Kragujevac J. Math. 24 (2002) 43–53.

A NOTE ON GRAPHS WITH TWO MAIN EIGENVALUES

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(Received February 25, 2002)

Abstract. Let G be a simple connected or disconnected graph which has exactly two main eigenvalues. Let $G^k = G \setminus k$ be the corresponding vertex deleted subgraph of G. If G^i and G^j are cospectral in this paper we prove that their complementary graphs $\overline{G^i}$ and $\overline{G_j}$ are also cospectral.

Let G be a simple graph of order n with vertex set $V(G) = \{1, 2, \ldots, n\}$. The spectrum of G consists of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of its ordinary adjacency matrix A and is denoted by $\sigma(G)$.

We say that an eigenvalue μ of G is main if and only if $\langle \mathbf{j}, \mathbf{Pj} \rangle = n \cos^2 \alpha \neq 0$, where $\mathbf{j}$ is the main vector (with coordinates equal to 1) and $\mathbf{P}$ is the orthogonal projection of the space $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}_A(\mu)$. The quantity $\beta = |\cos \alpha|$ is called the main angle of μ.

Let $A^k = [a_{ij}^{(k)}]$ for any non-negative integer k. The number N_k of all walks of length k in G equals $\mathbf{sum}\, A^k$, where $\mathbf{sum}\, M$ is the sum of all elements in a matrix M.

According to [2], [3], the generating function $H_G(t)$ is defined by $H_G(t) = \sum_{k=0}^{+\infty} N_k t^k$. Besides, it was proved in [2] that

$$H_G(t) = \frac{1}{t}\left[\frac{(-1)^n P_{\overline{G}}\big(-\frac{t+1}{t}\big)}{P_G\big(\frac{1}{t}\big)} - 1\right], \tag{1}$$

where $P_G(\lambda) = |\lambda I - A|$ is the characteristic polynomial of G and $\overline{G}$ its complementary graph. We also note that $H_G(t)$ can be represented in the form

$$H_G\big(\frac{1}{\lambda}\big) = \frac{n_1\lambda}{\lambda - \mu_1} + \frac{n_2\lambda}{\lambda - \mu_2} + \cdots + \frac{n_k\lambda}{\lambda - \mu_k}, \tag{2}$$

where $n_i = n\beta_i^2$ and $n_1 + n_2 + \cdots + n_k = n$; μ_i and β_i $(i = 1, 2, \ldots, k)$ stand for the main eigenvalues and main angles of G, respectively. Using this notation we can see that $N_m = n_1\mu_1^m + n_2\mu_2^m + \cdots + n_k\mu_k^m$ for any non-negative integer m.

In [1] was proved that the graph G and its complement $\overline{G}$ have the same number of main eigenvalues. We also know that $\lambda_1(G) + \lambda_1(\overline{G}) = n - 1$ if and only if G is regular. More generally, it was proved in [4] the following result.

Theorem 1. *Let $\mu_1, \mu_2, \ldots, \mu_k$ and $\overline{\mu}_1, \overline{\mu}_2, \ldots, \overline{\mu}_k$ be the main eigenvalues of the graph G and its complement $\overline{G}$, respectively. Then $\sum_{i=1}^{k}\big(\mu_i + \overline{\mu}_i\big) = n - k$.*

Let $G = G_1 \cup G_2$ be the union of two regular graphs G_1 and G_2 (not necessarily connected) of order n_1 and n_2 and degree r_1 and r_2 $(r_1 \neq r_2)$, respectively. Using the fact that $H_G(t) = H_{G_1}(t) + H_{G_2}(t)$, we have $H_G(\frac{1}{\lambda}) = \frac{n_1\lambda}{\lambda - r_1} + \frac{n_2\lambda}{\lambda - r_2}$, wherefrom we obtain that G has two main eigenvalues r_1 and r_2.

Let $\overline{r}_1$, $\overline{r}_2$ denote the main eigenvalues of $\overline{G}$ and let $\overline{n}_1 = n\overline{\beta}_1^2$, $\overline{n}_2 = n\overline{\beta}_2^2$, where $\overline{\beta}_1$ and $\overline{\beta}_2$ are the main angles of $\overline{r}_1$ and $\overline{r}_2$, respectively. For the graph $\overline{G}$ let $\overline{A}^k = [\overline{a}_{ij}^{(k)}]$ for any non-negative integer k, where $\overline{A}$ is the adjacency matrix of $\overline{G}$.

In the sequel, we shall use the following notations: $\overline{a}_{ij}^{(k,1)} = \overline{a}_{ij}^{(k)}$ if $i, j \in V(G_1)$; $\overline{a}_{ij}^{(k,2)} = \overline{a}_{ij}^{(k)}$ if $i, j \in V(G_2)$; and $\omega_{ij}^{(k)} = \overline{a}_{ij}^{(k)}$, otherwise. We can see that $\omega_{ij}^{(k)}$ is independent of the choice of vertices i and j. Consequently, in order to simplify the notation, we shall set $\omega_{ij}^{(k)} = \omega_k$. Let

$$\Delta_{k,\ell} = \frac{s_\ell^k + (-1)^{k-1}(r_\ell + 1)^k}{n_\ell} + n_{\overline{\ell}} \Big[\sum_{m=0}^{k-2} s_\ell^m \omega_{k-1-m} \Big] , \tag{3}$$

where $s_\ell = (n_\ell - 1) - r_\ell$ and $\overline{\ell} = 3 - \ell$ for $\ell = 1, 2$. By induction on k we can easily see that

$$\overline{a}_{ij}^{(k,\ell)} = \Delta_{k,\ell} + (-1)^k \sum_{m=0}^{k} \binom{k}{m} a_{ij}^{(m)} \qquad \big(i, j \in V(G_\ell)\big) , \tag{4}$$

by understanding that $a_{ij}^{(0)} = \delta_{ij}$, where δ_{ij} is the Kronecker delta symbol. Since the proof of relation (4) is trivial it will be omitted. We now note that

$$\sum_{\ell=1}^{2} \Big[\sum_{i\in V_\ell} \sum_{j\in V_\ell} \overline{a}_{ij}^{(k,\ell)} \Big] + 2n_1 n_2 \omega_k = \overline{n}_1 \overline{r}_1^k + \overline{n}_2 \overline{r}_2^k , \tag{5}$$

for any non-negative integer k. By a straightforward calculation, it is not difficult to show that the expression for

$$\overline{r}_1 = \frac{(\overline{n}_1 n - 2\overline{n}_1 + n) + (n - n_1 - \overline{n}_1)\, r_1 + (n_1 - \overline{n}_1)\, r_2}{2\overline{n}_1 - n} \tag{6}$$

and

$$\overline{r}_2 = \frac{(\overline{n}_1 n - 2\overline{n}_1 + n - n^2) + (n_1 - \overline{n}_1)\, r_1 + (n - n_1 - \overline{n}_1)\, r_2}{2\overline{n}_1 - n} , \tag{7}$$

can be obtained by solving the following system of equations

$$\begin{aligned} &n_1 s_1 + n_2 s_2 + 2n_1 n_2 = \overline{n}_1 \overline{r}_1 + \overline{n}_2 \overline{r}_2 , \\ &r_1 + r_2 + \overline{r}_1 + \overline{r}_2 = n - 2 . \end{aligned}$$

Observe that the first equation is obtained from relation (5) for $k = 1$, and the second one follows from Theorem for $k = 2$. Besides, setting $k = 2$ in relation (5) and using (3) and (4) we find that

$$\big[n_1 s_1^2 + n_1^2 n_2\big] + \big[n_2 s_2^2 + n_2^2 n_1\big] + 2n_1 n_2 \omega_2 = \overline{n}_1 \overline{r}_1^2 + \overline{n}_2 \overline{r}_2^2 ,$$

where $\omega_2 = s_1 + s_2$. Substituting $\overline{r}_1$ and $\overline{r}_2$ from (6) and (7) in the last relation, by a straightforward calculation we obtain the following quadratic equation:

$$\overline{n}_1^2 - n\overline{n}_1 + \frac{n_1(n - n_1)(r_1 - r_2)^2}{(r_1 - r_2 + n)^2 - 4n_1(r_1 - r_2)} = 0 .$$

Hence,

$$\overline{n}_{1,2} = \frac{n}{2} \pm \frac{n^2 + (n-2n_1)(r_1 - r_2)}{2\sqrt{\Delta}}\,, \tag{8}$$

where $\Delta = (r_1 - r_2 + n)^2 - 4n_1(r_1 - r_2)$. Substituting $\overline{n}_1$ back into (6) and (7), we obtain that

$$\overline{r}_{1,2} = \frac{n - 2 - r_1 - r_2}{2} \pm \frac{\sqrt{\Delta}}{2}\,. \tag{9}$$

Next, we have

$$n_1\omega_k = \sum_{i\in V_1}\Big[\sum_{j\in V_1}\overline{a}_{ij}^{(k-1,1)}\Big] + n_1 s_2 \omega_{k-1}\,;$$

$$n_2\omega_k = \sum_{i\in V_2}\Big[\sum_{j\in V_2}\overline{a}_{ij}^{(k-1,2)}\Big] + n_2 s_1 \omega_{k-1}\,,$$

from which an easy calculation yields $n\omega_k + \big[n_1 r_2 + n_2 r_1 + n\big]\omega_{k-1} = \overline{N}_{k-1}$, where $\overline{N}_k = \mathbf{sum}\,\overline{A}^k$. From the last difference equation, we get

$$\omega_k = \frac{1}{n}\sum_{i=0}^{k-1}(-1)^{k-1-i}\Big(\frac{\nabla}{n}\Big)^{k-1-i}\overline{N}_i\,,$$

where $\nabla = \big[n_1 r_2 + (n-n_1)\,r_1 + n\big]$. Since $\overline{N}_k = \overline{n}_1\,\overline{r}_1^k + \overline{n}_2\,\overline{r}_2^k$ the previous relation is transformed into

$$\begin{aligned}
\omega_k &= \frac{(-\nabla)^{k-1}}{n^k}\sum_{i=0}^{k-1}(-1)^i\Big[\overline{n}_1\Big(\frac{n\,\overline{r}_1}{\nabla}\Big)^i + \overline{n}_2\Big(\frac{n\,\overline{r}_2}{\nabla}\Big)^i\Big] \\
&= \frac{(-1)^{k-1}}{n^k}\sum_{\ell=1}^{2}\overline{n}_\ell\Big[\frac{\nabla^k + (-1)^{k+1}n^k\,\overline{r}_\ell^k}{\nabla + n\,\overline{r}_\ell}\Big] \\
&= \frac{\nabla^{(1)} + \nabla^{(2)}}{n^k\big[\nabla + n\overline{r}_1\big]\big[\nabla + n\overline{r}_2\big]}\,,
\end{aligned}$$

where $\nabla^{(1)} = (-1)^{k-1}n\nabla^k\big[\nabla + \overline{n}_1\,\overline{r}_2 + \overline{n}_2\,\overline{r}_1\big]$ and $\nabla^{(2)} = n^k\big[\nabla\overline{N}_k + n\,\overline{r}_1\,\overline{r}_2\,\overline{N}_{k-1}\big]$. We can easily verify that $\nabla^{(1)} = 0$ and

$$\frac{\nabla^{(2)}}{n^k} = \overline{n}_1\,\overline{r}_1\big[\nabla + n\,\overline{r}_2\big]\overline{r}_1^{k-1} + \overline{n}_2\,\overline{r}_2\big[\nabla + n\,\overline{r}_1\big]\overline{r}_2^{k-1}\,,$$

which results in

$$\omega_k = \Big[\frac{\overline{n}_1\,\overline{r}_1}{\nabla + n\,\overline{r}_1}\Big]\overline{r}_1^{k-1} + \Big[\frac{\overline{n}_2\,\overline{r}_2}{\nabla + n\,\overline{r}_2}\Big]\overline{r}_2^{k-1}\,.$$

By a straightforward calculation we find that

$$\left[\frac{\overline{n}_\ell\,\overline{r}_\ell}{\nabla + n\,\overline{r}_\ell}\right] = \frac{1}{2}\left[1 \pm \frac{n-2-r_1-r_2}{\sqrt{\Delta}}\right],$$

where '+' and '$-$' are related to $\ell = 1$ and $\ell = 2$, respectively. Using the last relation, we finally have

$$\omega_k = \frac{\overline{r}_1^k - \overline{r}_2^k}{\sqrt{\Delta}} \qquad \big(k = 0, 1, 2, \dots\big). \tag{10}$$

With regard to (3) and (10), we notice that $\Delta_{k,\ell}$ may be written in the following form:

$$\Delta_{k,\ell} = \frac{s_\ell^k + (-1)^{k-1}(r_\ell + 1)^k}{n_\ell} + \frac{n_{\overline{\ell}}}{\sqrt{\Delta}}\left[\frac{\overline{r}_1^k - s_\ell^k}{\overline{r}_1 - s_\ell} - \frac{\overline{r}_2^k - s_\ell^k}{\overline{r}_2 - s_\ell}\right]. \tag{11}$$

Further, let S be any (possibly empty) subset of the vertex set $V(G)$ and let G_S be the graph obtained from the graph G by adding a new vertex x ($x \notin V(G)$), which is adjacent exactly to the vertices from S.

For a square matrix M denote by $\{M\}$ the adjoint of M and for any two subsets $X, Y \subseteq V(G)$ define $\langle X, Y\rangle = \sum_{i\in X}\sum_{j\in Y} \mathbf{A_{ij}}$, where $\mathbf{A} = [\mathbf{A_{ij}}] = \{\lambda I - A\}$. The expression $\langle X, Y\rangle$ is called *the formal product* of the sets X and Y, associated with the graph G. For any two disjoint subsets $X, Y \subseteq V(G)$ let $X+Y$ denote the union of X and Y. Then $\langle X+Y, Z\rangle = \langle X, Z\rangle + \langle Y, Z\rangle$ for any $Z \subseteq V(G)$ and $\langle X, Y\rangle = \langle Y, X\rangle$ for any (not necessarily disjoint) $X, Y \subseteq V(G)$. According to [5],

$$P_{G_S}(\lambda) = P_G(\lambda)\left[\lambda - \frac{1}{\lambda}\,\mathfrak{F}_S\big(\frac{1}{\lambda}\big)\right] \quad \text{and} \quad \langle S, S\rangle = \frac{P_G(\lambda)}{\lambda}\,\mathfrak{F}_S\big(\frac{1}{\lambda}\big), \tag{12}$$

where $\mathfrak{F}_S(t) = \sum\limits_{k=0}^{+\infty} d^{(k)}t^k$ and $d^{(k)} = \sum\limits_{i\in S}\sum\limits_{j\in S} a_{ij}^{(k)}$ $(k = 0, 1, 2, \dots)$. More generally, we proved in [6] that

$$\langle X, Y\rangle = \frac{P_G(\lambda)}{\lambda}\,\mathfrak{F}_{X,Y}\big(\frac{1}{\lambda}\big) \qquad (X, Y \subseteq V(G)), \tag{13}$$

where $\mathfrak{F}_{X,Y}(t) = \sum\limits_{k=0}^{+\infty} e^{(k)}t^k$ and $e^{(k)} = \sum\limits_{i\in X}\sum\limits_{j\in Y} a_{ij}^{(k)}$ $(k = 0, 1, 2, \dots)$. Setting $S^\bullet = V(G)$ we obtain that $\langle S^\bullet, S^\bullet\rangle = \mathbf{sum}\,\{\lambda I - A\}$ and $\mathfrak{F}_{S^\bullet}(t) = H_G(t)$. We also note from (12) that for any $S \subseteq V(G)$,

$$P_{G_S}(\lambda) = \lambda P_G(\lambda) - \langle S, S\rangle\,, \tag{14}$$

where $\langle S, S\rangle$ is the formal product associated with G.

Let i be a fixed vertex from the vertex set $V(G)$ and let $G^i = G \setminus i$ be its corresponding vertex deleted subgraph.

Proposition 1 (Lepović [9]). *Let G be a connected or disconnected regular graph of order n and degree r. Then for any $i \in V(G)$ and any $S \subseteq V(G)$ we have:*

$$(1^0) \qquad P_{\overline{G^i}}(\lambda) = \frac{(-1)^{n-1}}{\lambda + r + 1} \Big[\big(\lambda - \overline{r}\big) P_{G^i}(\,\overline{\lambda}\,) - \frac{P_G(\,\overline{\lambda}\,)}{\lambda + r + 1} \Big] \, ;$$

$$(2^0) \qquad P_{G_T}(\lambda) = P_{G_S}(\lambda) - \frac{n - 2\,|S|}{\lambda - r} \, P_G(\lambda) \, ;$$

$$(3^0) \qquad P_{\overline{G_S}}(\lambda) = \frac{(-1)^{n+1}}{\lambda + r + 1} \Big[\big(\lambda - \overline{r}\big) P_{G_S}(\,\overline{\lambda}\,) + \frac{\big(\lambda + r + 1 - |S|\big)^2}{\lambda + r + 1} \, P_G(\,\overline{\lambda}\,) \Big] \, ,$$

where $\overline{r} = (n-1) - r$, $\overline{\lambda} = -\,\lambda - 1$ and $T = V(G) \setminus S$.

Let G be the union of any k (not necessarily connected) graphs $G^{(1)}, G^{(2)}, \ldots, G^{(k)}$ and let $S = S_1 \cup S_2 \cup \cdots \cup S_k \subseteq V(G)$, where $S_i \subseteq V(G^{(i)})$ for $i = 1, 2, \ldots, k$. In [7] it was proved that

$$P_{G_S}(\lambda) = \sum_{i=1}^{k} \Big[P_{G^{(i)}_{S_i}}(\lambda) \prod_{j \in \mathcal{V}_i} P_{G^{(j)}}(\lambda) \Big] - (k-1)\,\lambda \prod_{i=1}^{k} P_{G^{(i)}}(\lambda) \, , \tag{15}$$

where $\mathcal{V}_i = \{1, 2, \ldots, k\} \setminus \{i\}$. Using the last relation and Proposition (2^0), we easily obtain the following result.

Proposition 2. *Let G be the union of k regular graphs $G_1, G_2, \ldots, G_k$ of order $n_1, n_2, \ldots, n_k$ and degree $r_1, r_2, \ldots, r_k$, respectively. Then for any $S = \bigcup_{m=1}^{k} S_m \subseteq V(G)$, we have*

$$P_{G_T}(\lambda) = P_{G_S}(\lambda) - \Big[\sum_{m=1}^{k} \frac{n_m - 2\,|S_m|}{\lambda - r_m} \Big] P_G(\lambda) \, , \tag{16}$$

where $T = V(G) \setminus S$ and $S_m \subseteq V(G_m)$ for $m = 1, 2, \ldots, k$.

Let $\mathcal{M}(G) = \{\mu_1, \mu_2, \ldots, \mu_k\}$ be the set of all main eigenvalues of a graph G of order n. As is known, if $\lambda \in \sigma(G) \setminus \mathcal{M}(G)$ then $-\,\lambda - 1 \in \sigma(\,\overline{G}\,) \setminus \mathcal{M}(\overline{G})$, which provides the following relation

$$\Big[\prod_{m=1}^{k} \big(\lambda + \mu_m + 1\big) \Big] P_{\overline{G}}(\lambda) = \Big[\prod_{m=1}^{k} \big(\lambda - \overline{\mu}_m\big) \Big] (-1)^n P_G(-\,\lambda - 1) \, , \tag{17}$$

where $\overline{\mu}_m \in \mathcal{M}(\overline{G})$ for $m = 1, 2, \ldots, k$.

Further, let $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ denote a complete set of mutually orthogonal normalized eigenvectors of the adjacency matrix A corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of G, respectively. Let $X = [\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}] = [x_{ij}]$ denote the orthogonal matrix of eigenvectors $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$. Besides, let $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the diagonal matrix of the eigenvalues of G. Since $X^{-1} = X^\top$ and $A^k = X\,\Lambda^k\,X^\top$, where $X^\top$ is the transpose of X, we have

$$a_{ij}^{(k)} = \sum_{\nu=1}^{n} x_{i\nu}\, x_{j\nu}\, \lambda_\nu^k \qquad (i, j = 1, 2, \ldots, n)\,, \tag{18}$$

for any non-negative integer k.

Proposition 3. *Let G be the union of two regular graphs G_1 and G_2 of order n_1 and n_2 and degree r_1 and r_2, respectively. Then for any $S = S_1 \cup S_2 \subseteq V(G)$ we have*

$$(-1)^{n+1} P_{\overline{G_S}}(\lambda) = \Big[1 - \sum_{m=1}^{2} \frac{n_m}{\lambda + r_m + 1}\Big]\, P_{G_S}(\,\overline{\lambda}\,) + \Big[\sum_{m=1}^{2} \frac{|S_m|}{\lambda + r_m + 1}\Big]^2 P_G(\,\overline{\lambda}\,)$$
$$+ \Big[1 - \sum_{m=1}^{2} \frac{2\,|S_m|}{\lambda + r_m + 1}\Big]\, P_G(\,\overline{\lambda}\,)\,,$$

where $\overline{\lambda} = -\,\lambda - 1$ and $S_m \subseteq V(G_m)$ for $m = 1, 2$.

Proof. Using (12) and (18) we have $\mathfrak{F}_S(\frac{1}{\lambda}) = \lambda \big[\sum\limits_{\nu=1}^{n} \frac{d_\nu}{\lambda - \lambda_\nu}\big]$ where $d_\nu = \sum\limits_{i\in S} \sum\limits_{j\in S} x_{i\nu} x_{j\nu}$. Then we easily obtain

$$\sum_{i=1}^{n} \frac{d_i}{\lambda + \lambda_i + 1} = \big(\lambda + 1\big) + \frac{P_{G_S}(\,\overline{\lambda}\,)}{P_G(\overline{\lambda}\,)}\,. \tag{19}$$

Next, denote the formal generating function of $\overline{G}_S$ by $\mathfrak{F}_{\overline{S}}(t) = \sum_{k=0}^{+\infty} \overline{d}^{(k)} t^k$, where $\overline{d}^{(k)} = \sum_{i\in S} \sum_{j\in S} \overline{a}_{ij}^{(k)}$. Since $\langle S, S\rangle = \langle S_1, S_1\rangle + 2\,\langle S_1, S_2\rangle + \langle S_2, S_2\rangle$ and $\sqrt{\Delta} = \overline{r}_1 - \overline{r}_2$, using equations (4), (10), (11), (13), (18), we get

$$\mathfrak{F}_{\overline{S}}\Big(\frac{1}{\lambda}\Big) = \lambda\Big[\sum_{m=1}^{2}\frac{\big|S_m\big|^2}{\big(\lambda-s_m\big)\big(\lambda+r_m+1\big)}\Big] + \lambda\Big[\sum_{i=1}^{n}\frac{d_i}{\lambda+\lambda_i+1}\Big]$$

$$+\frac{\lambda}{\big(\lambda-\overline{r}_1\big)\big(\lambda-\overline{r}_2\big)}\Big[\sum_{\ell=1}^{2}\frac{n_{\overline{\ell}}\,\big|S_\ell\big|^2}{\lambda-s_\ell}\Big] + \frac{2\big|S_1\big|\big|S_2\big|\,\lambda}{\big(\lambda-\overline{r}_1\big)\big(\lambda-\overline{r}_2\big)}\,.$$

In view of (12), (14), (17), (19) and the previous relation, a straightforward calculation yields

$$(-1)^{n+1}P_{\overline{G}_S}(\lambda) = \Big[\Big(\sum_{\ell=1}^{2}\frac{n_{\overline{\ell}}\,\big|S_\ell\big|^2}{\lambda-s_\ell}\Big) + 2\big|S_1\big|\big|S_2\big|\Big]\frac{P_G(\overline{\lambda}\,)}{\big(\lambda+r_1+1\big)\big(\lambda+r_2+1\big)}$$

$$+\frac{\big(\lambda-\overline{r}_1\big)\big(\lambda-\overline{r}_2\big)}{\big(\lambda+r_1+1\big)\big(\lambda+r_2+1\big)}\Big[P_G(\overline{\lambda}\,) + P_{G_S}(\overline{\lambda}\,) +$$

$$+\sum_{m=1}^{2}\frac{\big|S_m\big|^2 P_G(\overline{\lambda}\,)}{\big(\lambda-s_m\big)\big(\lambda+r_m+1\big)}\Big]\,.$$

Since $(\lambda_1 - s_1)(\lambda - s_2) - (\lambda - \overline{r}_1)(\lambda - \overline{r}_2) = n_1 n_2$ the last relation is transformed in the form

$$(-1)^{n+1}P_{\overline{G}_S}(\lambda) = \Big[1-\sum_{m=1}^{2}\frac{n_m}{\lambda+r_m+1}\Big]\big(P_{G_S}(\overline{\lambda}\,)+P_G(\overline{\lambda}\,)\big)$$

$$+\Big[\sum_{m=1}^{2}\frac{\big|S_m\big|}{\lambda+r_m+1}\Big]^2 P_G(\overline{\lambda}\,)\,.$$

Finally, since $P_{\overline{G_S}}(\lambda) = P_{\overline{G}_T}(\lambda)$ where $T = T_1 \cup T_2$ and $T_m = V(G_m) \setminus S_m$, applying (16) to the previous relation we obtain that

$$(-1)^{n+1}P_{\overline{G_S}}(\lambda) = \Big[1-\sum_{m=1}^{2}\frac{n_m}{\lambda+r_m+1}\Big]\Big[\Big(\sum_{m=1}^{2}\frac{n_m-2\,\big|S_m\big|}{\lambda+r_m+1}\Big)P_G(\overline{\lambda}\,)+$$

$$+P_{G_S}(\overline{\lambda}\,)+P_G(\overline{\lambda}\,)\Big] + \Big[\sum_{m=1}^{2}\frac{n_m-\big|S_m\big|}{\lambda+r_m+1}\Big]^2 P_G(\overline{\lambda}\,)\,,$$

from which we find the proof.

Let G be a graph with k main eigenvalues $\mu_1, \mu_2, \ldots, \mu_k$ and let $(x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})$ denote the eigenvector of μ_m so that $\sum\limits_{i=1}^{n} x_i^{(m)} = \sqrt{n_m}$.

Proposition 4 (Lepović [10]). *Let G be a connected or disconnected graph of order n with exactly two main eigenvalues μ_1 and μ_2. Then $x_i^{(1)} = \frac{\deg(i)-\mu_2}{\sqrt{n_1}\,(\mu_1-\mu_2)}$ for $i = 1, 2, \ldots, n$.*

Proposition 5 (Lepović [10]). *Let G be any connected or disconnected graph of order n with k main eigenvalues $\mu_1, \mu_2, \ldots, \mu_k$. Then for any $i \in V(G)$ and any $S \subseteq V(G)$ we have:*

$$(-1)^{n+1} P_{\overline{G_T}}(\overline{\lambda}) = \Big[1 + \sum_{m=1}^{k} \frac{n_m}{\lambda - \mu_m}\Big]\big(P_{G_S}(\lambda) + P_G(\lambda)\big) + \Big[\sum_{m=1}^{k} \frac{\big|\mathbb{S}_m\big|}{\lambda - \mu_m}\Big]^2 P_G(\lambda)\,;$$

$$(-1)^{n+1} P_{\overline{G_S}}(\overline{\lambda}) = \Big[1 + \sum_{m=1}^{k} \frac{n_m}{\lambda - \mu_m}\Big] P_{G_S}(\lambda) + \Big[1 + \sum_{m=1}^{k} \frac{\big|\mathbb{S}_m\big|}{\lambda - \mu_m}\Big]^2 P_G(\lambda)\,;$$

$$(-1)^{n-1} P_{\overline{G^i}}(\overline{\lambda}) = \Big[1 + \sum_{m=1}^{k} \frac{n_m}{\lambda - \mu_m}\Big] P_{G^i}(\lambda) - \Big[\sum_{m=1}^{k} \frac{\big|\mathbb{I}_m^{(i)}\big|}{\lambda - \mu_m}\Big]^2 P_G(\lambda)\,,$$

where $\overline{\lambda} = -\lambda - 1$ and $T = V(G) \smallsetminus S$; $|\mathbb{S}_m| = \sqrt{n_m}\,\big[\sum_{i \in S} x_i^{(m)}\big]$ and $\mathbb{I}_m^{(i)}| = \sqrt{n_m}\, x_i^{(m)}$.

Theorem 2. *Let G be a connected or disconnected graph with exactly two main eigenvalues and let $P_{G^i}(\lambda) = P_{G^j}(\lambda)$. Then $P_{\overline{G^i}}(\lambda) = P_{\overline{G^j}}(\lambda)$.*

Proof. According to Proposition it suffices to show that $|\mathbb{I}_1^{(i)}| = |\mathbb{I}_1^{(j)}|$ and $|\mathbb{I}_2^{(i)}| = |\mathbb{I}_2^{(j)}|$. We note that $|\mathbb{I}_1^{(i)}| + |\mathbb{I}_2^{(i)}| = |\mathbb{I}_1^{(j)}| + |\mathbb{I}_2^{(j)}|$ (see also [10]). Since $\deg(i) = \deg(j)$ from Proposition it follows that $|\mathbb{I}_1^{(i)}| = |\mathbb{I}_1^{(j)}|$, which provides the proof.

Further, for any $S \subseteq V(G)$ denote by $G_{S,T}$ the graph obtained from G by adding two new non-adjacent vertices x, y, so that x is adjacent exactly to the vertices from S, and y is adjacent exactly to the vertices from $T = V(G) \smallsetminus S$. Besides, let $G_{\dot{S},\dot{T}}$ be the overgraph of G obtained by adding two new adjacent vertices x, y, so that x and y are adjacent in G exactly to the vertices from S and T, respectively.

Theorem 3 (Lepović [7]). *Let G be any graph of order n. Then for any $S \subseteq V(G)$ we have:*

$$P_{G_{S,T}}(\lambda) = \lambda\, P_{G_S}(\lambda) + (-1)^n P_{\overline{G_S}}(-\lambda-1) - (\lambda^2+\lambda)\, P_G(\lambda) +$$
$$+ (-1)^n(\lambda+1)\, P_{\overline{G}}(-\lambda-1) + (\lambda+1)\, P_{G_T}(\lambda)\,;$$
$$P_{G_{\dot{S},\dot{T}}}(\lambda) = (\lambda-1)\, P_{G_S}(\lambda) + (-1)^n P_{\overline{G_S}}(-\lambda-1) - (\lambda^2-\lambda)\, P_G(\lambda) +$$
$$+ (-1)^n \lambda\, P_{\overline{G}}(-\lambda-1) + \lambda\, P_{G_T}(\lambda)\,,$$

where $T = V(G) \setminus S$.

Proposition 6. *Let G be a connected or disconnected graph with k main eigenvalues $\mu_1, \mu_2, \ldots, \mu_k$. Then for any $S \subseteq V(G)$, we have:*

$$P_{G_{S,T}}(\lambda) = \Big[2\lambda - \sum_{m=1}^{k} \frac{n_m}{\lambda-\mu_m}\Big] P_{G_S}(\lambda) - \Big[\lambda - \sum_{m=1}^{k} \frac{|\mathbb{S}_m|}{\lambda-\mu_m}\Big]^2 P_G(\lambda)\,;$$

$$P_{G_{\dot{S},\dot{T}}}(\lambda) = \Big[2(\lambda-1) - \sum_{m=1}^{k} \frac{n_m}{\lambda-\mu_m}\Big] P_{G_S}(\lambda) - \Big[(\lambda-1) - \sum_{m=1}^{k} \frac{|\mathbb{S}_m|}{\lambda-\mu_m}\Big]^2 P_G(\lambda)\,,$$

where $T = V(G) \setminus S$ and $|\mathbb{S}_m| = \sqrt{n_m}\,\big[\sum_{\imath \in S} x_i^{(m)}\big]$ for $m = 1, 2, \ldots, k$.

Proof. Using Proposition 5 and Theorem 3 by an easy calculation we obtain the required statement.

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