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A GENERALIZATION OF THE GUMBEL DISTRIBUTION

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Abstract. We propose as a generalization of the Gumbel distribution, the asymptotic distribution of m-th extremes obtained by *(Gumbel,1934)*. Some of its properties are obtained. A t-approximation to its cumulative ditsribution function is also proposed.

1. INTRODUCTION

The probability density function of the Gumbel random variable also called the extreme value density Type I is given as

$$f(x) = e^{-x} \exp(-e^{-x}), \quad -\infty \ < \ x \ < \ \infty. \tag{0}$$

The corresponding characteristic function is given as

$$\Phi_x(t) = \Gamma(1 - it). \tag{1}$$

Recently, *(Ojo,2001)* obtained a generalization of the Gumbel distribution with probability density function, p.d.f,

$$g(y) \;=\; \frac{1}{\Gamma(k)} e^{-ky} \exp(-e^{-y}), \quad -\infty \;<\; y \;<\; \infty \tag{2}$$

where $k > 0$ is the shape parameter; and its characteristic function is given as

$$\Phi_Y(t) \;=\; \frac{\Gamma(k-it)}{\Gamma(k)}. \tag{3}$$

The author claimed that the distribution is a first generalization of the Gumbel distribution. In this paper, we propose as a generalization of the Gumbel distribution, the asymptotic distribution of the m-th extremes obtained by *(Gumbel,1934)* with density function

$$g(y) = \frac{m^m}{\Gamma(m)} e^{-my} \exp(-me^{-y}), \quad -\infty \;<\; y \;<\; \infty \tag{4}$$

For the purpose of this study, we shall deal with the distribution

$$f(y) = \frac{\lambda^\lambda}{\Gamma(\lambda)} e^{-\lambda y} \exp(-\lambda e^{-y}), \quad -\infty < y < \infty, \quad \lambda > \quad 0 \tag{5}$$

where $\lambda > 0$ is the shape parameter and when $\lambda = 1$, it reduces to (1.1). The corresponding characteristic function is given as

$$\Phi_{Y(t)} = \lambda^{it} \frac{\Gamma(\lambda - it)}{\Gamma(\lambda)}. \tag{6}$$

In section 2, we derive the moments of the distribution for various values of λ. In section 3, three characterizing theorems are proved. In the last section a t-approximation to the cumulative distribution function is proposed.

2. THE CUMULANTS

In what follows we derive the cumulants of the distribution. The cumulant generating function is given by

$$\ln M(t) = t\ln\lambda + \ln\Gamma(\lambda - t) - \ln\Gamma(\lambda)$$

The r-th cumulant is given by

$$\kappa_r = \frac{d^r}{dt^r}[t\ln\lambda]_{t=o} + \frac{d^r}{dt^r}[\ln\Gamma(\lambda - t)]_{t=0}$$

Using

$$\sum_{n=0}^{\infty}(n+m)^{-r} = \sum_{n=m}^{\infty} n^{-r} = \sum_{n=1}^{\infty} n^{-r} - \sum_{n=1}^{m-1} n^{-r}$$

Then the r-th cumulant becomes

$$\kappa_r = \frac{d^r}{dt^r}[t\ln\lambda]_{t=o} + (r-1)![\sum_{j=1}^{\infty} j^{-r} - \sum_{j=1}^{\lambda-1} j^{-r}$$

The above cumulants can be computed for positive integer values of λ only. In particular, the first four cumulants which will be subsequently used are given as

$$\kappa_1 = \ln\lambda + \gamma - \sum_{j=1}^{\lambda-1} j^{-1} \tag{7}$$

$$\kappa_2 = \frac{\pi^2}{6} - \sum_{j=1}^{\lambda-1} j^{-2} \tag{8}$$

$$\kappa_3 = 2[1.2021 - \sum_{j=1}^{\lambda-1} j^{-3}] \tag{9}$$

$$\kappa_4 = [\frac{\pi^4}{15} - \sum_{j=1}^{\lambda-1} j^{-4}] \tag{10}$$

where γ is the Euler's constant.

However, if $\lambda = m + \frac{1}{2}$, m being a positive integer, we can write the cumulants as

$$\kappa_r = (r-1)![(2^r - 1)\sum_{n=1}^{\infty} n^{-r} - 2^r \sum_{n=0}^{m-1}(2n+1)^{-r} \tag{11}$$

In particular, the first four cumulants for half plus positive integer are

$$\kappa_1 = \sum_{n=1}^{\infty} n^{-1} - 2\sum_{n=0}^{m-1}(2n+1)^{-1} + \ln(m + \frac{1}{2}) \tag{12}$$

$$\kappa_2 = \frac{\pi^2}{6} - 4\sum_{n=0}^{m-1}(2n+1)^{-2} \tag{13}$$

$$\kappa_3 = 14\sum_{n=1}^{\infty} n^{-3} - 8\sum_{n=0}^{m-1}(2n+1)^{-3} \tag{14}$$

$$\kappa_4 = \pi^4 - 16\sum_{n=0}^{m-1}(2n+1)^{-4} \tag{15}$$

The cumulants so far obtained can only be computed for positive integer and half plus positive integer λ. We hereby obtain approximations to the cumulants for $\lambda \in \Re^+$. By applying the Stirling's approximation to the gamma function and differentiating w.r.t t and putting $t = 0$, we have approximations to the first four cumulants as

$$\kappa_1 = \frac{1}{2\lambda} + \frac{1}{12\lambda^2} \tag{16}$$

$$\kappa_2 = \frac{1}{\lambda} + \frac{1}{2\lambda^2} + \frac{1}{6\lambda^3} \tag{17}$$

$$\kappa_3 = \frac{1}{\lambda^2} + \frac{1}{\lambda^3} + \frac{1}{2\lambda^4} \tag{18}$$

$$\kappa_4 = \frac{2}{\lambda^3} + \frac{3}{2\lambda^4} + \frac{1}{\lambda^5} \tag{19}$$

The approximation are quite close to the exact values as λ increases, as shown in the table in the appendix and can be used for application purposes, particularly when the cumulants are required for any positive value of λ.

3. CHARACTERIZATION

We prove three theorems that characterizes the generalized Gumbel distribution.

Theorem 3.1

The random variable $Y = -\ln X$ is generalized Gumbel with parameter λ if and only if X is an Erlang random variable with parameter λ

Proof of Theorem 3.1

The 'if' part is as follows. The density function of X is given as

$$f(x) = \frac{\lambda(\lambda x)^{\lambda-1}}{\Gamma(\lambda)}e^{-\lambda x}, \quad x > 0, \quad \lambda > 0$$

For $Y = -\ln X$, therefore,

$$f(y) = \frac{\lambda^\lambda}{\Gamma(\lambda)}e^{-\lambda y}e^{-\lambda e^{-y}}$$

Conversely, suppose $-\ln X$ has the generalized Gumbel distribution, but $f(x)$ is unknown. The moment generating function of y is given by

$$M_Y(t) = \frac{\lambda^t \Gamma(\lambda - t)}{\Gamma(\lambda)}$$

that is

$$E(e^{-\ln xt}) = \frac{\lambda^t \Gamma(\lambda - t)}{\Gamma(\lambda)}$$

That is

$$E(x^{-t}) = \int_{-\infty}^{\infty} x^{-t} f(x) dx$$

but $f(x)$ is unknown. The only function, $f(x)$, satisfying the above is

$$f(x) = \frac{\lambda(\lambda x)^{\lambda-1}}{\Gamma(\lambda)} e^{-\lambda x}$$

which is the p.d.f of an Erlang random variable with parameter λ.

Theorem 3.2

Let X_1 and X_2 be independent random variables with a common density. Then the random variable $Y = X_1 - X_2$ has the generalized logistic distribution with parameters α and β if X_1 and X_2 each has the generalized Gumbel distribution.

Proof of Theorem 3.2

Suppose X_1 and X_2 are independent with density functions

$$h_1(x_1) = \frac{\alpha^\alpha}{\Gamma(\alpha)} e^{-\alpha x_1} \exp(-\alpha e^{-x_1}), \quad -\infty < x_1 < \infty, \quad \alpha \ > \ 0$$

and

$$h_2(x_2) = \frac{\beta^\beta}{\Gamma(\beta)} e^{-\beta x_2} \exp(-\beta e^{-x_2}), \quad -\infty < x_2 < \infty, \quad \beta \ > \ 0$$

Then the characteristic function of $X_1 - X_2$ is given as

$$\begin{aligned} \Phi_{X_1 - X_2}(t) &= \Phi_{X_1} \times \Phi_{X_2}(-t) \\ &= \frac{\Gamma(\alpha - it)\Gamma(\beta + it)}{\Gamma(\alpha)\Gamma(\beta)} \end{aligned}$$

This is the characteristic function of the generalized logistic distribution with parameters α and β, and the theorem is proved.

Theorem 3.3

The random variable Y is generalized Gumbel with parameter λ if and only if the density function $f(y)$ satisfies the homogeneous differential equation

$$f' - \lambda f(e^{-y} - 1) = 0 \tag{20}$$

Proof of Theorem 3.3

Suppose Y is generalized Gumbel with density function

$$f(y) = \frac{\lambda^\lambda}{\Gamma(\lambda)} e^{-\lambda y} e^{-\lambda e^{-y}} \tag{21}$$

Differentiating (21) w.r.t. y and subtituting into (20) gives the proof of the first part. Conversely, suppose the density function (21) satisfies (20), by seperating variables and integrating we have,

$$f = K e^{-\lambda y} e^{-\lambda e^{-y}}$$

The normalising constant

$$K = \frac{\lambda^\lambda}{\Gamma(\lambda)}$$

and thus the proof is complete.

4. THE T-APPROXIMATION TO THE PROBABILITY FUNCTION

We hereby propose a t-approximation to the cumulative distribution function,c.d.f, of the generalized Gumbel distribution. The c.d.f of the generalized Gumbel distribution is given by

$$\begin{aligned} F(y) &= \frac{\lambda^\lambda}{\Gamma(\lambda)} \int_{-\infty}^{y} e^{-\lambda x} \exp(-\lambda e^{-x}) dx \\ &= \frac{1}{\Gamma(\lambda)} \int_{\lambda e^{-x}}^{\infty} p^{\lambda-1} e^{p} \, dp \end{aligned}$$

that is

$$F(y) = I_u(\lambda) \tag{22}$$

where $I(.)$ is the incomplete gamma function and $u = \lambda e^{-y}$. Since (22) is not in closed form, we propose a t-approximation to it by following the standard method used by *(Ojo,1988)*. To obtain the appropriate degree of freedom of the approximating t for given values of λ, we equate the coefficents of kurtosis of the two distributions. That is $\beta_2(y) = \beta_2(t)$. By using equation (8) and (10), we have

$$\beta_2(y) = \frac{(\frac{\pi^4}{15} - \sum_{j=1}^{\lambda-1} j^{-4})}{(\frac{\pi^2}{6} - \sum_{j=1}^{\lambda-1} j^{-2})^2}$$

By equating $\beta_2(y)$ to $\frac{6}{\nu-4}$, the coefficient of kurtosis of the t distribution with ν degrees of freedom, we have the relation

$$\nu = \frac{6 + 4\beta_2(y)}{\beta_2(y)}$$

from where we obtain the regression line of ν on λ to be

$$\nu = 2.93 + 3.12\lambda$$

Now if we let

$$Y_* = \frac{(Y - \kappa_1)}{\sqrt{\kappa_2}}$$

and

$$T = \frac{t}{\sigma_t}$$

where $\sigma_t^2 = \frac{\nu}{\nu-2}$ be the standardized Gumbel and the t random variables respectively, we propose the approximation

$$P(Y_* \leq y) \approx P(T \leq t) = F(y) \approx P(T \leq c(y - \kappa_1))$$

where

$$c = \frac{\sigma_t}{\sqrt{\kappa_2}}$$

Computations are carried out for Tail probabilities to show the quality of the approximation. It turns out that the t-approximation gives a good result as λ increases. This is substantiated by the error marging of the tail probabilities in the appendix.

References

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APPENDIX

TABLE OF ERROR OF APPROXIMATIONS TO CUMULANTS

λ	κ_1	Approximation	Absolute Error
1	0.5772	0.5833	0.0061
2	0.2703	0.2708	0.0005
3	0.1758	0.1759	0.0001
4	0.1302	0.1302	0.0000
5	0.1039	0.1039	0.0000
10	0.0508	0.0508	0.0000

Table 1:

λ	κ_2	Approximation	Absolute Error
1	1.6449	1.6667	0.0218
2	0.6449	0.6459	0.0010
3	0.3948	0.3951	0.0003
4	0.2838	0.2838	0.0000
5	0.2213	0.2213	0.0000
10	0.1052	0.1052	0.0000

Table 2:

λ	κ_3	Approximation	Absolute Error
1	2.4859	2.5000	0.0141
2	0.4071	0.4063	0.0008
3	0.1544	0.1543	0.0001
4	0.2832	0.2832	0.0000
5	0.0488	0.0488	0.0000
10	0.0111	0.0111	0.0000

Table 3:

λ	κ_4	Approximation	Absolute Error
1	6.4939	7.0000	0.5061
2	0.4939	0.5000	0.0061
3	0.1189	0.1193	0.0004
4	0.0448	0.0448	0.0000
5	0.0211	0.0211	0.0000
10	0.0022	0.0022	0.0000

Table 4:

TABLE OF ERROR OF APPROXIMATIONS TO PROBABILITY FUNCTION

$F(y)$ is the probability function of the generalized Gumbel distribution.
$T(y)$ is the t approximation to the probability function.

y	$F(y)$	$T(y)$	Absolute Error
2.365	0.9750	0.9577	0.0173
2.998	0.9900	0.9785	0.0115
3.499	0.9950	0.9874	0.0076
4.785	0.9990	0.9969	0.0021
5.408	0.9995	0.9984	0.0011

Table 5: $\lambda = 1, \nu = 7$

y	$F(y)$	$T(y)$	Absolute Error
2.262	0.9750	0.9654	0.0096
2.821	0.9900	0.9813	0.0087
3.257	0.9950	0.9895	0.0055
4.297	0.9990	0.9975	0.0015
4.781	0.9995	0.9987	0.0008

Table 6: $\lambda = 2, \nu = 9$

y	$F(y)$	$T(y)$	Absolute Error
2.179	0.9750	0.9674	0.0076
2.681	0.9900	0.9824	0.0076
3.055	0.9950	0.9900	0.0050
3.930	0.9990	0.9975	0.0015
4.318	0.9995	0.9987	0.0008

Table 7: $\lambda = 3, \nu = 12$

y	$F(y)$	$T(y)$	Absolute Error
2.131	0.9750	0.9644	0.0106
2.602	0.9900	0.9830	0.0070
2.947	0.9950	0.9904	0.0046
3.733	0.9990	0.9981	0.0009
4.073	0.9995	0.9987	0.0008

Table 8: $\lambda = 4, \nu = 15$

y	$F(y)$	$T(y)$	Absolute Error
2.101	0.9750	0.9653	0.0097
2.552	0.9900	0.9835	0.0065
2.878	0.9950	0.9916	0.0034
3.610	0.9990	0.9976	0.0014
3.922	0.9995	0.9987	0.0001

Table 9: $\lambda = 5, \nu = 18$