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POWER PRODUCT INEQUALITIES FOR THE GAMMA FUNCTION

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Abstract. In this note some bounds for the gamma function are given.

1. INTRODUCTION

The gamma function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \qquad (x > 0)$$

is an extension of the factorial of a natural number. The gamma function is one of the most important functions in analysis and its applications. There exists a rich literature on inequalities for the gamma function. For instance, see H. Alzer [1] and references therein.

In the paper [2] we find the following inequalities

$$\frac{n!\, n^x}{x(x+1)\cdots(x+n)} < \Gamma(x) < \frac{n!\, n^x}{x(x+1)\cdots(x+n)} \cdot \frac{n+x}{n} \cdot \exp \frac{1}{2(n-1)} \qquad (0 < x < 1).$$

Our aim is to give an upper and a lower power product estimate for the gamma function on the interval $[1, \infty)$.

2. RESULTS

Now, we can state and prove our result.

Theorem. *The inequalities*

$$\begin{aligned}
&\frac{n!}{x(x+1)\cdots(x+n-1)}\cdot\left(\frac{x+n}{n+1}\right)^{x+n}\cdot\exp(H(n)-1-C)(x-1)\\
\le\ &\Gamma(x)\le\frac{(n+1)!}{x(x+1)\cdots(x+n)}\cdot\left(\frac{x+n}{n+1}\right)^{x+n}\cdot\exp(H(n+1)-1-C)(x-1)
\end{aligned}\tag{1}$$

$(x\ge 1,\ n=1,2,\ldots)$ *hold, where* $C=0.57721...$ *is the Euler constant and* $H(n)=1+1/2+\ldots 1/n$. *Equality occurs if* $x=1$.

Proof. For the digamma function $\psi(x)=(\ln\Gamma(x))'$ $(x>0)$, $\psi(1)=-C$, we have

$$\psi'(x)=\sum_{i=0}^{\infty}\frac{1}{(x+i)^2}\qquad(x>0)$$

that is

$$\psi'(x)=\sum_{i=0}^{n-1}\frac{1}{(x+i)^2}+\sum_{i=0}^{\infty}\frac{1}{(x+n+i)^2}\,.\tag{2}$$

It is well known that if $g(t)$ is a strictly decreasing positive function with $\lim_{t\to\infty}g(t)=0$, then

$$\int_0^{\infty}g(t)\,dt<\sum_{i=0}^{\infty}g(i)<g(0)+\int_0^{\infty}g(t)\,dt\,.\tag{3}$$

Letting $g(t)=1/(x+n+t)^2$ and using $\int_0^{\infty}g(t)\,dt=1/(x+n)$ by virtue of (3) the equality (2) becomes

$$\sum_{i=0}^{n-1}\frac{1}{(x+i)^2}+\frac{1}{x+n}<\psi'(x)<\sum_{i=0}^{n}\frac{1}{(x+i)^2}+\frac{1}{x+n}\quad(x>0)\,.$$

Integrating these inequalities from 1 to x we obtain

$$\begin{aligned}
H(n)\ &-\ C+\ln\frac{x+n}{1+n}-\sum_{i=0}^{n-1}\frac{1}{x+i}\\
&\le\ \psi(x)\le H(n+1)-C+\ln\frac{x+n}{1+n}-\sum_{i=0}^{n}\frac{1}{x+i}\quad(x\ge 1)\,.
\end{aligned}$$

Finally, integrating the last inequalities from 1 to x, after simple calculations, we obtain our inequalities.

Remark. A simple calculation shows that the left and the right side of (1) tend to $\Gamma(x)$ as $n \to \infty$.

We are going to mention three simple consequences of Theorem.

Corollary 1. *In the case when* $n = 1$ *the inequalities* (1) *become*

$$\frac{1}{x}\left(\frac{x+1}{2}\right)^{x+1} \cdot \exp C(1-x) \le \Gamma(x) \le \frac{2}{x(x+1)} \cdot \left(\frac{x+1}{2}\right)^{x+1} \cdot \exp\left(C - \frac{1}{2}\right)(1-x),$$

where $x \ge 1$.

Corollary 2. *In the case when* $x = m+1$ *the inequalities* (1) *become*

$$\begin{aligned} & \frac{n!}{(m+1)(m+2)\cdots(m+n)} \cdot \left(1 + \frac{m}{n+1}\right)^{m+n+1} \cdot \exp(H(n) - 1 - C)m \le m! \\ \le \quad & \frac{(n+1)!}{(m+1)(m+2)\cdots(m+n+1)} \cdot \left(1 + \frac{m}{n+1}\right)^{m+n+1} \cdot \exp(H(n+1) - 1 - C)m, \end{aligned}$$

where $m = 0, 1, \ldots$.

Corollary 3. *In the case when* $x = 3/2$ *the inequalities* (1) *become*

$$\begin{aligned} & \frac{2^{n+1} n!}{3 \cdot 5 \cdots (2n+1)} \cdot \left(\frac{2n+3}{2(n+1)}\right)^{3/2+n} \cdot \exp\frac{1}{2}(H(n) - 1 - C) < \sqrt{\pi} \\ < \quad & \frac{2^{n+2}(n+1)!}{3 \cdot 5 \cdots (2n+3)} \cdot \left(\frac{2n+3}{2(n+1)}\right)^{3/2+n} \cdot \exp\frac{1}{2}(H(n+1) - 1 - C), \end{aligned}$$

where $n = 1, 2, \ldots$.

References

[1] H. Alzer, *On some inequalities for the gamma and psi functions,* Math. Comp. **66** (1997), No. 217, 373–389.

[2] I. B. Lazarević, A. Lupas, *Functional equations for Wallis and gamma functions,* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 461 – No. 497 (1974), 245–251.