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THE STAR IS THE TREE WITH GREATEST GREATEST LAPLACIAN EIGENVALUE

Ivan Gutman

Faculty of Science, P. O. Box 60, 34000 Kragujevac, Yugoslavia

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Abstract. It is shown that among all trees with a fixed number of vertices the star has the greatest value of the greatest Laplacian eigenvalue.

INTRODUCTION

Throughout this paper n denotes an integer greater than 1. All square matrices are assumed to be of order n and all vectors are assumed to be column-vectors of dimension n. If $\vec{C}$ is a column-vector, then $\vec{C}^t$ is its transpose, which is a row-vector of dimension n. The sum of all the n components of the vector $\vec{C}$ is denoted by $\sigma(\vec{C})$.

By I we denote the unit matrix and by J the square matrix whose all elements are unity. By $\vec{0}$ and $\vec{j}$ we denote the vector whose all components are respectively equal to zero and unity.

Let G be a graph on n vertices. Label the vertices of G by $v_1, v_2, \ldots, v_n$. Then the adjacency matrix $A(G)$ of G is a square matrix of order n, defined so that its (i,j)-entry is unity if the vertices v_i and v_j are adjacent and is zero otherwise.

The number of first neighbors of a vertex v is the degree of this vertex and is denoted by $d(v)$. Note that if v_i is a vertex of the graph G , then $d(v_i)$ is equal to the sum of the i-th row of the adjacency matrix $A(G)$.

Let $D(G)$ be a square matrix whose diagonal entries are $d(v_1), d(v_2), \ldots, d(v_n)$ and the off-diagonal elements are zero.

Then $L(G) = D(G) - A(G)$ is the Laplacian matrix of G .

The eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of $L(G)$ are called the Laplacian eigenvalues of the graph G and the respective eigenvectors $\vec{C}_1, \vec{C}_2, \ldots, \vec{C}_n$ the Laplacian eigenvectors of the graph G . Thus the equality $L(G)\,\vec{C}_i = \mu_i\,\vec{C}_i$ is obeyed for all $i = 1, 2, \ldots, n$. We label the Laplacian eigenvalues so that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.

Details of the theory of Laplacian spectra of graphs can be found in some of the numerous reviews published on this topic [2, 3, 4]. Here we only mention that it is always $\mu_n = 0$.

SOME AUXILIARY RESULTS

Lemma 1. *The vector $\vec{j}$ is a Laplacian eigenvector of any n-vertex graph, corresponding to the eigenvalue $\mu_n = 0$.*

Proof. Because the sum of any row of $L(G)$ is equal to zero, $L(G)\,\vec{j} = \vec{0} = 0 \cdot \vec{j}$.

In view of Lemma 1 we may choose $\vec{C}_n = \vec{j}$. In what follows we assume that the Laplacian eigenvectors $\vec{C}_i$, $i = 1, 2, \ldots, n-1$, are orthogonal to $\vec{C}_n$. If $\mu_{n-1} \neq 0$ then this orthogonality condition is automatically satisfied. If however, $\mu_{n-k} = 0$ for some $k \geq 1$, then the requirement that the eigenvectors $\vec{C}_{n-1}, \ldots, \vec{C}_{n-k}$ are chosen to be orthogonal to $\vec{C}_n$ must be additionally stipulated. Recall that $\mu_{n-1} \neq 0$ if and only if the graph G is connected [2, 3, 4].

From the fact that for any vector $\vec{C}$ the scalar product $\vec{j}^t \bullet \vec{C}$ is equal to $\sigma(\vec{C})$, it follows:

Lemma 2. *For any graph G and any $1 \leq i \leq n-1$, $\sigma(\vec{C}_i) = 0$.*

Denote by $\bar{G}$ the complement of the graph G and its Laplacian eigenvalues by $\bar{\mu}_1 \geq \bar{\mu}_2 \geq \cdots \geq \bar{\mu}_{n-1} \geq \bar{\mu}_n = 0$.

Lemma 3. *If $\vec{C}_i$ is a Laplacian eigenvector of the graph G, then $\vec{C}_i$ is a Laplacian eigenvector of the graph $\bar{G}$.*

Proof. For $i = n$ Lemma 3 follows from Lemma 1. Assume, therefore, that $1 \leq i \leq n-1$. Then, by Lemma 2, $\sigma(\vec{C}_i) = 0$.

From the construction of the complement of a graph it is clear that $L(G)+L(\bar{G}) = n\,I - J$. Consequently,

$$\begin{aligned} L(\bar{G})\,\vec{C}_i &= [n\,I - J - L(G)]\,\vec{C}_i \\ &= n\,I\,\vec{C}_i - J\,\vec{C}_i - L(G)\,\vec{C}_i \\ &= n\,\vec{C}_i - \sigma(\vec{C}_i)\,\vec{j} - \mu_i\,\vec{C}_i \\ &= (n - \mu_i)\,\vec{C}_i \end{aligned}$$

This not only proves that $\vec{C}_i$ is an eigenvalue of $L(\bar{G})$, but also shows the way in which the Laplacian eigenvalues of G and $\bar{G}$ are related:

Lemma 4. *For $i = n$, $\bar{\mu}_i = \mu_i = 0$. For $i = 1, 2, \ldots, n-1$, $\bar{\mu}_i = n - \mu_{n-i}$.*

As a direct consequence of Lemma 4 we have

Lemma 5. *If $\bar{G}$ is not connected, then $\mu_1 = n$. If $\bar{G}$ is connected, then $\mu_1 < n$.*

The Lemmas 4 and 5 are previously known results [2, 3, 4].

THE MAIN RESULT AND ITS PROOF

A tree is a connected acyclic graph. The star S_n is the n-vertex tree in which $n-1$ vertices are of degree 1 and one vertex is of degree $n-1$.

Lemma 6. *For any $n \geq 2$, the greatest Laplacian eigenvalue of the n-vertex star is equal to n.*

Proof. The complement of the star S_n is disconnected (and consists of the complete graph on $n-1$ vertices and an isolated vertex). Thus Lemma 6 follows from Lemma 5.

Lemma 7. *If T is any n-vertex tree, not isomorphic to S_n, then $\bar{T}$ is connected.*

Proof. A graph is connected if there is a path between any two of its vertices. Let u and v be two distinct vertices of T. We show that in $\bar{T}$ there always exists a path connecting u and v.

Case 1. Vertices u and v are not adjacent in T. Then these vertices are adjacent in $\bar{T}$ and are thus connected by an edge. We are done.

Case 2. Vertices u and v are adjacent in T. Then either
$d(u) = 1$ and $d(v) = 1$ (subcase 2.1), or
$d(u) = 1$ and $d(v) > 1$ (subcase 2.2), or
$d(u) > 1$ and $d(v) = 1$ (subcase 2.3), or
$d(u) > 1$ and $d(v) > 1$ (subcase 2.4).

Subcase 2.1 implies $T = S_2$, contradiction.

Subcase 2.2. If $d(u) = 1$ and $d(v) > 1$ then either (i) all neighbors of v are of degree 1, or (ii) at least one neighbor of v, say x, is of degree greater than one. If (i) holds, then T is a star, contradiction. It (ii) holds, then x has a further neighbor y. The vertex y differs from u, and y is not adjacent to either v or u, because otherwise T would possess a cycle. Then in $\bar{T}$, u and y are adjacent and v and y are adjacent. Therefore, in $\bar{T}$ there is a path (u, y, v) connecting u and v.

Subcase 2.3 is treated in a fully analogous manner.

Subcase 2.4. The vertex u has a neighbor x and the vertex v has a neighbor y. The vertices x and y are different and not adjacent, because otherwise T would possess a cycle. Then in $\bar{T}$, u is adjacent to y, v is adjacent to x, and x is adjacent to y. Consequently, in $\bar{T}$, the vertices u and v are joined bu the path (u, y, x, v).

By this all possibilities have been exhausted, and the general validity of Lemma 7 has been verified.

Combining Lemma 7 with Lemma 5 we conclude:

Lemma 8. *If T is any n-vertex tree, not isomorphic to S_n , then $\mu_1(T) < n$.*

Combining Lemmas 6 and 8 we reach our main result:

Theorem 1. *Among all trees with a fixed number of vertices the star has the greatest value of the greatest Laplacian eigenvalue.*

Although elementary, the result of Theorem 1 seems not to be previously reported in the mathematical literature. Even worse, in a recent paper [1] the property $\mu_1 = n$ was erroneously attributed only to the complete graph, from which it would (erroneously) follow that the greatest value of the greatest Laplacian eigenvalue of any n-vertex tree is less than n .

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