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THE PATH IS THE TREE WITH SMALLEST GREATEST LAPLACIAN EIGENVALUE

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Abstract. It is shown that among all trees with a fixed number of vertices the path has the smallest value of the greatest Laplacian eigenvalue.

INTRODUCTION

As in the preceding paper [3], by G is denoted a graph on n vertices, and by $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$ its Laplacian eigenvalues. Further, denote by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ the ordinary eigenvalues of G, i. e., the eigenvalues of the adjacency matrix $A(G)$ [1].

A tree is a connected acyclic graph. A tree with n vertices possesses $n-1$ edges. Recall that $n-1$ is the smallest number of edges in a connected n-vertex graph.

For $n \geq 2$, the n-vertex path P_n is the tree possessing exactly two vertices of degree one (and therefore $n-2$ vertices of degree 2). For $n = 1$, the path P_1 is defined as the graph consisting of one isolated vertex.

For additional details on the notation and terminology used here see the preceding paper [3].

Within certain recent investigations in theoretical chemistry [5, 6] the problem has been encountered to characterize the tree on n vertices, having the smallest μ_1-value among all n-vertex trees. Literature search revealed that the solution of this problem is not well known. In 1999 Li [7] claimed that the respective extremal tree was the path, but instead of proving this, he referred to an unpublished work by H. Yuan. This work by H. Yuan could not be recovered in 2002 in the richly supplied mathematical library of the Bielefeld University.

In view of all this, we decided to communicate the proof of the following:

Theorem 1. *Among all trees with a fixed number of vertices the path has the smallest value of the greatest Laplacian eigenvalue.*

PROOF OF THEOREM 1

The proof of Theorem 1 is easy, provided certain known results from the theory of Laplacian and ordinary graph spectra are taken into account. In Lemmas 1 and 2 we re-state two well known graph–spectral properties [1, 9, 10, 11].

Lemma 1. *Let G' be a graph obtained by deleting an edge from the graph G. Then for $i = 1, 2, \ldots, n-1$, $\mu_i(G) \geq \mu_i(G') \geq \mu_{i+1}(G)$.*

Lemma 2. *Let G' be a graph obtained by deleting an edge from the graph G. Then $\lambda_1(G') \leq \lambda_1(G)$. If G and G' are connected, then $\lambda_1(G') < \lambda_1(G)$.*

Lemma 3 is a less well known result. It was first communicated in [4] (see also [2]). Lemma 4 is due to Lovaśz and Pelikán [8].

Lemma 3. *Let T be an n-vertex tree and $\mathcal{L}(T)$ its line graph. Then, for $i = 1, 2, \ldots, n-1$, $\mu_i(T) = \lambda_i(\mathcal{L}(T)) + 2$.*

Lemma 4. *Among n-vertex trees, the path P_n has the smallest greatest ordinary eigenvalue.*

From Lemma 1 follows that among connected graphs, some graph with minimum number of edges, i. e., some tree, will have the smallest greatest Laplacian eigenvalue.

From Lemma 3 follows that the n-vertex tree with the smallest greatest Laplacian eigenvalue is the tree whose line graph has the smallest greatest ordinary eigenvalue. This line graph possesses $n-1$ vertices and is connected.

The line graphs of all n-vertex trees, except the path, possess cycles. The line graph of P_n is P_{n-1}.

From Lemma 2 follows that for any connected cycle–containing $(n-1)$-vertex graph G there is an $(n-1)$-vertex tree (namely any spanning tree of G), whose λ_1-value is smaller than $\lambda_1(G)$. By Lemma 4, of all these $(n-1)$-vertex trees the path P_{n-1} has the smallest λ_1-value.

Now, because P_{n-1} happens to be the line graph of a tree, namely of P_n, it follows that P_n has the smallest greatest Laplacian eigenvalue among all n-vertex trees, which is just what was claimed in Theorem 1.

DISCUSSION

Because Lemma 3 holds for all bipartite graphs [4], by deducing Theorem 1 we, in fact, proved a stronger result:

Theorem 2. *Among all connected bipartite graphs with a fixed number of vertices the path has the smallest value of the greatest Laplacian eigenvalue.*

The expression $\lambda_1(P_n) = 2\,\cos[\pi/(n+1)]$ is long known [1, 8]. Then in view of Lemma 3 we get

$$\mu_1(P_n) = 2 + 2\,\cos\frac{\pi}{n+1}\ .$$

Evidently, $\mu_1(P_n) < 4$.

Bearing in mind the classical result of Smith [12] (in which all graphs with $\lambda_1 < 2$ and $\lambda_1 = 2$ are characterized) we arrive at the following corollary of Theorem 2:

Theorem 3. *The only connected bipartite graphs whose greatest Laplacian eigenvalues are less than 4 are the paths* $P_n\ ,\ n = 1, 2, 3, \ldots$.

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