Kragujevac J. Math. 24 (2002) 95–105.

SOME GENERALIZATION OF WEIGHTED NORM INEQUALITIES FOR CERTAIN CLASS OF INTEGRAL OPERATORS

K. Rauf and C. O. Imoru

Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria

(Received June 20, 2001)

Abstract. A generalization is obtained for a non-negative weight function w for which there is a non-negative weight function $\nu < \infty$ μ-almost everywhere such that T maps $L^p(\nu)$ to $L^q(w)$, i. e.

$$\left[\int_X (Tf)^q w\, d\mu\right]^{1/q} \le C\left[\int_X f^p \nu\, d\mu\right]^{1/p} \quad \text{for all } f \ge 0 \tag{1.1}$$

and C is a constant depending on K, p, q but independent of f. Furthermore, for T sub-linear operator generalization is obtained for weight functions for which T is bounded from $L^q(\mathcal{R}^n, \omega\, dx)$ to $L^p(\mathcal{R}^n, \nu\, dx)$ for some nontrivial w.

1. INTRODUCTION

Let (X, A, μ) be a σ-finite measure space and let $K(x, y)$ be a non-negative and measurable on $X \times X$. Set $Tf(x) = \int_X K(x, y)f(y)\, dy$ and it's dual $T^*f(y) = \int_X K(y, x)f(y)\, dy$ for non-negative function f.

For $1 < p < \infty$, we shall consider the weighted norm inequality

$$\int_X (Tf)^p\, w\, d\mu \le C \int_X f^p \nu\, d\mu \qquad \text{for all } f \ge 0, \tag{1.2}$$

where w and ν are non-negative measurable weight functions on X.

In [4], R. Kerman and E. Sawyer proved the following theorem on weighted norm inequalities for positive linear operators.

Theorem 1.1. *Let $1 < p < \infty$ and suppose w is a weight on X. Then there is a weight ν, finite μ-almost everywhere on X, such that the weighted norm inequality (1.2) holds if and only if there exists a positive function Φ on X with*

$$\int_X (T\Phi)^p w \, d\mu < \infty \qquad \textit{or equaivalently} \tag{1.3}$$

$$\Phi^{1-p} T^* \Big((T\Phi)^{p-1} w \Big) < \infty \qquad \mu\textit{-almost everywhere.} \tag{1.4}$$

This theorem is known to have extended some earlier results of B. Muchenhoupt in [5]. The main objective of the present paper is to prove a result which is more general than Theorem 1.1.

Throughout this paper, p' denotes the conjugate index of p, $p \neq 0$ and is defined by $\frac{1}{p} + \frac{1}{p'}$ with $p' = \infty$ if $p = 1$, the conjugate of q is defined in the same way.

2. MAIN RESULTS

We state our main result.

Theorem 2.1. *Let $1 < p < \infty$ and suppose w is a weight on X. Define the sublinear operator T by*

$$T(f+g)(x) = \int_X K(x,y)(f+g)(y) \, d\mu(y). \tag{2.1}$$

Then, there exists a weight function ν, finite μ-almost everywhere on X such that

$$\int_X \{T(f+g)\}^p w \, d\mu \le C(K,p) \int_X (f^p + g^p) \nu \, d\mu \tag{2.2}$$

holds, for all $f, g > 0$, if and only if there is a positive function Φ and θ on X with

$$\int_X (T\Phi)^p w \, d\mu < \infty \qquad \textit{and} \tag{2.3a}$$

$$\int_X (T\theta)^p w\, d\mu < \infty \qquad \textit{or equivalently} \tag{2.3b}$$

$$\Phi^{1-p} T^* \Big((T\Phi)^{p-1} w \Big) < \infty \qquad \textit{and} \tag{2.4a}$$

$$\theta^{1-p} T^* \Big((T\theta)^{p-1} w \Big) < \infty \qquad \mu\textit{-almost everywhere}, \quad \textit{and} \tag{2.4b}$$

$$C(K,p) = \max\{C_1(K,p), C_2(K,p)\}$$

is a constant independent of f *and* g.

Indeed, the weighted inequality (2.2) holds with ν_1 and ν_2 equal to the weight in (2.4a) and (2.4b) respectively.

Proof. Let

$$I = \int_X (T(f+g))^p\, w\, d\mu.$$

Then

$$\begin{aligned}
I &= \int_X \left\{ \int_X K(x,y)(f+g)\, d\mu(y) \right\}^p w\, d\mu \\
&= \int_X \left\{ \int_X \left(K(x,y)f(y) + K(x,y)g(y) \right)\, d\mu(y) \right\}^p w\, d\mu \\
&\le \int_X \left\{ \int_X K(x,y)f(y)\, d\mu(y) \right\}^p w\, d\mu + \int_X \left\{ \int_X K(x,y)f(y)\, d\mu(y) \right\}^p w\, d\mu
\end{aligned}$$

by Minkowski's inequality

$$\begin{aligned}
&\le \int_X \left(\int K(x,y) f(y)^p \Phi^{-p/p'}\, d\mu(y) \right) \left(\int K(x,y)\Phi\, d\mu(y) \right)^{p/p'} w\, d\mu \\
&+ \int_X \left(\int K(x,y) g(y)^p \theta^{-p/p'}\, d\mu(y) \right) \left(\int K(x,y)\theta\, d\mu(y) \right)^{p/p'} w\, d\mu
\end{aligned}$$

by Holder's inequality

$$\begin{aligned}
&= \int_X \left\{ \int K(x,y) f(y)^p \Phi^{1-p}\, d\mu(y) \right\} \left(\int K(x,y)\Phi\, d\mu(y) \right)^{p-1} w\, d\mu \\
&+ \int_X \left\{ \int K(x,y) g(y)^p \theta^{1-p}\, d\mu(y) \right\} \left(\int K(x,y)\theta\, d\mu(y) \right)^{p-1} w\, d\mu \\
&= \int_X \left[(Tf^p\Phi^{1-p})(T\Phi)^{p-1} w \right]\, d\mu + \int_X \left[(Tg^p\theta^{1-p})(T\theta)^{p-1} w \right]\, d\mu
\end{aligned}$$

$$\begin{aligned}
&= \int_X f^p \Phi^{1-p} T^*(T\Phi)^{p-1} w\, d\mu + \int_X g^p \theta^{1-p} T^*(T\theta)^{p-1} w\, d\mu \\
&\leq C_1(K,p) \int_X f^p \nu_1\, d\mu + C_2(K,p) \int_X g^p \nu_2\, d\mu \\
&= C(K,p) \int_X (f^p + g^p) \nu\, d\mu\,,
\end{aligned}$$

where $\nu = \max\{\nu_1, \nu_2\}$ and $C(K,p) = \max\{C_1, C_2\}$ which yields (2.2) with ν equal to the weight in (2.4a) and (2.4b).

Conversely, assume (2.2) holds for some $\nu < \infty$ μ-almost everywhere. Using the σ-finiteness of μ. One can easily construct a positive functions Φ and θ such that

$$\int_X (\Phi^p + \theta^p)\, \nu\, d\mu < \infty$$

and hence such that (2.3) holds. Finally, suppose (2.3) holds and let ν denotes the weight in (2.4a) and (2.4b). Then

$$\begin{aligned}
&\int_X (\Phi^p + \theta^p)\, \nu\, d\mu = \int_X \Phi^p \nu_1\, d\mu + \theta^p \nu_2\, d\mu \\
&= \int_X \Phi^p \left(\Phi^{1-p} T^* \left[(T\Phi)^{p-1} w \right] \right) d\mu + \int_X \theta^p \left(\theta^{1-p} T^* \left[(T\theta)^{p-1} w \right] \right) d\mu \\
&= \int_X \Phi T^* \left[(T\Phi)^{p-1} w \right] d\mu + \int_X \theta T^* \left[(T\theta)^{p-1} w \right] d\mu \\
&= \int_X T\Phi (T\Phi)^{p-1} w\, d\mu + \int_X T\theta (T\theta)^{p-1} w\, d\mu \\
&= \int_X (T\Phi)^p w\, d\mu + \int_X (T\theta)^p w\, d\mu \\
&= \int_X T(\Phi + \theta)^p w\, d\mu < \infty \quad \text{by (2.3).}
\end{aligned}$$

Since $\Phi > 0, \theta > 0$, we conclude $\nu < \infty$, μ-almost everywhere and this completes the proof of the Theorem.

Remark 2.1. When $g \equiv 0$ on X. Theorem (2.1) reduces to Kerman and Sawyer result [4].

Theorem 2.2. *Let $1 < p < \infty$ and suppose w is a weight on X. Define the sublinear operator T^* by*

$$T^*(f+g)(x) = \int_X K(y,x)(f+g)(y)\, d\mu(y). \tag{2.6}$$

Then, there exists a weight function ν, finite μ-almost everywhere on X such that

$$\int_X \{T^*(f+g)\}^p\, w\, d\mu \le C(K,p) \int_X (f^p + g^p)\, \nu\, d\mu \tag{2.7}$$

holds, for all $f, g > 0$, if and only if there is a positive function Φ and θ on X with

$$\int_X (T^*\Phi)^p w\, d\mu < \infty \tag{2.8a}$$

$$\int_X (T^*\theta)^p w\, d\mu < \infty \qquad \text{or equivalently} \tag{2.8b}$$

$$\Phi^{1-p} T\left((T^*\Phi)^{p-1} w\right) < \infty \qquad \text{and} \tag{2.9a}$$

$$\theta^{1-p} T\left((T^*\theta)^{p-1} w\right) < \infty \qquad \mu\text{-almost everywhere, and} \tag{2.9b}$$

$$C(K,p) = \max\{C_1(K,p), C_2(K,p)\} \tag{2.10}$$

is a constant independent of f and g.

Indeed, the weighted inequality (2.7) holds with ν_1 and ν_2 equal to the weight in (2.9a) and (2.9b) respectively.

Proof. The proof is immediate from the proof of Theorem 2.1. by defining T^* as $(T^*f)(x) = \int_X K(y,x) f(y)\, dy$.

3. THE CASE $1 < p \le q \le \infty$

In this section, we shall obtain some weighted norm inequalities for mixed norm under more restricted condition on ν and w. See [1], [2], and [3] for related work.

Theorem 3.1. *Let $1 < p \le q = \infty$ and suppose $u = w^{1/q}$ is a weight on X, then there is a weight ν, finite μ-almost everywhere on X such that the weighted norm inequality:*

$$\left[\int_X (Tf)^q w\, d\mu\right]^{1/q} \le C \left[\int_X f^p \nu\, d\mu\right]^{1/p} \qquad \text{for all } f \ge 0, \tag{3.1}$$

holds, if and only if there is a positive function Φ on X satisfying

$$\Phi(y)^p \le \nu \tag{3.2}$$

and $C = C(K, p, q)$ is a constant independent of f.

Proof. Let
$$I = \left\{\int_X (Tf)^q w \, d\mu\right\}^{1/q}$$
Then
$$\begin{aligned}
I &= \left\{\int_X (uTf)^q \, d\mu\right\}^{1/q} \leq \sup_{x<\infty} \left\{u(x) \int_X K(x,y) f(y) \, dy\right\} \\
&= \sup_{x<\infty} \left\{u(x) \int_X K(x,y)^\beta f(y) K(x,y)^{1-\beta} \, dy\right\} \\
&\leq \sup_{z<x} \operatorname{ess} \, K(x,z)^\beta u(x) \int_X K(x,y)^{1-\beta} f(y) \, dy \\
&= \sup_{z<x} \operatorname{ess} \, K(x,z)^\beta u(x) \int_X K(x,y)^{1-\beta} \Phi(y)^{-1} f(y) \Phi(y) \, dy \\
&\leq \sup_{z<x} \operatorname{ess} \, K(x,z)^\beta u(x) \left\{\int_X K(x,y)^{(1-\beta)p'} \Phi(y)^{-p'} \, dy\right\}^{1/p'} \left\{\int f(y)^p \Phi(y)^p \, dy\right\}^{1/p}
\end{aligned}$$
by Holder's inequality.

The integral
$$\left\{\int_X K(x,y)^{(1-\beta)p'} \Phi(y)^{-p'} \, dy\right\}^{1/p'} \leq C \left\{\sup_{t>x} \operatorname{ess} \, K(t,x)^\beta u(t)\right\}^{-1}$$
since $u(x)$ and $\Phi(x)$ depend on p and q with constant C.

Hence,
$$I \leq C \sup_{z<x} \operatorname{ess} \, K(x,z)^\beta u(x) \left\{\sup_{t>x} \operatorname{ess} \, K(t,x)^\beta u(t)\right\}^{-1} \left\{\int f(y)^p \Phi(y)^p \, dy\right\}^{1/p}.$$

Now
$$\begin{aligned}
&\sup_{z<x} \operatorname{ess} \, K(x,z)^\beta u(x) \left\{\sup_{t>x} \operatorname{ess} \, K(t,x)^\beta u(t)\right\}^{-1} \\
\leq\; &\sup_{z<x} \operatorname{ess} \, K(x,z)^\beta u(t) \left\{\sup_{z>x} \operatorname{ess} \, K(t,z)^\beta u(t)\right\}^{-1} = 1
\end{aligned}$$
since $K(t, \cdot)$ is not-decreasing.

Therefore,
$$I = C \left(\int_X f(y)^p \Phi(y)^p \, dy\right)^{1/p} \leq C \left\{\int_X f(y)^p \nu \, d\mu(y)\right\}^{1/p}.$$

This completes the proof.

Remark 3.1. If $q = p$, the above Theorem reduces to the result obtained by Kerman and Sawyer [4].

Theorem 3.2. *Let $1 < p \leq q = \infty$ and suppose $u = w^{1/q}$ is a weight on X. Then there is a weight ν, finite μ-almost everywhere on X such that the weighted norm inequality:*

$$\left[\int_X (T^* f)^q w \, d\mu\right]^{1/q} \leq C \left[\int_X f^p \nu \, d\mu\right]^{1/p} \qquad \textit{for all } f \geq 0, \tag{3.3}$$

holds, if and only if there is a positive function Φ on X satisfying

$$\Phi(x)^p \leq \nu, \tag{3.4}$$

and $C = C(K, p, q)$ is a constant independent of f.

Proof. Follows directly from the prove of Theorem 3.1. by defining T^* as $(T^* f)(x) = \int_X K(y, x) f(y) \, dy$.

Theorem 3.3. *Let $1 < p \leq q < \infty$ and suppose $u = w^{1/q}$ is a weight on X. Then there is a weight ν, finite μ-almost everywhere on X such that the weighted norm inequality:*

$$\left[\int_X (T f)^q w \, d\mu\right]^{1/q} \leq C \left[\int_X f^p \nu \, d\mu\right]^{1/p} \qquad \textit{for all } f \geq 0, \tag{3.5}$$

holds, if and only if there is a positive function Φ on X satisfying

$$\Phi(x)^p \leq \nu, \tag{3.6}$$

with

$$s(x) \leq \left(\int_X K(y, z) \Phi(z)^{-p} \, dz\right)^{1/(p+1)},$$

and $C = C(K, p, q)$ is a constant independent of f.

Proof. Let

$$I = \int_X [T f]^q w \, d\mu(x).$$

Then

$$
\begin{aligned}
I &= \int_X [u(x)Tf]^q \, d\mu(x) = \int_X \left[u(x) \int_X K(x,y) f(y) \, d\mu(y) \right]^q \, d\mu(x) \\
&= \int_X u(x)^q \left[\left(\int_X K(x,y)^\beta f(y)\Phi(y)s(y)K(x,y)^{1-\beta}\Phi(y)^{-1}s(y)^{-1} \right) \, d\mu(y) \right]^q \, d\mu(x) \\
&\leq \int_X u(x)^q \left[\left(\int_X K(x,y)^{\beta p} \left(f(y)\Phi(y)s(y) \right)^p \, d\mu(y) \right)^{q/p} \right. \\
&\quad \left. \times \left(\int_X K(x,y)^{(1-\beta)p'} \Phi(y)^{-p'} s(y)^{-p'} \, d\mu(y) \right)^{q/p'} \right] \, d\mu(x)
\end{aligned}
$$

by Holder's inequality.

$$
\begin{aligned}
&= \int_X u(x)^q \left[\left(\int_X K(x,y) \left(f(y)\Phi(y)s(y) \right)^p \, d\mu(y) \right)^{q/p} \right. \\
&\quad \left. \times \left(\int_X K(x,y) \Phi(y)^{-p'} s(y)^{-p'} \, d\mu(y) \right)^{q/p'} \right] \, d\mu(x) \\
&\leq (p'+1)^{q/p'} \int_X \left[u(x)^q \left(\int_X K(x,y) \left(f(y)\Phi(y)s(y) \right)^p \, d\mu(y) \right)^{q/p} s(x)^{q/p'} \right] \, d\mu(x)
\end{aligned}
$$

by definition of $s(x)$

$$
\leq (p'+1)^{q/p'} \left\{ \int_X \left(\int_X K(x,y) u(x)^q s(x)^{q/p'} \, d\mu(x) \right)^{p/q} \left(f(y)\Phi(y)s(y) \right)^p \, d\mu(y) \right\}^{q/p}
$$

by Minkowski's integral inequality.

$$
\begin{aligned}
&= (p'+1)^{q/p'} \left\{ \int_X \left(\int_X K(x,y) u(x)^q \left(K(y,z)\Phi(z)^{-p'} \, dz \right)^{\frac{q}{p'(p'+1)}} \, d\mu(x) \right)^{p/q} \right. \\
&\quad \left. \times \left(f(y)\Phi(y)s(y) \right)^p \, d\mu(y) \right\}^{q/p}
\end{aligned}
$$

But,

$$
\int_X K(x,y)\Phi(x)^{-p'} \, dx \leq C^{p'} \left\{ \int_X K(z,y) u(z)^q \, dz \right\}^{-p'/q} .
$$

Since $u(x)$ and $\Phi(x)$ depend on p and q with constant C

$$
\begin{aligned}
&\leq (p'+1)^{q/p'} C^{q/(p'+1)} \left\{ \int_X \left(\int_X K(x,y) u(x)^q \left(\int_X K(z,y) u(z)^q \, dz \right)^{-1/(p'+1)} \, d\mu(x) \right)^{p/q} \right. \\
&\quad \left. \times \left(f(y)\Phi(y)s(y) \right)^p \, d\mu(y) \right\}^{q/p} \\
&\leq (p'+1)^{q/p'} C^{q/(p'+1)} \left(\frac{p'+1}{p'} \right)^{p/q} \left\{ \int_X \left(\int_X K(z,y) u(z)^q \, dz \right)^{pp'/(q(p'+1))} \right. \\
&\quad \left. \times \left(f(y)\Phi(y)s(y) \right)^p \, d\mu(y) \right\}^{q/p} .
\end{aligned}
$$

But, since $u(x)$ and $\Phi(x)$ depend on p and q with constant C

$$\begin{aligned} &\leq (p'+1)^{q/p'} C^{q/(p'+1)} \left(\frac{p'+1}{p'}\right)^{p/q} C^{(pp')/(p'+1)} \left\{\int_X \left(\int_X K(y,z)\Phi(z)^{-p}\,dz\right)^{-p/(p'+1)} \right. \\ &\quad \left. \times \left(f(y)\Phi(y)s(y)\right)^p \,d\mu(y)\right\}^{q/p} \\ &= (p'+1)^{q/p'} C^{q/(p'+1)} \left(\frac{p'+1}{p'}\right)^{p/q} C^{(pp')/(p'+1)} \left(\int_X s(y)^{-p} s(y)^p f(y)^p \Phi(y)^p \,d\mu(y)\right)^{q/p}. \end{aligned}$$

Therefore,

$$\begin{aligned} I^{1/q} &= (p'+1)^{\frac{q^2+pp'}{q^2p'}} (p')^{-p/q^2} C^{\frac{q+pp'}{q(p'+1)}} \left(\int_X f(y)^p \Phi(y)^p \,d\mu(y)\right)^{1/p} \\ &\leq (p'+1)^{\frac{q^2+pp'}{q^2p'}} (p')^{-p/q^2} C^{\frac{q+pp'}{q(p'+1)}} \left(\int_X f(y)^p \nu \,d\mu(y)\right)^{1/p}. \end{aligned}$$

This completes the proof.

Theorem 3.4. *Let $1 < p \leq q < \infty$ and suppose $u = w^{1/q}$ is a weight on X. Then there is a weight ν, finite μ-almost everywhere on X such that the weighted norm inequality:*

$$\left[\int_X (T^* f)^q w \,d\mu\right]^{1/q} \leq C \left[\int_X f^p \nu \,d\mu\right]^{1/p} \qquad \textit{for all } f \geq 0, \tag{3.7}$$

holds, if and only if there is a positive function Φ on X satisfying:

$$\Phi(x)^p \leq \nu \tag{3.8}$$

with $s(x)\left(\int_X K(y,z)\Phi(z)^{-p}\,dz\right)^{1/(p+1)}$ and $C = C(K,p,q)$ is a constant independent of f.

Proof. Follows directly from the proof of Theorem 3.2. by defining T^* as

$$(T^* f)(x) = \int_X K(y,x) f(y)\,dy.$$

Remark 3.2. If we put $q = p$ we obtain Kerman and Sawyer result [4]. Hence, our result gives a better bound than Theorem 1.1.

4. CONSEQUENCES OF OUR MAIN RESULTS

Corollary 4.1. *Suppose that $\Phi, w \geq 0$ are locally integrable with respect to Lebesgue measure on $\mathcal{R}^n$ and that $\Phi(x) = \Phi(|x|)$ is non-increasing as a function of $|x|$.*

Define the convolution operator T by

$$(Tf)(x) = (\Phi^* f)(x) = \int_{\mathcal{R}^n} \Phi(x-y) f(y)\, dy \tag{4.1}$$

for a fixed $p \in (1,\infty)$ and a constant C, depending on p and q. Then there exists $\nu(x) < \infty$ almost everywhere and $C > 0$ such that

$$\left[\int_{\mathcal{R}^n} (Tf)^q w\, dx\right]^{1/q} \leq \left[C \int_{\mathcal{R}^n} f^p \nu\, dx\right]^{1/p} \quad \textit{for all } f \leq 0 \tag{4.2}$$

holds if and only if for all $y \in \mathcal{R}^n$: $\Phi(y)^p \leq \nu$, $u = w^{1/q}$ and $C = C(K,p,q)$ is a constant independent of f.

Proof. The proof is immediate from Theorem 3.1. and Theorem 3.2, if we set $K(x,y) \equiv \Phi(x-y)$.

Corollary 4.2. *Suppose that $w \geq 0$ is locally integrable with respect to Lebesgue measure on $\mathcal{R}_+ = (0,\infty)$. Denote the Laplace transform (L) of f on $\mathcal{R}_+$ by*

$$(Lf)(x) = \int_0^\infty e^{-xy} f(y)\, dy, \quad x \in \mathcal{R}_+$$

for a fixed $p \in (1,\infty)$. Then there exists $\nu(x) < \infty$ almost everywhere and $C > 0$ such that

$$\left[\int_{\mathcal{R}^n} (Lf)^q w\, dx\right]^{1/q} \leq \left[C \int_{\mathcal{R}^n} f^p \nu\, dx\right]^{1/p} \qquad \textit{for all } f \leq 0$$

holds if and only if $(Lw)(x) < \infty, x \in \mathcal{R}_+$.

Comment. There is a similar result for the dual operator as defined in [1].

References

[1] K. F. Andersen and H. P. Heinig, *Weighted norm inequalities for certain integral operators,* Siam J. Math. Annual **14**(4) (1983), 834–844.

[2] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities,* Cambridge, 1959.

[3] C. O. Imoru, *Weighted norm inequalities for Riemann-Liouville Franctional Integral,* Ann. Soc. Sc. Bruxelles. T. **104** (2) (1990), 63–77.

[4] R. Kerman and E. Sawyer, *On weighted norm inequalities for positive linear operators,* Amer. Math. Soc. **25** (1989), 589–593.

[5] B. Muckenhoupt, *Hardy's inequality with weights,* Studia Math. **44** (1972), 31–38.