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NEW ITERATION METHODS FOR PSEUDOCONTRACTIVE AND ACCRETIVE OPERATORS IN ARBITRARY BANACH SPACES

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Abstract. 1. In [1] Owojori and Imoru [10] introduced a three-step iteration procedure and established some convergence results.

2. In this work, a more acceptable revised three-step iteration scheme is introduced as a generalization of the Ishikawa and Mann iteration schemes with errors given by Liu [11] and Xu [16]. Some new fixed point results are then established which improve the results of Owojori and Imoru [13] and are generalizations of the results of Ishikawa(1974), Chidume [1, 2, 3, 4], Liu [11], Xu [16], and Chidume and Osilike [5, 6] on fixed points (solutions) of pseudocontractive operators (accretive operator equations) in arbitrary Banach spaces

1. INTRODUCTION

In the last four decades, researchers have been investigating fixed points of nonlinear operators with one-step and two-step iteration schemes, for which the Mann and iteration schemes have been prominent . Liu [8] introduced the Ishikawa and Mann iteration schemes with errors as a generalization of the Mann and Ishikawa iteration schemes . He defined the Ishikawa scheme with errors iteratively for arbitrary $x_1 \in K$ by:

$$\left.\begin{array}{l} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n \\ y_n(1-\beta_n)x_n + \beta_n T x_n + v_n \end{array}\right\} n \geq 1 \tag{1.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences in [0,1] ,
and $\sum \|u_n\| < \infty$ and $\sum \|v_n\| < \infty$.
This was used by many researchers to approximate solutions of nonlinear operator equations for various contractive mappings. However, it was observed by some authors that the condition that the error terms be absolutely summable is rather too restrictive. Xu[15] later introduced a more acceptable Ishikawa iteration schemes with errors which he defined for an arbitrary $x_0 \in K$ - a nonempty convex subset of a normed space X by :

$$\left.\begin{array}{l} x_{n+1} = a_n x_n + b_n T y_n + c_n u_n \\ y_n a_n^{'} x_n + b_n^{'} T x_n + c_n^{'} v_n \end{array}\right\} n \geq 1 \tag{1.2}$$

where T is a selfmapping of K. $\{u_n\}$, $\{v_n\}$ are bounded sequences in K and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n^{'}\}$, $\{b_n^{'}\}$, $\{c_n^{'}\}$, are sequences in [0,1], satisfying

$$a_n + b_n + c_n = 1 = a_n^{'} + b_n^{'} + c_n^{'}, \quad \text{and} \quad \sum b_n = 0, \qquad \forall\, n \geq 1 .$$

The scheme (1.2) became more acceptable than (1.1) because of the less restrictive conditions on the error terms. However, in either case, when $b_n^{'} = c_n^{'} = 0$ for all $n \geq 0$ then the resulting scheme is called the Mann iteration scheme with errors. In particular, (1.2) reduces to

$$x_{n+1} = a_n x_n + b_n T x_n + c_n u_n \tag{1.3}$$

2. A GENERALIZED ISHIKAWA TYPE ITERATION SCHEME

We now consider a new improved iteration scheme which is a three-step iteration scheme, given by the following :

Definition 2.1 Let K be a nonempty compact and convex subset of a Banach space B. For arbitrary $x_1 \in K$, define sequence $\{x_n\}$ iteratively by :

$$\left.\begin{array}{r}x_{n+1} = a_n x_n + b_n T y_n + c_n S x_n \\ y_n = a_n^{'} x_n + b_n^{'} S z_n + c_n^{'} v_n \\ z_n a_n^{''} x_n + b_n^{''} T x_n + c_n^{''} \omega_n\end{array}\right\} n \geq 1 \tag{2.1}$$

where S, T are uniformly continuous self-mappings of K satisfying some contractive definitions, $\{v_n\}$, $\{\omega_n\}$ are arbitrary sequences in K and $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$, $\{b_n\}$, $\{b_n^{'}\}$, $\{b_n^{''}\}$, $\{c_n$, $\{c_n^{'}\}$, $\{c_n^{''}\}$, are real sequences in $[0, 1]$ satisfying

(i) $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{''} + b_n^{''} + c_n^{''} = 1$,

(ii) $\sum b_n = \infty$.

The scheme (2.1) is called the generalized Ishikawa type iteration scheme with errors.

Remark 2.2 When $S = T$ in the (2.1), we obtain a version given by :

$$\left.\begin{array}{l}x_{n+1} = a_n x_n + b_n T y_n + c_n T x_n \\ y_n = a_n^{'} x_n + b_n^{'} T z_n + c_n^{'} v_n \\ z_n = a_n^{''} x_n + b_n^{''} T x_n + c_n^{''} \omega_n\end{array}\right\} n \geq 1 \tag{2.2}$$

which is contained in (2.1) as a special case.

Since S is uniformly continuous and K is convex, then we can always find an $u_n \in K$ such that $Sx_n = u_n$. In this case, (2.1) reduces to :

$$\left.\begin{array}{l}x_{n+1} = a_n x_n + b_n T y_n + c_n u_n \\ y_n = a_n^{'} x_n + b_n^{'} S z_n + c_n^{'} v_n \\ z_n = a_n^{''} x_n + b_n^{''} T x_n + c_n^{''} \omega_n\end{array}\right\} n \geq 1 \tag{2.1a}$$

Now letting $S = T$ in (2.1a), we obtain the following

$$\left.\begin{array}{l}x_{n+1} = a_n x_n + b_n T y_n + c_n u_n \\ y_n = a_n^{'} x_n + b_n^{'} T z_n + c_n^{'} v_n \\ z_n = a_n^{''} x_n + b_n^{''} T x_n + c_n^{''} \omega_n\end{array}\right\} n \geq 1 \tag{2.3}$$

We observe that the iteration schemes (2.1), (2.2) and (2.3) are well defined and are generalizations of the Mann and Ishikawa types iteration schemes with errors in the sense of Liu[8] and Xu[15]. This is evident by specialising some of the parameters. Indeed, when $c_n^{''} = b_n^{''} = 0$, then (2.3) reduces to:

$$\left.\begin{array}{l} x_{n+1} = a_n x_n + b_n T y_n + c_n \omega_n \\ y_n a_n^{'} x_n + b_n^{'} T x_n + c_n^{'} u_n \end{array}\right\} n \geq 1$$

which is the Ishikawa iteration scheme with errors in the sense of Xu [16]. Furthermore, if in addition, $c_n = c_n^{'} = 0$, then it becomes the Ishikawa iteration scheme. When $b_n^{''} c_n^{''} = 0$ and $c_n^{'} = b_n^{'} = 0$, then (2.3) will reduce to the Mann iteration scheme with errors in the sense of Xu [16]. Hence the revised generalized Ishikawa type iteration scheme (2.1) and its special cases (2.2) and (2.3) include the Mann and Ishikawa iteration schemes with errors in the sense of Liu [11] and Xu [16] as special cases.

3. FIXED POINTS OF PSEUDOCONTRACTIVE OPERATORS

In this section we establish the convergence of the revised three-step Ishikawa type iteration scheme (2.3) to the fixed point of uniformly continuous and strongly pseudocontractive operators in arbitrary Banach spaces. Then the convergence of the slightly more general scheme (2.2) to solution of the uniformly continuous and strongly accretive operator equation $Tx = f$, for a given $f \in K$, is established.
Let K be a nonempty subset of Banach space B. An operator $T : D(T) \to B$, where $D(T)$ is a proper subset of a Banach space B, is called pseudo-contractive if for all $r > 0$, the inequality

$$\|x - y\| \;\leq\; \|(1 + r)(x - y) - r(Tx - Ty)\| \tag{3.1}$$

holds for each pair of points $x, y \in D(T)$.

Also, an operator T with domain $D(T)$ and range $R(T)$ in a real Banach space B is called accretive if the inequality

$$\|x - y\| \leq \|x - y + r(Tx - Ty)\| \tag{3.2}$$

holds for each pair of points $x, y \in D(T)$ and for all $r > 0$.
An operator T is strongly pseudocontrctive (strongly accretive), if there exists a real number $k \in (0,1)$ such that $(T - kI)$ is pseudocontractive (accretive).

In the sequel, we shall require the following result:

Lemma 3.1 (L. Qihou [14]) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers satisfying the following inequality:

$$x_{n+1} \ \leq \ \omega x_n \ + \ \sigma_n, \qquad n \geq 1 \tag{3.3}$$

where $x_n \geq 0, \quad \sigma_n \geq 0$ and $\lim\limits_{n\to\infty} \sigma_n = 0, \quad 0 \leq \omega < 1$. Then $x_n \to 0$, as $n \to \infty$.

Our main result is the following:

Theorem 3.2 Let K be a nonempty closed bounded and convex subset of an arbitrary real Banach space B and suppose T is a uniformly continuous and strongly pseudocontractive self-mapping of K. Define sequence $\{x_n\}$ iteratively for arbitrary $x_1 \in K$ by:

$$\left.\begin{array}{l} x_{n+1} = a_n x_n + b_n T y_n + c_n u_n \\ y_n = a_n^{'} x_n + b_n^{'} T z_n + c_n^{'} v_n \\ z_n = a_n^{''} x_n + b_n^{''} T x_n + c_n^{''} \omega_n \end{array}\right\}$$

where $\{u_n\}$, $\{v_n\}$ and $\{\omega_n\}$ are bounded sequences in K and $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$, $\{b_n\}$,$\{b_n^{'}\}$, $\{b_n^{''}\}$, $\{c_n\}$, $\{c_n^{'}\}$, $\{c_n^{''}\}$, are real sequences in $[0,1]$ satisfying:
(i) $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{''} + b_n^{''} + c_n^{''} = 1$,
(ii) $\sum b_n = \infty$,
(iii) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} b_n^{'} \lim\limits_{n\to\infty} b_n^{''} = 0$,
(iv) $\alpha_n := b_n + c_n, \beta_n := b_n{'} + c_n{'}, \gamma_n : b_n" + c_n"$,
$\lim\limits_{n\to\infty} \frac{1}{1+k\alpha_n} = 0, \quad \text{for all} \quad k \in (0,1)$
Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. The existence of the unique fixed point of T follows from Deimling [5]. Let x^* be the unique fixed point of T, i.e. $Tx^* = x^*$. The operator T is strongly pseudocontractive implies that $(I-T)$ is strongly accretive and therefore $(I-T)-kI = (I-T-kI)$ is accretive. Therefore, for all $r > 0$ and $k \in (0,1)$, we have

$$\|x-y\| \leq \|x-y+r[(I-T-kI)x-(I-T-kI)y] \tag{3.4}$$

From our hypothesis, we obtain the following estimates.

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + b_nTy_n + c_nu_n \\ & x_n - \alpha_nx_n + b_nTy_n + c_nu_n \end{aligned}$$

So that

$$x_n = x_{n+1} + \alpha_nx_n - b_nTy_n - c_nu_n \tag{3.5}$$

From (3.5), we have

$$\begin{aligned} x_n &= x_{n+1} + \alpha_nx_n - b_nTy_n - c_nu_n \\ &= x_{n+1} + \alpha_nx_{n+1} - \alpha_nx_{n+1} + \alpha_nx_n - b_nTy_n - c_nu_n \\ &= (1+\alpha_n)x_{n+1} + \alpha_n(I-T-kI)x_{n+1} - \alpha_n(I-T-kI)x_{n+1} \\ &- \alpha_n(x_{n+1}-x_n) - b_nTy_n - c_nu_n \\ &= (1+\alpha_n)x_{n+1} + \alpha_n(I-T-kI)x_{n+1} - \alpha_n(I-kI)x_{n+1} \\ &+ \alpha_nTx_{n+1} + \alpha_n(x_n - x_{n+1}) - b_nTy_n - c_nu_n \end{aligned} \tag{3.6}$$

Since x^* is a fixed point of T, we can also write

$$x^* = (1+\alpha_n)x^* + \alpha_n(I-T-kI)x^* - \alpha_n(1-k)x^* \tag{3.7}$$

Subtracting (3.7) from (3.6) yields

$$\begin{aligned} x_n - x^* &= (1+\alpha_n)(x_{n+1}-x^*) + \alpha_n[(I-T-kI)x_{n+1} - (I-T-kI)x^*] \\ &- \alpha_n(1-k)(x_{n+1}-x^*) + [\alpha_n(T-I)x_{n+1} - b_nTy_n] \\ &+ [\alpha_nx_n - c_nu_n] \end{aligned} \tag{3.8}$$

Therefore,

$$\begin{aligned} \|x_n - x^*\| &= \|(1+\alpha_n)[(x_{n+1}-x^*) + \tfrac{\alpha_n}{1+\alpha_n}\{(I-T-kI)x_{n+1} - (I-T-kI)x^*\}] \\ &- \alpha_n(1-k)(x_{n+1}-x^*) + [\alpha_n(T-I)x_{n+1} - c_nu_n] \\ &+ [\alpha_nx_n - b_nTy_n]\| \\ &\geq (1+\alpha_n)\|[x_{n+1} - x^* + \tfrac{\alpha_n}{1+\alpha_n}\{(I-T-kI)x_{n+1} - (I-T-kI)x^*\}] \\ &- \alpha_n(1-k)\|x_{n+1}-x^*\| - \|\alpha_n(T-I)x_{n+1} - c_nu_n\| \\ &- \|\alpha_nx_n - b_nTy_n\| \end{aligned} \tag{3.9}$$

But T is strongly pseudocontractive, then (3.9) yields

$$\begin{aligned}\|x_n - x^*\| &\geq (1+\alpha_n)\|x_{n+1} - x^*\| - \ \alpha_n(1-k)\|x_{n+1} - x^*\| \\ &-\|\alpha_n(T-I)x_{n+1} - c_n u_n\| \\ &- \ \|\alpha_n x_n - b_n T y_n\| \\ &(1+k\alpha_n)\|x_{n+1} - x^*\| - \|\alpha_n(T-I)x_{n+1} - c_n u_n\| - \|\alpha_n x_n - b_n T y_n\|\end{aligned}$$

Therefore,

$$\|x_{n+1}-x^*\| \leq \frac{1}{1+k\alpha_n}[\|x_n-x^*\|+\|\alpha_n(T-I)x_{n+1}-c_n u_n\|+\|\alpha_n x_n-b_n T y_n\|] \quad (3.10)$$

Since T is uniformly continuous on the bounded set K, there exists a positive real number $M < \infty$ such that
$\|\alpha_n(T-I)x_{n+1} - c_n u_n\| \leq M/2$ and $\|\alpha_n x_n - b_n T y_n\| \leq M/2$. Thus (3.10) reduces to

$$\|x_{n+1} - x^*\| \leq \frac{1}{1+\alpha_n k}\|x_n - x^*\| + \frac{1}{1+\alpha_n k}M \quad (3.11)$$

Now, put $\delta_n \frac{1}{1+\alpha_n k}$, $\quad \sigma_n = \delta_n M \quad$ and

$$\rho_n = \|x_n - x^*\|$$

Then (3.11) reduces to

$$\rho_{n+1} \leq \delta_n \rho_n + \sigma_n$$

Clearly, $0 \leq \delta_n < 1$ and $\lim_{n\to\infty} \sigma_n = 0$ since $\lim_{n\to\infty} \delta_n = 0$. Therefore, by Lemma 3.1 (Qihou [11]), we have

$$\lim_{n\to\infty} \rho_n = 0$$

which implies that the sequence $\{x_n\}$ converges strongly to x^*. This completes the proof.

Remark. Theorem 3.2 above is a genralization of Theorem 1 of Ishikawa [9], Chidume [1, 2] and Chidume and Osilike [3, 4], Ishikawa [7] and others to the more general Ishikawa type iteration scheme and also to continuous pseudocontractive operators in arbitrary real Banach spaces.

Corollary 3.3 Let B be a Banach space and K a nonempty closed bounded convex subset of B. Suppose T is a continuous pseudocontractive selfmapping of K and define sequence $\{x_n\}$ iteratively for arbitrary $x_1 \in K$ by

$$\left.\begin{array}{l} x_{n+1} = a_n x_n + b_n T y_n + c_n T x_n \\ y_n = a_n^{'} x_n + b_n^{'} T z_n + c_n^{'} u_n \\ z_n = a_n^{''} x_n + b_n^{''} T x_n + c_n^{''} v_n \end{array}\right\} n \geq 1$$

where $\{u_n\}$, $\{v_n\}$ are bounded sequences in K and $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$, $\{b_n\}$,$\{b_n^{'}\}$, $\{b_n^{''}\}$, $\{c_n$, $\{c_n^{'}\}$, $\{c_n^{''}\}$, are real sequences in $[0,1]$ satisfying the following conditions:

$$\begin{array}{l} (i) a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{''} + b_n^{''} + c_n^{''} = 1, \\ (ii) \sum b_n = \infty \\ (iii) \alpha_n := b_n + c_n, \beta_n = b_n^{'} + c_n^{'}, \gamma_n = b_n^{''} + c_n^{''}. \end{array}$$

Then the sequence $\{x_n\}$ converges strongly to the fixed point of T.

Proof. The proof follows directly by following exactly the same procedure as in the proof of the Theorem (with u_n replaced by Tx_n).

4. APPROXIMATION OF SOLUTIONS OF ACCRETIVE OPERATOR EQUATIONS

In this section, the convergence of the slightly more general Ishikawa type iteration scheme to the solution of accretive operator equations in Banach spaces is established. Our main result here is the following

Theorem 4.1 Let T be uniformly continuous and strongly accretive selfmapping of a closed convex bounded subset K of an arbitrary real Banach space B . Define a mapping $R : K \to K$ by $Rx = x - Tx + f$ for some $f \in B$. Consider the sequence $\{x_n\}$ defined iteratively for arbitrary $x_1 \in K$ by

$$\left.\begin{array}{l} x_{n+1} = a_n x_n + b_n R y_n + c_n T x_n \\ y_n = a_n^{'} x_n + b_n^{'} R z_n + c_n^{'} v_n \\ z_n a_n^{''} x_n + b_n^{''} T x_n + c_n^{''} \omega_n \end{array}\right\} n \geq 1 \tag{4.1}$$

where $\{u_n\}$, $\{v_n\}$ are bounded sequences in K and $\{a_n\}$, $\{a_n^{'}\}$, $\{a_n^{''}\}$, $\{b_n\}$, $\{b_n^{'}\}$, $\{b_n^{''}\}$, $\{c_n\}$, $\{c_n^{'}\}$, $\{c_n^{''}\}$, are real sequences in $[0,1]$ satisfying the following conditions

(i) $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = a_n^{''} + b_n^{''} + c_n^{''} = 1,$

(ii) $\sum b_n = \infty,$

(iii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_n^{'} \lim_{n\to\infty} b_n^{''} = 0$

(iv) $\alpha_n := b_n + c_n, \beta_n := b_n{'} + c_n{'}, \gamma_n := b_n" + c_n"$

and $\lim_{n\to\infty} \frac{1}{1+k\alpha_n} = 0, \quad$ for all $\quad k \in (0,1)$

Then the sequence $\{x_n\}$ converges strongly to the solution of the equation $Tx = f$.

Proof. The existence of the unique solution of the equation $Tx = f$ follows from Deimling [7]. Let p be the solution. Then from the definition of R, p is a fixed point of R. We observe that R, T are uniformly continuous and for any given $f \in K$,

$$(I - R)x = x - f + Tx - x = Tx - f$$

Since T is strongly accretive, it follows that $T - kI$ is accretive. Therefore, for all $k \in (0,1)$ and $x, y \in K$,

$$\|x - y\| \leq \|x - y + r[(T - kI)x - (T - kI)y]\|$$

for all $r > 0$. Thus, we have

$$\begin{aligned}
&\|x - y + r[(I - R - kI)x - (I - R - kIy)]\| \\
&= \|x - y + r[(I - R)x - kIx - (I - R)y - kIy]\| \\
&= \|x - y + r[(Tx - f) - kIx - (Ty - f) - kIy] \\
&= \|x - y + r[(T - kI)x - (T - kI)y] \\
&\geq \|x - y\|
\end{aligned}$$

Hence, the

inequality

$$\|x - y\| \leq \|x - y + r[(I - R - kI)x - (I - R - kIy)]\|y \tag{4.2}$$

holds when T is strongly accretive.

Following the same procedure as in the proof of Theorem 3.2, we have

$$\begin{aligned}
x_{n+1} &= a_n x_n + b_n R y_n + c_n T x_n \\
&= (1 - \alpha_n) x_n + b_n R y_n + c_n T x_n \\
&= x_n - \alpha_n x_n + b_n R y_n + c_n T x_n
\end{aligned}$$

Therefore,

$$\begin{aligned} x_n &= x_{n+1} + \alpha_n x_n - b_n R y_n - c_n T x_n \\ &= x_{n+1} + \alpha_n x_{n+1} + {\alpha_n}^2 x_n - \alpha_n b_n R y_n - \alpha_n c_n T x_n \\ &- b_n R y_n - c_n T x_n \\ &= (1+\alpha_n) x_{n+1} + \alpha_n (I - R - kI) x_{n+1} - \alpha_n (I - kI) x_{n+1} \\ &+ \alpha_n R x_{n+1} + {\alpha_n}^2 x_n - (1+\alpha_n) b_n R y_n - (1+\alpha_n) c_n T x_n \end{aligned} \tag{4.3}$$

where k is a real constant in $(0, 1)$.

Since p is a fixed point of R, we can also write

$$p = (1+\alpha_n)p + \alpha_n(I - R - kI)p - \alpha_n(I - kI)p$$

for all $k \in (0, 1)$. Then, we have

$$\begin{aligned} x_n - p &= (1+\alpha_n)(x_{n+1} - p) + \alpha_n(I - R - kI)(x_{n+1} - p) \\ &- \alpha_n(I - kI)(x_{n+1} - p) + \alpha_n R x_{n+1} + {\alpha_n}^2 x_n \\ &-(1+\alpha_n) b_n R y_n - (1+\alpha_n) c_n T x_n \\ &= (1+\alpha_n)[(x_{n+1} - p) - \tfrac{\alpha_n}{1+\alpha_n}(I - R - kI)(x_{n+1} - p)] \\ &- \alpha_n(I - kI)(x_{n+1} - p) + [{\alpha_n}^2 x_n - (1+\alpha_n) c_n T x_n] \\ &+ [\alpha_n R x_{n+1} - (1+\alpha_n) b_n R y_n] \\ &\geq (1+\alpha_n)\|(x_{n+1} - p) - \tfrac{\alpha_n}{1+\alpha_n}(I - R - kI)(x_{n+1} - p)\| \\ &- \alpha_n(I - kI)\|x_{n+1} - p\| - \|{\alpha_n}^2 x_n - (1+\alpha_n) c_n T x_n\| \\ &- \|\alpha_n R x_{n+1} - (1+\alpha_n) b_n R y_n]\| \\ &\geq (1+\alpha_n)\|x_{n+1} - p\| - \alpha_n(I - kI)\|x_{n+1} - p\| \\ &- \|{\alpha_n}^2 x_n - (1+\alpha_n) c_n T x_n\| - \|\alpha_n R x_{n+1} - (1+\alpha_n) b_n R y_n\| \end{aligned} \tag{3.15}$$

But $b_n \leq 1$ and $c_n \leq 1$. Therefore

$$\begin{aligned} \|x_n - p\| &\geq (1+\alpha_n)\|x_{n+1} - p\| - \alpha_n(I - kI)\|x_{n+1} - p\| \\ &- \|{\alpha_n}^2 x_n - (1+\alpha_n) T x_n\| - \|\alpha_n R x_{n+1} - (1+\alpha_n) R y_n\| \\ &= (1+\alpha_n k)\|x_{n+1} - p\| - \|{\alpha_n}^2 x_n - (1+\alpha_n) T x_n\| \\ &- \|\alpha_n R x_{n+1} - (1+\alpha_n) R y_n\| \end{aligned} \tag{4.4}$$

This implies that,

$$\begin{aligned} (1+\alpha_n k)\|x_{n+1} - p\| &\leq \|x_n - p\| + \|{\alpha_n}^2 x_n - (1+\alpha_n) T x_n\| \\ &+ \|\alpha_n R x_{n+1} - (1+\alpha_n) R y_n\| \end{aligned}$$

Therefore,

$$\begin{aligned} \|x_{n+1} - p\| &\leq \tfrac{1}{1+\alpha_n k}\|x_n - p\| + \tfrac{1}{1+\alpha_n k}[\|{\alpha_n}^2 x_n - (1+\alpha_n) T x_n\| \\ &+ \|\alpha_n R x_{n+1} - (1+\alpha_n) R y_n\|] \end{aligned} \tag{4.5}$$

Now, let $\delta_n = \frac{1}{1+\alpha_n k}$. Observe that the continuity of R and T on the bounded set K implies that there exists a real numbers $M < \infty$ such that,
$\|{\alpha_n}^2 x_n - (1+\alpha_n)Tx_n\| \leq M$ and $\|\alpha_n R x_{n+1} - (1+\alpha_n)Ry_n\| \leq M$.
Substituting into (4.5) gives

$$\|x_{n+1} - p\| \leq \delta_n \|x_n - p\| + 2\delta_n M \tag{4.6}$$

Put $\rho_n = \|x_n - p\|$ and $\sigma_n = 2\delta_n M$ in (4.6), we have

$$\rho_{n+1} \ \leq \ \delta_n \rho_n \ + \ \sigma_n$$

It is clear that $0 \leq \delta_n \leq 1$ and $\sigma_n = o(\delta_n)$.
Also, $\lim\limits_{n\to\infty} \sigma_n = 0, \quad$ since $\lim\limits_{n\to\infty} \delta_n = 0$.
Hence, by Lemma 3.1, we have

$$\lim_{n\to\infty} \rho_n = 0$$

which implies that the sequence $\{x_n\}$ converges strongly to p, the unique solution of the operator equation $Tx = f$. This completes the proof.

Remark 4.2 Theorem 4.1 is clearly an extension of the related results of Chidume [3], Chidume and Osilike [5], Liu [11], and Xu [16] to the more general Ishikawa type iteration sheme with errors. Furthermore, this result is also valid for the iteration scheme (3.6) above.

5. GENERALIZED MANN ITERATION SCHEME IN BANACH SPACES

Consider the following iteration procedure for two nonlinear operators in Banach spaces.

Definition 5.1 Let K be a nonempty compact convex subset of an arbitrary Banach space and suppose $T : K \to K$ and $S : K \to K$ are uniformly continuous, nonlinear operators. Define sequence $\{x_n\}$ for arbitrary $x_1 \in K$ by

$$x_{n+1} = a_n x_n + b_n T x_n + c_n S x_n, n \geq 1 \tag{5.1}$$

where $\{a_n\}, \ \{b_n\}, \ \{c_n\}$ are real sequences in [0,1] satisfying
$a_n + b_n + c_n = 1$ and $\sum b_n = \infty$.

The sequence $\{x_n\}$ generated by (5.1) is called the generalized Mann type iteration procedure.

Considering the generalized Mann iteration scheme, we have the following results.

Theorem 5.2 Let K be a nonempty closed bounded convex subset of an arbitrary real Banach space B and T a uniformly continuous strongly pseudocontractive selfmapping of K. Define sequence $\{x_n\}$ iteratively for arbitrary $x_1 \in K$ by

$$x_{n+1} = a_n x_n + b_n T x_n + c_n S x_n. \qquad n \geq 1 \tag{5.2}$$

where S is uniformly continuous selfmapping of K, $\{a_n\}, \ \{b_n\}, \ \{c_n\}$ are real sequences in [0,1] satisfying
(i) $a_n + b_n + c_n = 1$ and $\sum b_n = \infty$
(ii) $\alpha_n := b_n + c_n$ and $\lim_{n\to\infty} \frac{1}{1+k\alpha_n} = 0$, for all $k \in (0,1)$
Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. The proof follows directly by replacing u_n with Sx_n in the proof of Theorem 3.2.

Theorem 5.3 Let K, B, S be as in Theorem 5.2 above. Define a mapping $R : K \to K$ by

$$Rx = x - Tx + f$$

for a given $f \in B$, where T is a uniformly continuous strongly accretive selfmapping of K. Define sequence $\{x_n\}$, defined iteratively for arbitrary $x_1 \in K$, by

$$x_{n+1} = a_n x_n + b_n R x_n + c_n S x_n. \qquad n \geq 1 \tag{5.2}$$

where $\{a_n\}, \ \{b_n\}, \ \{c_n\}$ are real sequences in [0,1] satisfying
(i) $a_n + b_n + c_n = 1$ and $\sum b_n = \infty$
(ii) $\alpha_n := b_n + c_n$ and $\lim_{n\to\infty} \frac{1}{1+k\alpha_n} = 0$, for all $k \in (0,1)$
Then the sequence $\{x_n\}$ converges strongly to the solution of the equation $Tx = f$.

Proof. The proof follows directly by replacing Tx_n with Sx_n in the proof of Theorem 4.1.

Remark 5.4 It is clear that Theorem 2.5.2 is a generalization of the results of Chidume [2] and Schu [15] on fixed points of pseudocontractive operators in Banach spaces. Also, Theorem 5.3 is also a generalization of previous results on solutions of accretive operator equations by Mann iteration procedures.

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