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UNIVARIATE SHEPARD-LAGRANGE INTERPOLATION

Radu T. Trîmbiţaş

Faculty of Mathematics and Computer Science, "Babeş-Bolyai" University, Cluj-Napoca, Romania

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Abstract. In this paper we study the univariate Shepard-Lagrange interpolation operator

$$S_{n,\mu}^{L,m}(Y;f;x) := S_{n,\mu}^{L,m}(f;x) = \frac{\sum\limits_{k=0}^{n} |x - y_{n,k}|^{-\mu}(L_m f)(x; y_{n,k})}{\sum\limits_{k=0}^{n} |x - y_{n,k}|^{-\mu}}$$

where $(y_{n,k})$ are the interpolation nodes and $(L_m f)(x; y_{n,k})$ is the Lagrange interpolation polynomial with nodes $y_{n,k}, y_{n,k+1}, \ldots, y_{n,k+m}$. Then we give error estimations for various distribution of interpolation nodes.

1. INTRODUCTION

Let $Y = (y_{n,i} \in I,\ i = \overline{0,n},\ n \in \mathbb{N})$ be an infinite matrix where each row is a set of distinct nodes in $I = [-1, 1]$. For $f \in C^r(I)$ the Shepard-Taylor operator is defined by

$$S_{n,\mu}^{r}(Y;f;x) = \frac{\sum\limits_{k=0}^{n} \left(|x - y_{n,k}|^{-\mu} \sum\limits_{j=0}^{r} f^{(j)}(y_{n,k})/j!(x - y_{n,k})^j \right)}{\sum\limits_{k=0}^{n} |x - y_{n,k}|^{-\mu}},$$

where $\mu = r + \alpha$, and $\alpha > 1$ fixed.

The operator $S^r_{n,\mu}$ was introduced by Shepard in [9]. The operator $S^r_{n,\mu}$ and his properties were studied in [4, 6, 2, 5].

We put $y_{n,n+k} = y_{n,n-m+k-1}$ for $k = \overline{1,n}$.

The aim of this paper is to study the univariate Shepard-Lagrage operator

$$S^{L,m}_{n,\mu}(Y;f;x) := S^{L,m}_{n,\mu}(f;x) = \frac{\sum\limits_{k=0}^{n} |x - y_{n,k}|^{-\mu}(L_m f)(x;y_{n,k})}{\sum\limits_{k=0}^{n} |x - y_{n,k}|^{-\mu}} \tag{1}$$

where $m \in \mathbb{N}$, $m < n$ is prescribed and $(L_m f)(x;y_{n,k})$ is the Lagrange interpolation polynomial with the nodes $y_{n,k}, y_{n,k+1}, \ldots, y_{n,k+m}$.

A bivariate variant of this operator was treated in [1].

The Shepard-Lagrange operator has the following properties

$$\begin{aligned} \left(S^{L,m}_{n,\mu}\right)(f;y_{n,k}) &= f(y_{n,k}), \quad k = \overline{0,n}, \\ \left(S^{L,m}_{n,\mu}\right)(e_i;x) &= e_i(x), \end{aligned}$$

where $e_i(x) = x^i$, for $i = \overline{0,m}$.

Gopengauz proved in [7], that for $|a| < 1$ there exists a continuous function on I for which there are no algebraic polynomial P_n of degree less than or equal n such that

$$|f(x) - P_n(x)| = O\left\{\omega\left(f; \frac{\sqrt{1-x^2}}{n}\varepsilon\left(1-x^2\right) + \frac{\delta(n^{-1})}{n^2}\right)\right\} \tag{2}$$

and

$$|f(x) - P_n(x)| = O\left\{\omega\left(f; \frac{\varepsilon\left(|x-a|\right) + \delta(n^{-1})}{n}\right)\right\} \tag{3}$$

$\forall n \in \mathbb{N}$, $\forall x \in I$, where $\varepsilon(u) \downarrow 0$ and $\delta(u) \downarrow 0$, when $u \to 0$.

Della Vechia and Mastroianni have obtained in [4] estimations of above type for Shepard-Taylor operators. We shall prove such estimations hold for Shepard-Lagrange operators.

2. ERROR ESTIMATION FOR SHEPARD-LAGRANGE OPERATOR

If $f \in C^p[a,b]$, $p \in \mathbb{N}$, $p < n$, and $(L_n f)$ is the n-th degree Lagrange interpolation polynomial with nodes $x_0, \dots, x_n$, and $x, x_0, \dots, x_n \in [a,b]$, we have the following estimation of the interpolation error (see [3])

$$|f(x) - (L_n f)(x)| \le M^{p+1}(1+\Lambda_n)\frac{(b-a)^p}{n(n-1)\dots(n-p+1)}\omega\left(f^{(p)}; \frac{b-a}{n-p}\right), \qquad (4)$$

where $\omega(f; .)$ is the usual modulus of continuity for f, and Λ_n is the Lebesgue constant associated to the points x_k and to the interval $[a,b]$.

Let $y_{n,d}$ be the closest point to x. We have from (1) and (4)

$$\begin{aligned}
&\left|f(x) - S_{n,\mu}^{L,m}(f;x)\right| \le \sum_{k=0}^{n}\left|\frac{x-y_{n,d}}{x-y_{n,k}}\right|^{\mu}|f(x)-(L_m f)(x;y_{n,k})| \\
&\le \frac{M^{p+1}(1+\Lambda_m)}{m(m-1)\dots(m-p+1)}\left(\sum_{k=0}^{d-m}\left|\frac{x-y_{n,d}}{x-y_{n,k}}\right|^{\mu}|x-y_{n,k}|^p\omega\left(f^{(p)};\frac{|x-y_{n,k}|}{m-p}\right)+\right. \\
&\sum_{k=d-m+1}^{d}\left|\frac{x-y_{n,d}}{x-y_{n,k}}\right|^{\mu}|y_{n,k+m}-y_{n,k}|^p\omega\left(f^{(p)};\frac{|y_{n,k+m}-y_{n,k}|}{m-p}\right)+ \qquad (5)\\
&\left.\sum_{k=d+1}^{n}\left|\frac{x-y_{n,d}}{x-y_{n,k}}\right|^{\mu}|x-y_{n,k+m}|^p\omega\left(f^{(p)};\frac{|x-y_{n,k+m}|}{m-p}\right)\right) \\
&\le \frac{M^{p+1}(1+\Lambda_m)}{m(m-1)\dots(m-p+1)}(S_1+S_2+S_3).
\end{aligned}$$

Remark 1. Since m is fixed, Λ_m is bounded.

In the sequel one gives estimations of the approximation error for various distribution of knots.

2.1. THE CASE OF ZEROS OF ORTHOGONAL POLYNOMIALS

This case is treated for the simple Shepard operator $S_{n,\mu}^0$ in [2], and the proof follows the ideas from that paper.

Let $(p_n(w,.))$ be the sequence of orthogonal polynomials on I with respect to the weight w defined by

$$w(x) = \psi(x) \prod_{k=0}^{s+1} |x - t_k|^{\gamma_k}, \quad x \in I,$$

where $-1 = t_0 < t_1 < \ldots < t_s < t_{s+1} = 1,\ \gamma_k > -1,\ k = \overline{0, s+1}$, and the function ψ is such that $\int_0^1 \omega(\psi, \delta)\delta^{-1} < \infty$. Let $x_{n,i} = x_{n,i}(w)$ be the zeros of $p_n(w,.)$ and we suppose $x_{n,1} < x_{n,2} < \ldots < x_{n,n}$. We set $x_{n,i} = \cos\theta_{n,i}$, $i = \overline{0, n+1}$, where $x_{n,0} = -1$, $x_{n,n+1} = 1$ and $\theta_{n,i} \in [0, \pi]$. We shall use a result of Nevai [8, page 166]

$$\theta_{n,i} - \theta_{n,i+1} \sim \frac{1}{n}. \tag{6}$$

The matrix Y has on his rows the zeros of $(1 - x^2)p_n(w; x)$.

Theorem 2. *If $f \in C^p(I)$ and $\mu > p + 1$, we have*

$$\left| f - \left(S_{n,\mu}^{L,m}\right)(f; x)\right| \le \frac{(1 - x^2)^p}{n^{\mu-1}} \int_{1/n}^{1} \frac{\omega(f^{(p)}, t\sqrt{1 - x^2})}{t^{\mu-p}} dt. \tag{7}$$

Proof. The relation (6) implies

$$\begin{aligned} |x - y_{n,d}| &\le \frac{\text{const}}{n}\sqrt{1 - x^2}, \\ |x - y_{n,k}| &\ge \frac{\text{const}}{n}|k - d|\sqrt{1 - x^2}. \end{aligned}$$

Also, for $\delta_2 \ge \delta_1$, we have

$$\frac{\omega(f; \delta_2)}{\delta_2} \le 2\frac{\omega(f; \delta_1)}{\delta_1}.$$

Now it follows estimations for S_1, S_2 and S_3(introduced in (5)).

$$S_1 \le C\left(1 + \frac{1}{m}\right)\left(\frac{\sqrt{1 - x^2}}{n}\right)^p \sum_{k=0}^{d-m} \frac{1}{|k - d|^{\mu-p}} \omega\left(f^{(p)}; \frac{|k - d|}{n}\sqrt{1 - x^2}\right) \tag{8}$$

$$S_2 \le Cm^p(m - 1)\left(\frac{\sqrt{1 - x^2}}{n}\right)^p \omega\left(f^{(p)}; \frac{\sqrt{1 - x^2}}{n}\right). \tag{9}$$

Since $|x - y_{n,k+m}| \le |x - y_{n,k}| + c\dfrac{m}{n}$ and

$$\left|\frac{x - y_{n,d}}{x - y_{n,k}}\right|^\mu |x - y_{n,k+m}|^p \le \sum_{l=0}^{p} \binom{p}{l} \frac{\left(\frac{m}{n}\sqrt{1 - x^2}\right)^{l+\mu}}{\left(\dfrac{|k - d|}{n}\sqrt{1 - x^2}\right)^{\mu-p+l}}$$

$$\leq \left(\frac{\sqrt{1-x^2}}{n}\right)^p m^p \frac{(p+1)}{|k-d-1|^{\mu-p}},$$

we have

$$\begin{aligned} S_3 \;\; &\leq \left(\frac{\sqrt{1-x^2}}{n}\right)^p m^p(p+1) \sum_{k=d+1}^{n} \frac{1}{|k-d-1|^{\mu-p}} \left(\left(1+\frac{1}{m}\right)\omega(f^{(p)};|x-y_{n,k}|\right) \\ &\quad +\omega\left(f^{(p)};\frac{\sqrt{1-x^2}}{n}\right). \end{aligned}$$

But, for $\mu - p > 1$, $\sum_{k=d+1}^{n} |k-d-1|^{p-\mu}$ is bounded and we obtain

$$\begin{aligned} S_3 \;\; &\leq \left(\frac{\sqrt{1-x^2}}{n}\right)^p \left(C_1 \sum_{k=d+1}^{n} \frac{1}{|k-d-1|^{\mu-p}} \left(\omega(f^{(p)};|x-y_{n,k}|\right) + \right. \\ &\quad \left. C_2\omega\left(f^{(p)};\frac{\sqrt{1-x^2}}{n}\right)\right). \end{aligned} \tag{10}$$

Since

$$\omega\left(f^{(p)};\frac{\sqrt{1-x^2}}{n}\right) \leq \frac{1}{n^{\mu-p-1}} \int_{1/n}^{1} \frac{\omega(f^{(p)};t\sqrt{1-x^2})}{t^{\mu-p}}dt$$

and

$$\begin{aligned} S_1+S_2+S_2 \;\; &\leq \;\; \left(\frac{\sqrt{1-x^2}}{n}\right)^p \left[C_1\omega\left(f^{(p)};\frac{\sqrt{1-x^5}}{n}\right) + \right. \\ &\quad \left. C_2\sum_{k=2}^{n} \frac{1}{|k-d|^{\mu}}\omega\left(f^{(p)};\frac{|k-d|}{n}\sqrt{1-h^2}\right)\right], \end{aligned}$$

(7) follows.

2.2. OTHER DISTRIBUTIONS

We consider the distribution given by $x = x_p : [0,1] \mapsto [-1,1]$

$$x = x(\theta) = \begin{cases} (2\theta)^{2p+1} - 1, & \theta \in [0, \frac{1}{2}] \\ -(2-2\theta)^{2p+1} + 9, & \theta \in [\frac{1}{2}, 1] \end{cases}, \tag{11}$$

and $X = \left(y_{n,k} = x(k/n),\ k = \overline{0,n},\ n \in \mathbb{N}\right)$.

This distribution is considered in [4] and we follow the ideas from that paper.

Theorem 3. *If $f \in C^q(I)$ and $\mu > q+\alpha$, $\alpha > 1$, then*

$$\left|f - \left(S_{n,\mu}^{L,m}\right)(f;x)\right| \leq A\frac{\left[(1-x^2)^{2p/(2p+1)}\right]^q}{n^{\mu-1}}\int\limits_{1/n}^{1}\frac{\omega(f^{(q)},t(1-x^2)^{2p/(6p+1)})}{t^{\mu-q}}dt. \qquad (12)$$

where A is a constant depending on p, q, μ, m and α.

Proof. The function x given by (11) is increasing on [0,1] and x' is convex increasing on $[0,1/2]$ and convex decreasing on $[1/2,0]$. It holds

$$x'(\theta) \leq 2(2p+1)(1-x^2)^{2p/(2p+1)}. \qquad (13)$$

Because the points are symmetric with respect to the origin, we need to prove the theorem only for $x > 0$ and $x \neq y_{n,k}$, $k = \overline{0,n}$. Thus, $x = x(\theta)$, $\theta \in [\frac{1}{2}, 1]$ and we shall suppose $y_{n,d-1} < x < y_{n,d}$, $y_{n,d}$ being the closest point to x. Also,

$$|x - y_{n,d}| = \left|\int_0^{\frac{d}{n}} x'(u)du\right| \leq \frac{x'(\theta)}{n}. \qquad (14)$$

Let us estimate now the error.

$$\left|f(x) - S_{n,\mu}^{L,m}(X;f;x)\right| \leq \frac{M^{q+1}(1+\Lambda_m)}{m(m-1)\dots(m-q+1)}(S_1+S_3+S_3+S_4),$$

where

$$\begin{aligned}
S_1 &= \sum_{0<y_{n,k}\leq y_{n,d-m-1}} \left|\frac{x-y_{n,d}}{x-y_{n,k}}\right|^\mu |x-y_{n,k}|^q \omega\left(f^{(q)}; \frac{|x-y_{n,k}|}{m-q}\right), \\
S_2 &= \sum_{y_{n,d-m-1}<y_{n,k}\leq y_{n,d-1}} \left|\frac{x-y_{n,d}}{x-y_{n,k}}\right|^\mu |y_{n,k+m}-y_{n,k}|^q \omega\left(f^{(q)}; \frac{|y_{n,k+m}-y_{n,k}|}{m-q}\right), \qquad (15) \\
S_3 &= \sum_{y_{n,d}<y_{n,k}\leq y_{n,n}} \left|\frac{x-y_{n,d}}{x-y_{n,k}}\right|^\mu |x-y_{n,k+m}|^q \omega\left(f^{(q)}; \frac{|x-y_{n,k+m}|}{m-q}\right)
\end{aligned}$$

and

$$S_4 = \sum_{y_{n,k}\leq 0} \left|\frac{x-y_{n,d}}{x-y_{n,k}}\right|^\mu |x-y_{n,k}|^q \omega\left(f^{(q)}; \frac{|x-y_{n,k}|}{m-q}\right).$$

When $0 < x_k \leq x_{d-m-1}$ we have

$$|x - y_{n,k}| = \int_{\frac{k}{n}}^{\theta} x'(u) \geq x'(\theta)\frac{d-1-k}{n} \qquad (16)$$

and

$$\frac{\omega\left(f^{(q)};\frac{|x-y_{n,k}|}{m-p}\right)}{|x-y_{n,k}|}\leq 2\left(1+\frac{1}{m-p}\right)\frac{\omega\left(f^{(q)};\frac{x'(\theta)(d-k-1)}{n}\right)}{\frac{x'(\theta)(d-k-1)}{n}}$$

and thus

$$S_1\leq C_1\sum_{k=[n/2]+1}^{d-m-1}\left(\frac{x'(\theta)}{n}\right)^q\frac{\omega\left(f^{(q)};\frac{x'(\theta)(d-k-1)}{n}\right)}{(d-k-1)^{\mu-q}},\tag{17}$$

where C_1 is a constant which depends on p,q,μ,m and α.

To estimate S_2 we note that

$$|y_{n,k+m}-y_{n,k}|=\int_{\frac{k}{n}}^{\frac{k+m}{n}}x'(u)du\leq\frac{m}{n}x'\left(\frac{k}{n}\right)\leq\frac{m}{n}x'(\theta)$$

and

$$\omega\left(f^{(q)};\frac{|y_{n,k+m}-y_{n,k}|}{m-1}\right)\leq 2\left(1+\frac{1}{m-p}\right)\omega\left(f^{(q)};\frac{x'(\theta)}{n}\right).$$

These inequalities lead us to

$$S_2\leq\left(\frac{x'(\theta)}{n}\right)^q C_2(m-1)\omega\left(f^{(q)};\frac{x'(\theta)}{n}\right).\tag{18}$$

For S_3 we have $y_{n,k}\geq y_{n,d}$ and

$$\begin{aligned}|x-x_k| &= \left|\int_\theta^{k/n}x'(u)du\right|>\int_\theta^{(\theta+k/n)/2}x'(u)du\\ &= \frac{k/n-\theta}{2}x'\left(\frac{k/n+\theta}{2}\right)>\frac{k/n-\theta}{2}x'\left(\frac{\theta+1}{2}\right)\\ &= \left(\frac{k}{n}-\theta\right)2^{-2p-1}x'(\theta)\end{aligned}$$

(see also [4] for this estimation).

Hence

$$|x-x_k|>\frac{k-d}{n}2^{-2p-1}x'(\theta).$$

But

$$\begin{aligned}\left|\frac{x-y_{n,d}}{x-y_{n,k}}\right|^\mu|x-y_{n,k+m}| &\leq \left|\frac{x-y_{n,d}}{x-y_{n,k}}\right|^\mu\sum_{l=0}^{q}\binom{q}{l}|x-y_{n,k}|^{q-l}|y_{n,k}-y_{n,k+m}|^l\\ &\leq\left(\frac{x'(\theta)}{n}\right)^\mu\sum_{l=0}^{q}\binom{q}{l}\frac{\left(\frac{m}{n}x'(\theta)\right)}{\left|\frac{k-d}{n}x'(\theta)\right|^{\mu-q+l}}\\ &\leq C_3\left(\frac{x'(\theta)}{n}\right)^q\frac{1}{\left|\frac{k-d}{n}\right|^{\mu-q}};\end{aligned}$$

so

$$S_3 \leq \left(\frac{x'(\theta)}{n}\right)^q \left(C_3 \sum_{k=d}^{n} \frac{\omega\left(f^{(q)}; \frac{|k-d|}{n} x'(\theta)\right)}{|k-d|^{\mu-q}} + C_4 \omega\left(f^{(q)}; \frac{x'(\theta)}{n}\right) \sum_{k=d}^{n} \frac{1}{|k-d|^{\mu-q}}\right).$$

Because $\mu - q = \alpha > 1$, $\sum_{k=d}^{n} |k-d|^{\mu-q}$ is bounded and

$$S_3 \leq \left(\frac{x'(\theta)}{n}\right)^q \left(C_3 \sum_{k=d}^{n} \frac{\omega\left(f^{(q)}; \frac{|k-d|}{n} x'(\theta)\right)}{|k-d|^{\mu-q}} + K\omega\left(f^{(q)}; \frac{x'(\theta)}{n}\right)\right).$$

We can estimate S_4 as we do for S_1, S_2 and S_3 (because of symmetry reason). Finally, because

$$S_1 + S_2 + S_3 + S_4 \leq \left(\frac{x'(\theta)}{n}\right)^q \left(K_1 \sum_{k=2}^{n} \frac{\omega\left(f^{(q)}; \frac{k}{n} x'(\theta)\right)}{\left(\frac{k}{n}\right)^{\mu-q}} + K_2 \omega\left(f^{(q)}; \frac{x'(\theta)}{n}\right)\right)$$

and

$$\omega\left(f^{(q)}; \frac{x'(\theta)}{n}\right) \leq \int_{1/n}^{1} \frac{\omega\left(f^{(q)}; \frac{x'(\theta)}{n}\right)}{t^{\mu-q}} dt,$$

the conclusion of Theorem results immediately using (13).

For Shepard-Lagrange operators there is also an analogous of (12) for some interior points. Della Vechia and Mastroianni have shown in [4] such a result for Shepard-Taylor operators. Let be now the distribution given by

$$z(\theta) = (2\theta - 1)^{2p+1}, \quad p \in \mathbb{N}, k = \overline{0, n}, \ n \text{ even},$$

and the matrix

$$Z = (z_{n,k} = z(k/n)). \tag{19}$$

We have

Theorem 4. *If $f \in C^q(I)$ then*

$$\left|f(x) - S_{n,\mu}^{L,q}(Z; f; x)\right| \leq A \frac{\left[|x|^{2p/(2p+1)}\right]^q}{n^{\mu-1}} \int_{1/n}^{1} \frac{\omega(f^{(q)}; t|x|^{2p/(2p+1)})}{t^{\mu-q}} dt.$$

Proof. z is increasing on $[0,1]$ and z' is convex decreasing on $[0,\frac{1}{2}]$ and convex increasing on $[\frac{1}{2},1]$. For symmetry reason we need to prove the theorem for $x<0$, $x\neq z_{n,k}$, $k=\overline{0,n}$.

We have

$$|x-z_{n,d}|=\left|\int_{\theta}^{d/n} z'(u)du\right|\leq\frac{z'(\theta)}{n},$$

and (13) is replaced by

$$z'(\theta)\leq 2(2p+1)|x|^{2p/(2p+1)};$$

the proof proceeds as for the previous theorem.

In [5] the authors give more general matrix of nodes. Using (4) and techniques from the last-cited paper we can prove analogous of Theorem for those matrices.

Remark 5. In (19), for $p=0$, we obtain equispaced nodes.

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