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# THE INNER MAPPINGS OF EXTRA LOOPS

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**Abstract.** The present paper is concerned with an investigation of the translations of extra loops. Some of the elements of the multiplication group of an extra loop are proved to be elements of the Bryant–Schneider group of an extra loop. It is established here that every extra loop is embeddable into its Bryant–Schneider group.

## 1. INTRODUCTION

A considerable amount of work has been done on extra loops. It includes those of Fenyves [2], Goodaire and Robinson [3], Solarin and Chiboka [6], Robinson [4, 5], Chiboka [1]. The present work deviates from the previous investigations in the sense that the present concetrates on the translations of extra loops.

**Definition 1.1.** *Let $(G, \cdot)$ be a loop. The translation maps $R(x)$ and $L(x)$ for $x$ in $G$ are defined by $aR(x) = ax$ and $aL(x) = xa$ for all $a, x$ in $G$*

**Definition 1.2.** *A loop $(G, \cdot)$ is an extra loop if $(xy \cdot z)x = x(y \cdot zx)$ for all $x, y, z$ in $G$*

**Definition 1.3.** *Let* $(G,\cdot)$ *be a loop, and* $BS(G,\cdot)$ *the set of all permutations,* $\alpha$*, of* $G$ *such that* $< \alpha R(a)^{-1}, \alpha L(b)^{-1}, \alpha >$ *is an autotopism of* $G$ *for some* $a, b$ *in* $G$*. Then* $BS(G,\cdot)$ *is called Bryant–Schneider group of the loop* $(G,\cdot)$*.*

**Definition 1.4.** *The inner mapping group* $I^*$ *on a loop* $G$ *is defined to be the set of all elements* $U$ *of* $M(G,\cdot)$ *such that* $1U = 1$*, where* $M(G,\cdot)$ *is the group generated by all the permutations* $R(x)$ *and* $L(x)$*. It is known that* $I^*$ *is generated by the permutations*

$$T(x) = R(x)L(x)^{-1};\ R(x,y) = R(x)R(y)R(x,y)^{-1}; L(x,y)L(x)L(y)(yx)^{-1}\,.$$

## 2. MAIN RESULTS

**Theorem 2.1.** *Let* $(G,\cdot)$ *be an extra loop. For each fixed* $a$ *in* $G$*, define the principal isotope* $(G,\circ)$ *of* $(G,\cdot)$*, by* $x \circ y = xR(a)\cdot yL(a)^{-1}$ *for all* $x, y$*, in* $G$*. Let* $\theta_a$ *be the mapping defined by* $x\theta_a = R(a)L(a)^{-1}$ *for all* $x$ *in* $G$*. Then* $\theta_a$ *is an element of* $BS(G,\cdot)$*.*

**Proof.** $(xy)\theta_a = (xy)R(a)L(a)^{-1}$ for all $x, y$ in $G$. Now

$$\begin{aligned} x\theta_a \circ y\theta_a &= x\theta_a R(a)\cdot y\theta_a L(a)^{-1} \\ &= xR(a)L(a)^{-1}R(a)\cdot yR(a)L(a)^{-1}L(a)^{-1} = (a^{-1}\cdot xa)a\cdot a^{-1}(a^{-1}\cdot ya). \end{aligned}$$

Since $G$ is an $M$-loop, we have $a^{-1}(a^{-1}\cdot ya) = ((a^{-1})^2 y)a$, for all $a, y$ in $G$, so

$$(a^{-1}\cdot xa)a\cdot a^{-1}(a^{-1}\cdot ya) = (a^{-1}\cdot xa)a\cdot (a^{-1})^2\cdot y)a.$$

Since $G$ is an extra loop, we have

$$\begin{aligned} (a^{-1}\cdot xa)a\cdot((a^{-1})^2\cdot y)a &= ((a^{-1}\cdot xa)\cdot a((a^{-1})^2 y))a \\ &= ((a^{-1}\cdot xa)\cdot a^{-1}y)a = (a^{-1}\cdot xa)a^{-1}\cdot ya = a^{-1}x\cdot ya \end{aligned}$$

for all $a, x, y$ in $G$. Hence, $x\theta_a \circ y\theta_a = xL(a)^{-1} \cdot yR(a)$ for all $x, y$ in $G$.

$G$ is conjugacy closed and has the inverse properties, so the autotopism

$$A =< R(a), R(a)L(a)^{-1}, R(a) >$$

implies that

$$B =< JR(a)J, R(a), R(a)L(a)^{-1} >=< L(a)^{-1}, R(a), R(a)L(a)^{-1} >$$

is an autotopism of $G$. So $xL(a)^{-1} \cdot yR(a) = (xy)R(a)L(a)^{-1}$ for all $x, y$ in $G$, and $(x \cdot y)\theta_a = x\theta_a \circ y\theta_a$ for all $x, y$ in $G$. Thus $\theta_a$ is a homomorphism from $(G, \cdot)$ to $(G, \circ)$.

Suppose $x\theta_a = y\theta_a$, then

$$xR(a)L(a)^{-1} = yR(a)L(a)^{-1}$$

that is

$$a^{-1} \cdot xa = a^{-1} \cdot ya$$

for all $x, y$ in $G$. This gives $xa = ya$, hence $x = y$ and $\theta_a$ is 1-1.

For each $x$ in $G$, we have $x = a^{-1}[(ax \cdot a^{-1})a] = (ax \cdot a^{-1})R(a)L(a)$.

So $x = (ax \cdot a^{-1})\theta_a$, hence $\theta_a$ is onto. Consequently, $\theta_a$ is an isomorphism and hence $\theta_a$ is an element of $BS(G, \cdot)$.

**Theorem 2.2.** *Let $(G, \cdot)$ be an extra loop, then the inner mapping $R(a)L(a)^{-1}$ is an inner automorphism of $G$ if and only if $a$ is in the nucleus, $N$ of $G$.*

**Proof.** If $a$ is in $N$, then $xy - a = x \cdot ya$ for all $x, y$ in $G$, so

$$\begin{aligned} xR(a)L(a)^{-1} \quad \cdot \quad & yR(a)L(a)^{-1} = (a^{-1} \cdot xa) \cdot (a^{-1} \cdot ya) \\ = \quad & a^{-1}[(xa \cdot a^{-1}) \cdot ya] = a^{-1}[x \cdot ya = a^{-1}[xy \cdot a] = (xy)R(a)L(a)^{-1}. \end{aligned}$$

So, $R(a)L(a)^{-1}$ is an automorphism of $G$.

On the other hand, suppose $R(a)L(a)^{-1}$ is an automorphism of $G$, then

$$(xy)R(a)L(a)^{-1} = xR(a)L(a)^{-1} \cdot yR(a)L(a)^{-1}$$

for all $x, y$ in $G$, that is, for all $x, y$ in $G$, we have

$$a^{-1}[xy \cdot a] = (a^{-1} \cdot xa) \cdot (a^{-1} \cdot ya) = a^{-1}(xa \cdot a^{-1}) \cdot ya] = a^{-1}(x \cdot ya).$$

So, $xy \cdot a = x \cdot ya$ for all $x$, in $G$, hence $a$ is in $N$.

**Theorem 2.3.** *Let $(G, \cdot)$ be an extra loop, then $(G, \cdot)$ is embeddable into a subgroup of $BS(G, \cdot)$.*

**Proof.** Let $RL(G, \cdot) = \{T_a | \; a \in G\}$. Let GRL be the group generated by $RL(G, \cdot)$ for each $a$ in $G$, $T_a$ defines an isomorphism from, $(G, \cdot)$ to $(G, \circ)$ by Theorem 2.1., so $GRL$ is a subgroup of $BS(G, \cdot)$. We associate with each $a$ in $G$, the principal isotope given by $x \circ y = xR(a) \cdot yL(a)^{-1}$ for all $x, y$ in $G$, then $T_a = R(a)L(a)^{-1}$.

Define a binary operation $^*$ on $RL(G, \cdot)$ by $T_a^* T_b = T_{ab}$ for all $a, b$ in $G$. So $x(T_a^* T_b) = xT_{ab}$.

We now show that the set $RL(G, \cdot),^* )$ is a loop

(i) By definition, $T_a^* T_a = T_{ab}$ is in $RL(G, \cdot)$ since $a$ and $b$ in $G$ implies that $ab$ is in $G$.

(ii) Suppose $T_a^* T_b = T_a^* T_c$, then $T_{ab} = T_{ac}$. So $ab = ac$ and $b = c$. Hence $T_b = T_c$.

(iii) $T_e^* T_a = T_{ea} = T_{ae} = T_a^* T_e = T_a$. So $T_e$ is the identity element in $((RLG, \cdot),^* )$.

(iv) If $(T_a^* T_b)^* T_c = T_a^* (T_b^* T_c)$ for all $a, b, c$ in $G$, Then $T_{ab \cdot c} = T_{a \cdot bc}$ gives $ab \cdot c = a \cdot bc$ for all $a, b, c$ in $G$. So $(G, \cdot)$ is a group.

Consequently, $(RL(G, \cdot),^* )$ is a loop.

Define a mapping $\alpha : G \mapsto RL(G, \cdot)$ by a $\alpha = T_a = R(a)L(a)^{-1}$ for all $a$ in $G$. Then $\alpha$ is clearly 1-1 and onto. Now,

$$(ab)\alpha = T_{ab} = T_a^* T_b = a\alpha^* b\alpha.$$

So, $\alpha$ is a homomorphism from $(G, \cdot)$ into $(RL(G, \cdot),^* )$. But the group, $GRL$, generated by the elements of $RL(G, \cdot)$ is a subgroup of $BS(G, \cdot)$, hence the proof is complete.

**Corollary 2.1.** *If $(G, \cdot)$ is an extra loop and $(K, \cdot)$ is a sub-loop of $(G, \cdot)$,* *then $(K,\cdot)$ is embeddable into the group generated by the set $\{R(a)L(a)|\ a\in K\} = TL(K,\cdot)$.*

**Proof.** Let $KRL$ be the group generated by $RL(K,\cdot)$ then $KRL$ is a subgroup of $BS(G,\cdot)$. By the restriction of $\alpha$ to $K$, we obtain an isomorphism as in Theorem 2.3. above.

**Theorem 2.4.** *Let $(G,\cdot)$ be an extra loop. The inner mapping $R(x)R(y)R(xy)^{-1}$ is an automorphism if $x^2 = e$.*

**Proof.** For any $a, b$ in $G$, we have

$$aR(x)R(y)R(xy)^{-1}\cdot bR(x)R(y)R(xy)^{-1} = (ax\cdot y)(xy)^{-1}\cdot(bx\cdot y)(xy)^{-1}$$

$$\begin{aligned}
&= (ax\cdot y)\cdot(y^{-1}x^{-1})\cdot(bx\cdot y)\cdot(y^{-1}x^{-1}) = [(ax\cdot y)\cdot(y^{-1}x^{-1})(bx\cdot y)](y^{-1}x^{-1})\\
&= [(ax\cdot y)\cdot(y^{-1}\cdot x^{-1}(bx))y](y^{-1}x^{-1}) = [(ax\cdot y(y^{-1}\cdot x^{-1}(bx))]y\cdot(y^{-1}x^{-1})\\
&= [(ax)\cdot x^{-1}(bx)]y\cdot y^{-1}x^{-1}\,. \qquad (1)
\end{aligned}$$

On the other hand,

$$(ab)R(x)R(y)R(xy)^{-1} = (ab\cdot x)y\cdot(xy)^{-1} = (ab\cdot x)y\cdot y^{-1}x^{-1}\,.$$

Suppose $x^2 = e$, then $x^{-1} = x$, so (1) becomes

$$\begin{aligned}
[ax\cdot x(bx)]y\cdot y^{-1}x^{-1} &= [ax\cdot(xb)x)]y\cdot y^{-1}x^{-1} = [a\cdot x(xb)x]y\cdot y^{-1}x^{-1}\\
&= [(a\cdot x^2b)x]y\cdot y^{-1}x^{-1} = [ab\cdot x]y\cdot y^{-1}x^{-1}\,.
\end{aligned}$$

Consequently,

$$aR(x)R(y)R(xy)^{-1}\cdot bR(x)R(y)R(xy)^{-1} = (ab)R(x)R(b)R(xy)^{-1}$$

for all $a, b$ in $G$, hence $R(x)R(y)R(xy)^{-1}$ is an automorphism.

# References

[1] V. O. Chiboka, *The Bryant–Schneider group of an extra loop,* Zb. Radova (Kragujevac) **18** (1996), 9–20.

[2] F. Fenyves,*Extra loops I ,* Publ. Math. Debrecen **15** (1968), 235–238.

[3] E. G. Goodaire and D. A. Robinson, *Some special conjugacy closed loops,* Canad Math. Bull. **33**(1) (1990), 73–78.

[4] D. A. Robinson, *The Bryant–Schneider group of a Bol loop,* Ann de la Sociate scientifique de Bruxelles **94** II-III (1980), 69–81.

[5] D. A. Robinson, *Some Moufang G-loops in a Matrix setting,* Comm. Alg. **17**(4) (1998), 843–853.

[6] A. R. T. Solarin and V. O. Chiboka, *A note on G-loops,* Zb. Radova (Kragujevac) **17** (1995), 17–26.