t + \frac{B}{A^2-B^2}R_{jt}J_i^t\Big) - \frac{C}{A+B}F_{ji}.
\end{aligned}
$$

Substituting this into (3.14), we find

$$
\begin{aligned}
p_{ji} =& -\Big[\frac{(3f+f^2)A + (f-f^2)B}{A^2-B^2}\widetilde{R}_{ji} - \frac{(3f+f^2)B + (f-f^2)A}{A^2-B^2}\widetilde{R}_{pq}J_i^pJ_j^q\Big]\\
&+ \Big[\frac{(3f+f^2)A + (f-f^2)B}{A^2-B^2}R_{ji} - \frac{(3f+f^2)B + (f-f^2)A}{A^2-B^2}R_{pq}J_i^pJ_j^q\Big]\\
&+ \Big[\frac{2(f+f^2)C}{A+B} + (f - \frac{1}{2}f^2)\sigma_t\sigma^t\Big]g_{ij},\\
q_{ji} =& (f+f^2)[-(\widetilde{R}_{ti}J_j^t - \widetilde{R}_{tj}J_i^t) + (R_{ti}J_j^t - R_{tj}J_i^t)]\\
&+ \Big[\frac{2(f+f^2)C}{A+B} - \frac{1}{2}f^2\sigma_t\sigma^t\Big]F_{ji},\\
\alpha_{kj} =& \frac{2(f+f^2)}{A+B}[(\widetilde{R}_{tj}J_k^t - \widetilde{R}_{tk}J_j^t) - (R_{tj}J_k^t - R_{tk}J_j^t)]\\
&+ [-4(f+f^2)\frac{C}{A+B} + 2f^2\sigma_t\sigma^t]F_{kj},\\
\beta_{ih} =& -\frac{2(f+f^2)}{A+B}[(\widetilde{R}_{ti}J_h^t - \widetilde{R}_{th}J_i^t) - (R_{ti}J_h^t - R_{th}J_i^t)] + \frac{4(f+f^2)C}{A+B}F_{hi}.
\end{aligned}
$$

To determine $\sigma_t \sigma^t$ we transvect (5.3) with $g^{ij} = e^{2p}\widetilde{g}^{ij}$. Then, putting

$$\widetilde{R} = \widetilde{R}_{ij}\widetilde{g}^{ij}, \quad R = R_{ij}g^{ij},$$

we get

$$\sigma_t \sigma^t = \frac{e^{2p}\widetilde{R} - R}{4\varphi}, \tag{5.5}$$

where

$$\varphi = (n+1)(n+2)f^2 - (2n^2 - 3n - 2)f. \tag{5.6}$$

We note that, in view of (5.3), $\widetilde{R}_{ij}$ is the symmetric tensor. Also, we note that

$$A + B = 4(n+2)(f + f^2), \quad A - B = 8(n-1)f,$$

$$A^2 - B^2 = 32(n-1)(n+2)f(f + f^2).$$

Thus, if $f \neq -1$, we finally have

$$\begin{aligned}
p_{ji} =& -\Big(\frac{2n+1}{4(n-1)(n+2)}\widetilde{R}_{ji} - \frac{3}{4(n-1)(n+2)}\widetilde{R}_{pq}J_j^p J_i^q\Big) \\
&+ \Big(\frac{2n+1}{4(n-1)(n+2)}R_{ji} - \frac{3}{4(n-1)(n+2)}R_{pq}J_j^p J_i^q\Big) \\
&+ \Big(\frac{1}{2}f^2 - \frac{n-4}{n+2}f\Big)\Big(\frac{e^{2p}\widetilde{R} - R}{4\varphi}\Big)g_{ji}, \\
q_{ji} =& -\frac{1}{4(n+2)}(\widetilde{R}_{ti}J_j^t - \widetilde{R}_{tj}J_i^t) + \frac{1}{4(n+2)}(R_{ti}J_j^t - R_{tj}J_i^t) \\
&+ \Big[\frac{1}{2}f^2 - \frac{2(n-1)}{n+2}f\Big]\Big(\frac{e^{2p}\widetilde{R} - R}{4\varphi}\Big)F_{ji}, \\
\alpha_{kj} =& \frac{1}{2(n+2)}(\widetilde{R}_{tj}J_k^t - \widetilde{R}_{tk}J_j^t\Big) - \frac{1}{2(n+2)}(R_{tj}J_k^t - R_{tk}J_j^t) \\
&+ \frac{4(n-1)f}{n+2}\Big(\frac{e^{2p}\widetilde{R} - R}{4\varphi}\Big)F_{kj}, \\
\beta_{ih} =& -\frac{1}{2(n+2)}(\widetilde{R}_{ti}J_h^t - \widetilde{R}_{th}J_i^t) + \frac{1}{2(n+2)}(R_{ti}J_h^t - R_{th}J_i^t) \\
&+ \frac{1}{n+2}[2(n+2)f^2 - 4(n-1)f]\Big(\frac{e^{2p}\widetilde{R} - R}{4\varphi}\Big)F_{hi}.
\end{aligned}$$

Substituting this into (2.5), we get

$$e^{-2p}\widetilde{W}_{kjih} = W_{kjih} \tag{5.7}$$

where

$$
\begin{aligned}
W_{kjih} &= R_{kjih} \\
&+ \frac{2n+1}{4(n-1)(n+2)}(g_{jh}R_{ki} + g_{ki}R_{jh} - g_{kh}R_{ji} - g_{ji}R_{kh}) \\
&- \frac{3}{4(n-1)(n+2)}(g_{jh}R_{st}J_k^sJ_i^t + g_{ki}R_{st}J_j^sJ_h^t - g_{kh}R_{st}J_j^sJ_i^t \\
&- g_{ji}R_{st}J_k^sJ_h^t) \\
&+ \frac{1}{4(n+2)}[F_{jh}(R_{ti}J_k^t - R_{tk}J_i^t) + F_{ki}(R_{th}J_j^t - R_{tj}J_h^t) \\
&- F_{kh}(R_{ti}J_j^t - R_{tj}J_i^t) - F_{ji}(R_{th}J_k^t - R_{tk}J_h^t)] \\
&- \frac{1}{2(n+2)}[F_{kj}(R_{ti}J_h^t - R_{th}J_i^t) + F_{ih}(R_{tk}J_j^t - R_{tj}J_k^t)] \\
&+ \frac{R}{4\varphi}\Big[\Big(f^2 - \frac{2(n-4)}{n+2}f\Big)(g_{kh}g_{ji} - g_{jh}g_{ki}) \\
&+ \Big(f^2 - \frac{4(n-1)f}{n+2}\Big)(F_{kh}F_{ji} - F_{jh}F_{ki} - 2F_{kj}F_{ih})\Big],
\end{aligned}
\tag{5.8}
$$

and $\widetilde{W}_{kjih}$ is constructed in the same manner, but using the curvature tensor $\widetilde{R}_{kjih}$ and the metruc $\widetilde{g}$.

We can easely see that the Ricci tensor of the tensor (5.8), $W_{ji} = W_{kjih}g^{kh}$, vaniches. Thus, and in view of Theorem 1, we can state

Theorem 3. *Let (M, g, J) be a locally conformal Kähler manifold and let $\widetilde{R}$ be the curvature tensor of the complex conformal connection (1.2) on it. If $\widetilde{R}$ is an algebraic curvature tensor and $f \neq -1$, then for the tensor (5.8) and the tensor $\widetilde{W}$ constructed in the same manner but using $\widetilde{R}$ and the metric $\widetilde{g}$ instead of R and g, (5.7) holds. The Ricci tensor of the tensor (5.8) vanishes.*

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