Kragujevac J. Math. 24 (2002) 107–121.

3-TYPE CURVES IN THE EUCLIDEAN SPACE $\mathbf{E}^5$

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(Received February 25, 2002)

Abstract. In [1] D. Blair gave a complete classification of 3–type curves in the space E^3. In a recent paper [8] we gave a complete classification of 3–type curves in the space E^4. In this paper a complete classification of 3–type curves in the space E^5 is given.

The notion of finite type curves was introduced by B. Y. Chen around 1980. A closed curve γ in a Euclidean space E^n is of finite type (type k, $k \in N$) if its Fourier series expansion with respect to an arclength parameter is finite (has exactly k nonzero terms).

It is proved in [3] that a closed curve $\gamma\colon [0, 2\pi r] \mapsto E^n$ is of k–type ($k \in N$) if and only if there is a vector $A_0 \in E^n$, natural numbers $p_1 < p_2 < \ldots < p_k$, and vectors $A_1, \ldots, A_k, B_1, \ldots, B_k \in E^n$ such that $\|A_i\|^2 + \|B_i\|^2 \neq 0$ $(i = 1, \ldots, k)$ and

$$\gamma(s) = A_0 + \sum_{i=1}^{k} (A_i \cos \frac{p_i s}{r} + B_i \sin \frac{p_i s}{r}).$$

It is shown in [5] that every curve of the type k in the space E^n lies in an affine subspace of the dimension $2k$. Hence, the only interesting case is $n \leq 2k$.

In particular, 3–type curves in the space E^3, have been investigated several times in the literature (see e.g. [1], [9], [10]). One of the most important papers in that direction is the paper [1] by D. Blair, where a complete classification of such curves in the space E^3 is given. In [8] we have classified all 3–type curves in the space E^4.

In this paper we will go a step further, and classify 3–type curves in the space E^5. In a subsequent paper we shall also consider 3–type curves in the space E^6. This will obviously complete the investigation of 3–type curves in Euclidean spaces.

Our paper is close to the paper [1], but it is not a simple imitation of this paper. Namely, in the space E^{k+1} $(k \in N)$ we often really meet some cases and situations which are contradictory in the space E^k. This paper is also very closed to the paper [8], in particular the methods which we use are similar to the corresponding methods in the paper [8]. In view of these similarities, we very often mention only results omitting the proofs.

It is also important to say that, referring to the parameters p_1, p_2, p_3 which are involved for all 3–type curves in any Euclidean spaces, some cases which are impossible in the space E^4 become quite possible in the space E^5, of one dimension more.

We also need to mention here that by usual lifting $(x, y, z, t) \mapsto (x, y, z, t, 0)$ of the space E^4 in E^5, every 3–type curve in the space E^4 becomes a 3–type curve in the space E^5. So, it is interesting to search only for 3–type curve in the space E^5 which are not of such a type.

By the general statement, we have that a curve $\gamma \subseteq E^5$ is of 3–type if there are natural numbers $p_1 < p_2 < p_3$ (frequency numbers of the curve) such that $\gamma\colon [0, 2\pi r] \mapsto E^5$ has the form

$$\gamma(s) = A_0 + \sum_{i=1}^{3}(A_i \cos\frac{p_i s}{r} + B_i \sin\frac{p_i s}{r}),$$

where $A_0 \in E^5$ and $A_1, A_2, A_3, B_1, B_2, B_3 \in E^5$ are such that $\|A_i\|^2 + \|B_i\|^2 \neq 0$ for each $i = 1, 2, 3$.

It is proved in [3] that the last condition is equivalent to the following system of equations:

$$(O) \qquad \sum_{i=1}^{3} p_i^2 D_{ii} = 2r^2,$$

$$I(l) \qquad \sum_{\substack{i=1\\2p_i=l}}^{3} p_i^2 A_{ii} + 2 \sum_{\substack{i,j=1\\i>j\\p_i+p_j=l}}^{3} p_i p_j A_{ij} - \sum_{\substack{i,j=1\\i>j\\p_i-p_j=l}}^{3} p_i p_j D_{ij} = 0,$$

$$\overline{I}(l) \qquad \sum_{\substack{i=1\\2p_i=l}}^{3} p_i^2 \overline{A}_{ii} + 2 \sum_{\substack{i,j=1\\i>j\\p_i+p_j=l}}^{3} p_i p_j \overline{A}_{ij} - \sum_{\substack{i,j=1\\i>j\\p_i-p_j=l}}^{3} p_i p_j \overline{D}_{ij} = 0,$$

where

$$A_{ij} = \langle A_i, A_j\rangle - \langle B_i, B_j\rangle, \quad \overline{A}_{ij} = \langle A_i, B_j\rangle + \langle A_j, B_i\rangle,$$
$$D_{ij} = \langle A_i, A_j\rangle + \langle B_i, B_j\rangle, \quad \overline{D}_{ij} = \langle A_i, B_j\rangle - \langle A_j, B_i\rangle,$$

$(i, j = 1, 2, 3)$, and l runs the set

$$\mathcal{A} = \{2p_1, 2p_2, 2p_3, p_1 + p_2, p_1 + p_3, p_2 + p_3, p_2 - p_1, p_3 - p_1, p_3 - p_2\}.$$

The main theorem of this paper is the following.

Theorem. *If $\gamma(s)$ is a 3–type curve in the Euclidean space E^5, then $\gamma(s)$ belongs to a k–parameter family of curves where k is one of the numbers $4, 6, 9, 11, 13$, and we have one of equalities $p_2 = 3p_1$, $p_3 = 3p_1, 3p_2, 2p_1 + p_2, 2p_2 + p_1, 2p_2 - p_1$.*

The proof of this theorem follows from a series of propositions which we are going to prove.

In the sequel, the most important thing is to differ the cases when all indices in the set $\mathcal{A}$ are distinct, or some of them coincide.

The complete classification of all these cases is as follows.

$$p_2 \neq 3p_1,\ p_3 \neq 3p_1, 3p_2, p_2 + 2p_1, 2p_2 \pm p_1 \tag{1^0}$$

$$p_2 = 3p_1,\ p_3 \neq 5p_1, 7p_1, 9p_1 \tag{2^0}$$

$$p_2 \neq 2p_1,\ p_3 = 3p_1 \tag{3^0}$$

$$p_2 \neq 3p_1,\ p_3 = 3p_2 \tag{4^0}$$

$$p_2 \neq 3p_1,\ p_3 = p_2 + 2p_1 \tag{5^0}$$

$$p_2 = 3p_1,\ p_3 = 5p_1 \tag{6^0}$$

$$p_2 \neq 3p_1,\ p_3 = p_1 + 2p_2 \tag{7^0}$$

$$p_2 = 3p_1,\ p_3 = 7p_1 \tag{8^0}$$

$$p_2 \neq 2p_1,\ 3p_1, \quad p_3 = 2p_2 - p_1 \tag{9^0}$$

$$p_2 = 2p_1,\ p_3 = 3p_1 \tag{10^0}$$

$$p_2 = 3p_1,\ p_3 = 9p_1. \tag{11^0}$$

We shall discuss all these cases separately. First introduce the following notations:

$$A_1 = (a_{11}, a_{12}, a_{13}, a_{14}, a_{15}), \quad B_1 = (b_{11}, b_{12}, b_{13}, b_{14}, b_{15}),$$
$$A_2 = (a_{21}, a_{22}, a_{23}, a_{24}, a_{25}), \quad B_2 = (b_{21}, b_{22}, b_{23}, b_{24}, b_{25}),$$
$$A_3 = (a_{31}, a_{32}, a_{33}, a_{34}, a_{35}), \quad B_3 = (b_{31}, b_{32}, b_{33}, b_{34}, b_{35}),$$

If some index in the set $\mathcal{A}$ differs of all other indices in this set , we shall call it "single". The set $\mathcal{A}$ obviously has at least two single indices, namely $2p_3$ and p_2+p_3. These indices are evidently the greatest in $\mathcal{A}$.

Lemma 1. *By a suitable change of the coordinate system, we can assume that*

$$A_3 = (\mu, 0, 0, 0, 0), \quad B_3 = (0, \mu, 0, 0, 0) \qquad (\mu \neq 0).$$

In this system we have $b_{21} = -a_{22}, b_{22} = a_{21}$, *thus* $B_2 = (-a_{22}, a_{21}, b_{23},\ b_{24}, b_{25})$.

We omit the proof since it is quite similar to the corresponding proof of Lemma 1 in [8]. The similar is true in the next lemma.

Lemma 2. *If* $2p_2$ *and* $p_3 - p_2$ *are single parameters, then by a suitable change of coordinate system we can assume that*

$$A_2 = (0, 0, \nu, 0, 0), B_2 = (0, 0, 0, \nu, 0),$$

for some $\nu \neq 0$.

Proposition 1. *The case* (1^0) *is impossible.*

We again omit the proof.

Proposition 2. *In the case* (2^0) *a curve* $\gamma(s)$ *has the type 3 if and only if, in a coordinate system, we have*

$$A_1 = (0, 0, a_{13}, a_{14}, a_{15}), \quad B_1 = (0, 0, -a_{14}, a_{13}, b_{15}),$$
$$A_2 = (0, 0, \nu, 0, 0), \quad B_2 = (0, 0, 0, \nu, 0) \qquad (\nu \neq 0),$$
$$A_3 = (\mu, 0, 0, 0, 0), \quad B_3 = (0, \mu, 0, 0, 0) \qquad (\mu \neq 0),$$

where

$$a_{13} = \frac{a_{15}^2 - b_{15}^2}{6\nu}, \qquad a_{14} = -\frac{a_{15}b_{15}}{3\nu}.$$

Note that the case (2^0) is impossible in the space E^4. But in the space E^5 this case obviously generates the whole family of curves. Namely, we can obviously take that a_{15}, b_{15}, μ, ν are arbitrary parameters such that $\mu, \nu, a_{15}^2 + b_{15}^2 \neq 0$. Hence we get a 4–parameter family of curves.

Proposition 3. *In the case* (3^0) *a curve* $\gamma(s)$ *has the type 3 if and only if, in a coordinate system, we have*

$$A_1 = (a_{11}, a_{12}, 0, 0, a_{15}), \quad B_1 = (-a_{12}, a_{11}, 0, 0, b_{15}),$$
$$A_2 = (0, 0, \nu, 0, 0), \quad B_2 = (0, 0, 0, \nu, 0) \qquad (\nu \neq 0),$$
$$A_3 = (\mu, 0, 0, 0, 0), \quad B_3 = (0, \mu, 0, 0, 0) \qquad (\mu \neq 0),$$

where

$$a_{11} = \frac{a_{15}^2 - b_{15}^2}{6\mu}, \qquad a_{12} = -\frac{a_{15}b_{15}}{3\mu}.$$

The case (3^0) is also impossible in the space E^4. But in the space E^5 it also generates the whole family of curves. Namely, we can take that a_{25}, b_{25}, μ, ν are arbitrary parameters such that $\mu, \nu, a_{25}^2 + b_{25}^2 \neq 0$, while a_{21}, a_{22} are expressed by them. Hence we again get a 4–parameter family of curves.

Proposition 4. *In the case* (4^0) *a curve* $\gamma(s)$ *has the type 3 if and only if, in a coordinate system, we have*

$$A_1 = (0, 0, \nu, 0, 0), \quad B_1 = (0, 0, 0, \nu, 0) \qquad (\nu \neq 0),$$
$$A_2 = (a_{21}, a_{22}, 0, 0, a_{25}), \quad B_2 = (-a_{22}, a_{21}, 0, 0, b_{25}),$$
$$A_3 = (\mu, 0, 0, 0, 0), \quad B_3 = (0, \mu, 0, 0, 0) \qquad (\mu \neq 0),$$

where

$$a_{21} = \frac{a_{25}^2 - b_{25}^2}{6\mu}, \qquad a_{22} = -\frac{a_{25}b_{25}}{3\mu}$$

and $a_{21}^2 + a_{22}^2 \neq 0$*.*

Note that the case (4^0) is also impossible in the space E^4. But in the space E^5 it generates a 6–parameter family of curves.

Proposition 5. (Case (5^0)) $(p_2 \neq 3p_1, p_3 = p_2 + 2p_1)$. *In this case a curve* $\gamma(s)$ *has the type 3 if and only if in a coordinate system we have*

$$A_1 = (a_{11}, a_{12}, a_{13}, a_{14}, a_{15}), \quad B_1 = (-a_{12}, a_{11}, b_{13}, b_{14}, b_{15}),$$
$$A_2 = (a_{21}, a_{22}, \nu, 0, 0), \quad B_2 = (-a_{22}, a_{21}, 0, \nu, 0)$$
$$A_3 = (\mu, 0, 0, 0, 0), \quad B_3 = (0, \mu, 0, 0, 0),$$

where $\mu \neq 0$, $a_{21}^2 + a_{22}^2 + \nu^2 \neq 0$ *and*

$$a_{11}a_{21} + a_{12}a_{22} = -\frac{\nu}{2}\,(a_{13} + b_{14}), \tag{1}$$
$$a_{11}a_{22} - a_{12}a_{21} = \frac{\nu}{2}\,(a_{14} - b_{13}), \tag{2}$$
$$\sum_{i=3}^{5}(a_{1i}^2 - b_{1i}^2) = \frac{2p_2p_3}{p_1^2}\mu a_{21}, \tag{3}$$
$$\sum_{i=3}^{5} a_{1i}b_{1i} = -\frac{p_2p_3\,\mu\,a_{22}}{p_1^2}, \tag{4}$$

$$\nu(a_{13} - b_{14}) = \frac{p_3\,\mu\,a_{11}}{p_2}, \tag{5}$$

$$a_{11}a_{22} - a_{12}a_{21} + \nu\, b_{13} = -\frac{p_3\,\mu\,a_{12}}{p_2}. \tag{6}$$

Since the case (5^0) generates a whole family of curves in the space E^4, the similar is true in the space E^5.

Note that in the system of equations (1)–(6) we can take $\mu, \nu, a_{11}, a_{12}, a_{21}, a_{22}$ as parameters, while the other entries $a_{13}, a_{14}, a_{15}, b_{13}, b_{14}, b_{15}$ can be expressed explicitly by them. Namely, first observe that by (2) one can write equation (6) as

$$a_{14} + b_{13} = -\frac{2p_3\mu}{p_2\nu}a_{12}. \tag{7}$$

Therefore, equations (2) and (7) easily give a_{14} and b_{13} as functions of $a_{11}, a_{12}, a_{21}, a_{22}, \nu$. Similarly, equations (1) and (5) give a_{13} and b_{14} as functions of $a_{11}, a_{12}, a_{21}, a_{22}, \nu$. Next, substitute the obtained values for $a_{13}, a_{14}, b_{13}, b_{14}$ in equations (3) and (4), and consider the last equations as equations in $a_{15} = x, b_{15} = y$. Observe that they are of the form

$$x^2 - y^2 = \alpha, \quad xy = \beta.$$

Since the last system has an explicit solution in x, y for any value of real parameters α, β, we conclude that one can express a_{15}, b_{15} in a explicit form as functions of the entries $a_{11}, a_{12}, a_{21}, a_{22}, \mu, \nu$. Therefore the considered system defines a 6–parameter family of curves in the space E^5.

A similar situation is true in cases (6^0)–(11^0), so in each of these cases there is at least one 3–type curve (and evenmore the whole family of such curves) in the space E^5.

Proposition 6. (Case (6^0)) $(p_1 : p_2 : p_3 = 1 : 3 : 5)$ *In this case a curve $\gamma(s)$ has the type 3 if and only if, in a coordinate system we have*

$$A_1 = (a_{11}, a_{12}, a_{13}, a_{14}, a_{15}), \quad B_1 = (b_{11}, b_{12}, b_{13}, b_{14}, b_{15}),$$
$$A_2 = (a_{21}, a_{22}, a_{23}, a_{24}, a_{25}), \quad B_2 = (-a_{22}, a_{21}, b_{23}, b_{24}, b_{25}),$$
$$A_3 = (\mu, 0, 0, 0, 0), \quad B_3 = (0, \mu, 0, 0, 0),$$

where $\mu \neq 0$ *and*

$$\sum_{i=1}^{5}(a_{1i}^2 - b_{1i}^2) = 3\big[\sum_{i=1}^{5} a_{1i}a_{2i} - b_{11}a_{22} + b_{12}a_{21} + \sum_{i=3}^{5} b_{1i}b_{2i}\big] + 30\,\mu a_{21} \tag{1}$$

$$\sum_{i=1}^{5} a_{1i}b_{1i} = 1,5\,\sum_{i=1}^{5}(a_{2i}b_{1i} - a_{1i}b_{2i}) - 15\mu\,a_{22} \tag{2}$$

$$\sum_{i=1}^{5} a_{1i}a_{2i} - (-a_{22}b_{11} + a_{21}b_{12} + \sum_{i=3}^{5} b_{1i}b_{2i}) = \frac{5\,\mu}{6}\,(a_{11} + b_{12}) \tag{3}$$

$$\sum_{i=1}^{5} b_{1i}a_{2i} - a_{11}a_{22} + a_{12}a_{21} + \sum_{i=3}^{5} a_{1i}b_{2i} = \frac{5\,\mu}{6}\,(b_{11} - a_{12}) \tag{4}$$

$$\sum_{i=3}^{5}(a_{2i}^2 - b_{2i}^2) = -\frac{10}{9}\,\mu\,(a_{11} - b_{12}) \tag{5}$$

$$\sum_{i=3}^{5} a_{2i}b_{2i} = -\frac{5}{9}\,\mu\,(a_{12} + b_{11}). \tag{6}$$

It can be proved that in this case the considered system of equations defines a 13–parameter family of curves.

Proposition 7. (Case (7^0)) ($p_2 \neq 3p_1, p_3 = p_1 + 2p_2$). *In this case a curve $\gamma(s)$ is of type 3 if and only if, in a coordinate system we have*

$$A_1 = (a_{11}, a_{12}, \nu, 0, 0), \quad B_1 = (-a_{12}, a_{11}, 0, \nu, 0),$$
$$A_2 = (a_{21}, a_{22}, a_{23}, a_{24}, a_{25}), \quad B_2 = (-a_{22}, a_{21}, b_{23}, b_{24}, b_{25}),$$
$$A_3 = (\mu, 0, 0, 0, 0), \quad B_3 = (0, \mu, 0, 0, 0),$$

where $\mu \neq 0$, $a_{11}^2 + a_{12}^2 + \nu^2 \neq 0$ and

$$a_{11}a_{21} + a_{12}a_{22} = -\frac{\nu}{2}(a_{23} + b_{24}), \tag{1}$$
$$a_{11}a_{22} - a_{12}a_{21} = \frac{\nu}{2}(b_{23} - a_{24}), \tag{2}$$
$$\sum_{i=3}^{5}(a_{2i}^2 - b_{2i}^2) = \frac{2p_1p_3\mu}{p_2^2}\, a_{11}, \tag{3}$$
$$\sum_{i=3}^{5} a_{2i}b_{2i} = -\frac{p_1p_3\mu}{p_2^2}\, a_{12}, \tag{4}$$
$$\nu\,(a_{23} - b_{24}) = \frac{p_3\,\mu}{p_1} a_{21}, \tag{5}$$
$$\nu\,(a_{24} + b_{23}) = -\frac{p_3\mu}{p_1} a_{22}. \tag{6}$$

We shall prove that the parameters $a_{23}, a_{24}, a_{25}, b_{23}, b_{24}, b_{25}$ can be expressed by $a_{11}, a_{12}, a_{21}, a_{22}, \mu, \nu$.

Indeed, by equations (1) and (5) we can obviously express a_{23}, b_{24} by a_{11}, a_{12}, a_{21}, a_{22}, μ, ν. Similarly, by equations (2) and (6) we can express a_{24}, b_{23} by the same parameters. Substituting next the obtained values for $a_{23}, a_{24}, b_{23}, b_{24}$ in the equations (3),(4) and putting $a_{25} = x, b_{25} = y$, we obviously get a system of the form as in Proposition 5, where α, β are the explicit functions of $a_{11}, a_{12}, a_{21}, a_{22}, \mu, \nu$. Since such a system always has an explicit solution in x, y, we get a_{25}, b_{25} as some explicit functions of a_{1i}, a_{2i}, μ, ν $(i = 1, 2)$. Therefore the considered system (1)–(6) defines a 6–parameter family of curves in the space E^5.

Proposition 8. (Case (8^0)) ($p_1 : p_2 : p_3 = 1 : 3 : 7$). *In this case a curve $\gamma(s)$ is of type 3 if and only if, in a coordinate system, we have*

$$A_1 = (a_{11}, a_{12}, a_{13}, a_{14}, a_{15}), \quad B_1 = (-a_{12}, a_{11}, b_{13}, b_{14}, b_{15}),$$
$$A_2 = (a_{21}, a_{22}, a_{23}, a_{24}, a_{25}), \quad B_2 = (-a_{22}, a_{21}, b_{23}, b_{24}, b_{25}),$$
$$A_3 = (\mu, 0, 0, 0, 0), \quad B_3 = (0, \mu, 0, 0, 0),$$

where $\mu \neq 0$ and

$$\sum_{i=3}^{5}(a_{2i}^2 - b_{2i}^2) = \frac{14}{9}\,\mu\, a_{11}, \tag{1}$$

$$\sum_{i=3}^{5} a_{2i}b_{2i} = -\frac{7}{9}\,\mu\, a_{12}, \tag{2}$$

$$\sum_{i=3}^{5}(a_{1i}a_{2i} - b_{1i}b_{2i}) = 7\,\mu\, a_{21}, \tag{3}$$

$$\sum_{i=3}^{5}(a_{1i}b_{2i} + b_{1i}a_{2i}) = -7\,\mu\, a_{22}, \tag{4}$$

$$\sum_{i=3}^{5}(a_{1i}^2 - b_{1i}^2) = 6a_{11}a_{21} + 6a_{12}a_{22} + 3\sum_{i=3}^{5}(a_{1i}a_{2i} + b_{1i}b_{2i}), \tag{5}$$

$$\sum_{i=3}^{5} a_{1i}b_{1i} = 3a_{11}a_{22} - 3a_{12}a_{21} + 1,5\sum_{i=3}^{5}(b_{1i}a_{2i} - a_{1i}b_{2i}). \tag{6}$$

We shall prove that in the obtained system (1)–(6) parameters a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, b_{13} can be expressed explicitly by parameters $a_{14}, a_{15}, b_{14}, b_{15}, a_{23}, a_{24}, a_{25}$, $b_{23}, b_{24}, b_{25}, \mu$.

First since $\mu \neq 0$, by equations (1)–(4) we can express $a_{11}, a_{12}, a_{21}, a_{22}$ by a_{1i}, b_{1i}, a_{2i}, b_{2i} $(i = 3, 4, 5)$. Substituting this in the equations (5),(6) and putting $a_{13} = x, b_{13} = y$, the last equations get the form

$$ax^2 + by^2 + cx + dy + e = 0, \tag{7}$$

$$\alpha xy + \beta x + \gamma y + \delta = 0, \tag{8}$$

where $a, b, c, d, e, \alpha, \beta, \gamma, \delta$ are some expressions in $a_{14}, a_{15}, b_{14}, b_{15}, a_{23}, a_{24}, a_{25}, b_{23}$, b_{24}, b_{25}, μ.

Since the system (7)–(8) is of the 4 order in x (and in y) (the polynomial equations of the 4 order), it can be solved explicitly in x, y.

Hence the considered system (1)–(6) defines a 11–parameter family of curves in the space E^5.

Proposition 9. (Case (9^0)) $(p_2 \neq 2p_1, 3p_1,\ p_3 = 2p_2 - p_1)$. *In this case a curve $\gamma(s)$ is of type 3 if and only if in a coordinate system we have*

$$A_1 = (a_{11}, a_{12}, \nu, 0, 0), \quad B_1 = (a_{12}, -a_{11}, 0, \nu, 0),$$

$$A_2 = (a_{21}, a_{22}, a_{23}, a_{24}, a_{25}), \quad B_2 = (-a_{22}, a_{21}, b_{23}, b_{24}, b_{25}),$$

$$A_3 = (\mu, 0, 0, 0, 0), \quad B_3 = (0, \mu, 0, 0, 0),$$

where $\mu \neq 0$, $a_{11}^2 + a_{12}^2 + \nu^2 \neq 0$ and

$$a_{11}a_{21} + a_{12}a_{22} = \frac{\nu}{2}\,(b_{24} - a_{23}), \tag{1}$$

$$a_{11}a_{22} - a_{12}a_{21} = \frac{\nu}{2}\,(a_{24} + b_{23}), \tag{2}$$

$$\sum_{i=3}^{5}(a_{2i}^2 - b_{2i}^2) = -\frac{4p_1p_3\,\mu}{p_2^2}\,a_{11}, \tag{3}$$

$$\sum_{i=3}^{5} a_{2i}b_{2i} = -\frac{2p_1p_3\,\mu}{p_2^2}\,a_{12}, \tag{4}$$

$$\nu\,(a_{23}+b_{24}) = -\frac{2p_3}{p_1}\,\mu\,a_{21}, \tag{5}$$

$$\nu\,(a_{24}-b_{23}) = \frac{2p_3}{p_1}\,\mu\,a_{22}. \tag{6}$$

In this case we shall prove that parameters $a_{23}, a_{24}, a_{25}, b_{23}, b_{24}, b_{25}$ can be explicitly expressed by the parameters $a_{11}, a_{12}, a_{21}, a_{22}, \mu, \nu$.

First by equations (1),(5) we see that parameters a_{23}, b_{24} can be obviously explicitly expressed by $a_{11}, a_{12}, a_{21}, a_{22}, \mu, \nu$. Similarly holds for a_{24}, b_{23} using the equations (2),(6).

Substituting now the obtained values for $a_{23}, a_{24}, b_{23}, b_{24}$ in equations (3),(4) we obviously get a system of the form

$$a_{25}^2 - b_{25}^2 = \alpha, \quad a_{25}b_{25} = \beta,$$

which can be explicitly solved in a_{25}, b_{25}.

Hence the considered system (1)–(6) defines a 6–parameter family of curves in the space E^5.

Proposition 10. (Case (10^0)) $(p_1 : p_2 : p_3 = 1 : 2 : 3)$. *In this case $\gamma(s)$ is of type 3 if and only if, in a coordinate system, we have*

$$A_1 = (a_{11}, a_{12}, a_{13}, a_{14}, a_{15}), \quad B_1 = (b_{11}, b_{12}, b_{13}, b_{14}, b_{15}),$$

$$A_2 = (a_{21}, a_{22}, a_{23}, a_{24}, a_{25}), \quad B_2 = (-a_{22}, a_{21}, b_{23}, b_{24}, b_{25}),$$

$$A_3 = (\mu, 0, 0, 0, 0), \quad B_3 = (0, \mu, 0, 0, 0),$$

where $\mu \neq 0$ *and*

$$\sum_{i=1}^{5} a_{1i}a_{2i} = -b_{11}a_{22} + a_{21}b_{12} + \sum_{i=3}^{5} b_{1i}b_{2i}, \tag{1}$$

$$\sum_{i=1}^{5} b_{1i}a_{2i} = a_{11}a_{22} - a_{12}a_{21} - \sum_{i=3}^{5} a_{1i}b_{2i}, \tag{2}$$

$$\sum_{i=1}^{5}(a_{1i}^2 - b_{1i}^2) = 3\,\mu\,(a_{11} + b_{12}), \tag{3}$$

$$\sum_{i=1}^{5} a_{1i}b_{1i} = 1.5\,\mu\,(b_{11} - a_{12}), \tag{4}$$

$$\sum_{i=1}^{5} a_{1i}a_{2i} - b_{11}a_{22} + b_{12}a_{21} + \sum_{i=3}^{5} b_{1i}b_{2i} = -6\,\mu a_{21}, \tag{5}$$

$$\sum_{i=1}^{5} a_{2i}b_{1i} + a_{11}a_{22} - a_{12}a_{21} - \sum_{i=3}^{5} a_{1i}b_{2i} = 6\mu\, a_{22}, \tag{6}$$

$$\sum_{i=3}^{5}(a_{2i}^2 - b_{2i}^2) = -1.5\,\mu\,(a_{11} - b_{12}), \tag{7}$$

$$\sum_{i=3}^{5} a_{2i}b_{2i} = -\frac{3}{4}\,\mu\,(b_{11} + a_{12}). \tag{8}$$

In this case one can prove that the above system defines a 11–parameter family of curves.

Proposition 11. (Case (11^0)) $(p_1 : p_2 : p_3 = 1 : 3 : 9)$. *In this case a curve $\gamma(s)$ has the type 3 if and only if in a coordinate system we have*

$$A_1 = (0, 0, a_{13}, a_{14}, a_{15}), \quad B_1 = (0, 0, b_{13}, b_{14}, b_{15}),$$
$$A_2 = (a_{21}, a_{22}, a_{23}, a_{24}, a_{25}), \quad B_2 = (-a_{22}, a_{21}, b_{23}, b_{24}, b_{25}),$$
$$A_3 = (\mu, 0, 0, 0, 0), \quad B_3 = (0, \mu, 0, 0, 0),$$

where $\mu \neq 0$ and

$$\sum_{i=3}^{5} (a_{1i}a_{2i} - b_{1i}b_{2i}) = 0, \tag{1}$$
$$\sum_{i=3}^{5} (a_{1i}b_{2i} + a_{2i}b_{1i}) = 0, \tag{2}$$

$$\sum_{i=3}^{5} (a_{1i}^2 - b_{1i}^2) = 3\sum_{i=3}^{5} (a_{1i}a_{2i} + b_{1i}b_{2i}), \tag{3}$$
$$\sum_{i=3}^{5} a_{1i}b_{1i} = 1,5\sum_{i=3}^{5} (b_{1i}a_{2i} - a_{1i}b_{2i}), \tag{4}$$
$$\sum_{i=3}^{5} (a_{2i}^2 - b_{2i}^2) = 6\mu\, a_{21}, \tag{5}$$
$$\sum_{i=3}^{5} a_{2i}b_{2i} = -3\mu\, a_{22}. \tag{6}$$

Finally, in this case one can prove that the above system defines a 9–parameter family of curves.

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