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INVESTIGATION OF STABILITY ON NONLINEAR OSCILATIONS WITH ONE DEGREE OF FREEDOM

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Abstract. This paper presents a phase portrait of differential equation of vehicle vertical oscillations when establishing force in springs is changed under nonlinear laws:

$$F_c = c(x + \alpha x^2) \quad \text{and} \quad F_c = c(x + \alpha x^3), \quad c > 0,\ \alpha \neq 0.$$

Analysis of nonlinear oscillation stability has been done using the phase portrait analyses.

The differential equation of vehicle vertical oscillations with one degree of freedom when on the vehicle suspension mass m is acted the F_c establishing force spring that is changed under the law:

$$F_c = c(x + \alpha x^2), \quad c > 0,\ \alpha \neq 0,$$

where x is an generated coordinate of the following form:

$$mx'' + c(x + \alpha x^2) = 0, \tag{1}$$

where $m > 0,\ c > 0,\ \alpha \neq 0$ are real parameters.

The differential equation (1) is equal to the following system of equations

$$\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= -\frac{c}{m}(x + \alpha x^2)\,.\end{aligned} \tag{2}$$

The differential equation of the system phase trajectory (2) is:

$$ydy = -\frac{c}{m}(x + \alpha x^2)\,dx.$$

The phase trajectories are integral curves of the differential equation. After integration within the limits from t_0 to t with starting conditions $x(t_0) = x_0,\ y(t_0) = y_0$ we reach the following ([3]):

$$\Pi(x) = \frac{c}{6m}(3x^2 + 2\alpha x^3) \quad \text{and} \quad h = \frac{1}{2}y_0 + \frac{c}{6m}(3x_0^2 + 2\alpha x_0^3).$$

The phase trajectories are the following curves:

$$y = \pm\sqrt{2(h - \Pi(x))} \quad \text{or} \quad y = \pm\sqrt{2\left(h - \frac{c}{6m}(3x^2 + 2\alpha x^3)\right)}\,. \tag{3}$$

Draw the graphic of functions (3) and test the stability of vehicle motion in the case of rigid springs when $\alpha > 0$ (if $\alpha = 0.1$), and in the case of soft springs when $\alpha < 0$ (if $\alpha = -0.1$).

Theorem 1. *For the equation* (1) *the following is taken:*

(I) *For* $\alpha = 0.1$, *position of the systems equilibrium are points* $(0, 0)$ *and* $(-10, 0)$.

For $x = 0$ *the function* $\Pi(x)$ *has its minimum, it means that the point* $(0, 0)$ *is a center.*

For $x = -10$ *the function* $\Pi(x)$ *has its maximum, it means that the point* $(-10, 0)$ *is a saddle (fig. 1)*

(II) *For* $\alpha = -0.1$ *the equilibrium positions are the points* $(0, 0)$ *and* $(10, 0)$.

For $x = 0$ *the function* $\Pi(x)$ *has its minimum, it means that the point* $(0, 0)$ *is a center.*

For $x = 10$ *the function* $\Pi(x)$ *has its maximum, so the point* $(10, 0)$ *is a saddle (fig. 2).*

fig. 1

a) Change of pot. energy of rigid springs

b) Phase trajectories

fig. 2

a) Change of pot. energy of soft springs

b) Phase trajectories

For $\alpha > 0$ and $\alpha < 0$ there are two characteristic points, center O and saddle A. The system motion is stable at center O environment only, i.e. for the starting value of potential energy $\Pi(x_0) = h_0 = 0$.

For $h > h_0$ the phase point moves along the phase trajectories that are moved away from the equilibrium positions being suitable to unstable oscillatory vehicle motion.

2) The differential equation of vertical vehicle oscillations with one degree of freedom, when F_c spring establishing force affects the suspension mass m of the vehicle that changes under the law:

$$F_c = c(x + \alpha x^3), \quad c > 0,\ \alpha \neq 0$$

it reaches the following

$$mx'' + c(x + \alpha x^3) = 0. \tag{4}$$

Define the phase trajectories if $\alpha > 0$ (rigid springs) and if $\alpha < 0$ (soft springs).

The potential energy:

$$\Pi(x) = \frac{c}{4m}(2x^2 + \alpha x^4).$$

The phase trajectories are curves

$$y = \pm\sqrt{2\left(h - \frac{c}{4m}(2x^2 + 2\alpha x^4)\right)}.$$

If $\alpha > 0$ ($\alpha = 0.1$) the equilibrium position $O(0,0)$ is a center. The phase trajectories are ellipses. Starting value of potential energy is $\Pi(x_0)$ $h(x_0) : \Pi(x_0) = h_0 = 0$ (fig. 3).

fig. 3

a) Change of potential energy of rigid springs

b) Phase trajectories

The theorem is acceptable as follows:

Theorem 2. *For $\alpha > 0$ the differential equation* (4) *has a stable periodical solution.*

The vehicle oscillatory motion at arbitrary starting conditions is stable.

(II) Now look at the vehicle oscillatory motion for soft springs ($\alpha = -0.1$). The points $(0,0), (-\sqrt{10}, 0), (\sqrt{10}, 0)$ – are equilibrium positions of the system. The curve $\Pi(x)$ has an extreme values for $x = 0, x = -\sqrt{10}$ and $x = \sqrt{10}$. For $x = 0$ it has a minimum, for $x = -\sqrt{10}$ and $x = \sqrt{10}$ it has a maximum h_0.

The point $O(0,0)$ – is center, and points $A_1(-\sqrt{10},0),\ A_2(\sqrt{10},0)$ – are saddles (fig. 4).

fig. 4

a) Change of potential energy of soft springs

b) Phase trajectories

The point O is suitable for oscillatory vehicle motion. Points A_1 and A_2 are suitable for four half-trajectories. For values h being close to h_0 values, integral curves are similar to hyperbola form. Around the points A_1 and A_2 the phase plane is divided into four parts. Look at the point A_1.

If $h > h_0$ the phase trajectories are at upper I range and lower IV range. In the first case we have a stable periodical solution that is suitable to free harmonic vehicle oscillations. If the phase point is close to the equilibrium position and if it is on II and IV phase trajectories, it will asymptotically become closer to the equilibrium position. If the phase point is on the I and III phase trajectories branches and on the phase trajectories of hyperbole type left to the point A_1 – it will be moved away from the equilibration position. Unstable equilibration position of the system is suitable for the point A_1. If the starting potential energy has value of h_1, that crosses the curve $\Pi(x)$ nowhere nor concerns it, then the phase trajectories consist of two symmetric curves relating to the x axis that are moved away from both sides to the infinity. Due to the symmetry, the same could be concluded for the point A_2.

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