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GRAPHS WITH SMALLEST SUM OF SQUARES OF VERTEX DEGREES

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Abstract. Graphs with n vertices, m edges, and with smallest sum of squares of the vertex degrees are characterized. These are the graphs in which the degree of any vertex is equal to either $\lfloor 2m/n \rfloor$ or $\lceil 2m/n \rceil$. Such graphs exist for all $n \geq 1$ and $0 \leq m \leq n(n-1)/2$.

INTRODUCTION

Motivated by a recent work of Das [1] we present here an alternative way for characterizing graphs in which the sum of the squares of the vertex degrees is minimum.

In this paper we are concerned with simple graphs, possessing no multiple or directed edges. Let G be such a graph, and let it possess n vertices and m edges. The considerations in this paper apply to any value of n and m, i. e., to $n \geq 1$ and $0 \leq m \leq n(n-1)/2$. It is, however, assumed that the values of n and m are fixed.

Graphs possessing n vertices and m edges will be referred to as (n,m)-graphs.

Let the vertices of the graph G be $v_1, v_2, \ldots, v_n$. The number of first neighbors of the vertex v_i is called the *degree* of this vertex and is denoted by δ_i, $i = 1, 2, \ldots, n$. We are interested in the sum of the squares of the vertex degrees, namely in the quantity

$$M = \sum_{i=1}^{n} (\delta_i)^2 \ . \tag{1}$$

For previous mathematical studies of M see the paper [1] and the references quoted therein.

Formula (1) should be compared with the well known relation

$$\sum_{i=1}^{n} \delta_i = 2m \tag{2}$$

which holds for any (n, m)-graph.

The starting point in our considerations is the identity

$$M = 2(a+b)m - a\,b\,n + \sum_{i=1}^{n} (a - \delta_i)(b - \delta_i) \tag{3}$$

which is satisfied by any real (or even complex!) numbers a and b.

In order to deduce (3) note that

$$(a - \delta_i)(b - \delta_i) = (\delta_i)^2 - (a+b)\delta_i + a\,b$$

sum it over all i and take into account (1) and (2).

We state now a simple result which implies the existence of the (below specified) (n, m)-graphs with minimum M.

Lemma 1. For any $n \geq 1$ and $0 \leq m \leq n(n-1)/2$ there exist (n, m)-graphs in which the degree of any vertex is either $\lfloor 2m/n \rfloor$ or $\lceil 2m/n \rceil$.

Proof: by construction. Start with a graph on n vertices without edges. Insert to it the edges, one–by–one, always connecting two nonadjacent vertices with smallest degrees. After adding m edges, a graph with the required property is obtained. $\square$

GRAPHS WITH MINIMUM M

We have to distinguish between two cases: when $2m/n$ is an integer and when it is not.

Theorem 2a. If $2m/n$ is an integer, then among all (n, m)-graphs, the regular graphs of degree $2m/n$ have minimum M.

Proof. The special case of relation (3), for $a = b$ is

$$M = 4a\,m - a^2\,n + \sum_{i=1}^{n}(a - \delta_i)^2\ . \tag{4}$$

Each term in the summation on the right–hand side of (4) is non-negative. Therefore the right–hand side of (4) will be minimal if all these terms are equal to zero, i. e., if $\delta_i = a$ holds for all i, i. e., if G is regular of degree a. Because of (2) it must be $n\,a = 2m$ i. e., $a = 2m/n$. $\square$

The case when $2m/n$ is not an integer is somewhat more complicated. Because $2m/n$ is the average vertex degree, we see that in this case not all vertex degrees can be mutually equal.

Theorem 2b. If $2m/n$ is not an integer, then among all (n, m)-graphs, the graphs in which the degree of any vertex is either $\lfloor 2m/n \rfloor$ or $\lceil 2m/n \rceil$ have minimum M.

Proof. Set in (3) $b = a + 1$ and choose a to be an integer. If $\delta_i < a$ or if $\delta_i > b = a+1$, then the term $(a - \delta_i)(b - \delta_i)$ is positive–valued. If $\delta_i = a$ or if $\delta_i = b$, then the term $(a - \delta_i)(b - \delta_i)$ is equal to zero. In view of this, if a is integer and $b = a + 1$, then the right–hand side of (3) will be minimal if $\delta_i = a$ or $\delta_i = a + 1$ holds for all i. Because the left–hand side of (3) is independent of the choice of the parameter a, we see that the condition $\delta_i = a$ or $\delta_i = a + 1$ for all i determines the graphs with minimum M.

Because $2m/n$ is the average vertex degree, it must be

$$a < \frac{2m}{n} < a + 1$$

i. e.,

$$\frac{2m}{n} - 1 < a < \frac{2m}{n}$$

i. e.,

$$a = \left\lfloor \frac{2m}{n} \right\rfloor \qquad \text{and} \qquad a + 1 = \left\lceil \frac{2m}{n} \right\rceil\ .$$

This completes the proof of Theorem 2b. $\square$

If $2m/n$ is an integer, then

$$\left\lfloor \frac{2m}{n} \right\rfloor = \left\lceil \frac{2m}{n} \right\rceil = \frac{2m}{n}$$

Bearing this in mind, we can state Theorems 2a and 2b jointly as:

Theorem 2. Among all (n, m)-graphs, for any $n \geq 1$ and $0 \leq m \leq n(n-1)/2$, the graphs in which the degree of any vertex is either $\lfloor 2m/n \rfloor$ or $\lceil 2m/n \rceil$ have minimum M.

Lemma 1 guarantees that graphs specified in Theorem 2 exist for all n and m.

Corollary 2.1. For any (n, m)-graph,

$$M \geq 2m \left(\left\lfloor \frac{2m}{n} \right\rfloor + \left\lceil \frac{2m}{n} \right\rceil \right) - \left\lfloor \frac{2m}{n} \right\rfloor \left\lceil \frac{2m}{n} \right\rceil n$$

with equality attained if and only if the graph has the property specified in Theorem 2.

Corollary 2.1 is, of course, equivalent to the bound reported by Das [1].

References

[1] K. C. Das, Sharp bounds for the sum of the squares of the degrees of a graph, Kragujevac J. Math. **25** (2003), 31–49, preceding paper.