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SOME RELATIONSHIPS BETWEEN THE GENERALIZED GUMBEL AND OTHER DISTRIBUTIONS

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Abstract. The Gumbel distribution also called the extreme value distribution has recently been generalized. Some relationships between this generalized distribution and other distributions are established in this paper.

1. INTRODUCTION

The probability density function of the Gumbel random variable also called the extreme value density is given as

$$f(x) = e^{-x} exp - e^{-x}, \quad -\infty \ < \ x \ < \ \infty. \tag{1.1}$$

The corresponding characteristic function is given as

$$\Phi_x(t) = \Gamma(1 - it). \tag{1.2}$$

This distribution has been generalized for the first time in a recent paper (Ojo [2]). The generalized version of the distribution is given as

$$g(y) \ = \ \frac{1}{\Gamma(p)} e^{-py} exp - e^{-y}, \quad -\infty \ < \ y \ < \ \infty \tag{1.3}$$

where $p > 0$ is the shape parameter; and its characteristic function is given as

$$\Phi_Y(t) \;=\; \frac{\Gamma(p-it)}{\Gamma(p)}. \tag{1.4}$$

In this paper, relationships between this generalized distribution and some commonly encountered statistical disrtributions are established.

2. CHARACTERIZATION THEOREMS

In what follows we prove some theorems that relate the generalized Gumbel distribution to other distributions.

Theorem 2.1. *Let X be a continuously distributed random variable with density function $f(x)$ with $Pr(X>0)=1$. Then the random variable $Y=-\log X$ has the generalized Gumbel distribution if and only if X has the gamma distribution.*

Proof. Suppose X has the gamma density with parameter p. The characteristic function of $Y=-\log X$ is given as

$$\Phi_Y(t) = E(X^{-it}) = \frac{1}{\Gamma(p)}\int_0^\infty x^{p-it-1}e^{-x}dx = \frac{\Gamma(p-it)}{\Gamma(p)}$$

which is the characteristic function of the generalized Gumbel distribution. Conversely, suppose $-\log X$ has the generalized Gumbel distribution, then

$$E(X^{-it}) \;=\; \frac{\Gamma(p-it)}{\Gamma(p)}.$$

That is

$$\int_0^\infty x^{-it}f(x)dx \;=\; \frac{\Gamma(p-it)}{\Gamma(p)}. \tag{2.1}$$

Obviously, the unique function $f(x)$ satisfying (2.1) is given as

$$f(x) \;=\; \frac{1}{\Gamma(p)}x^{p-1}e^{-x}, \quad x>0.$$

Before we state and prove the next theorem, we re-introduce the generalized logistic distribution for the purpose of this paper. The random variable X has been said to have the generalized logistic distribution if its density function is defined as

$$g(x) \;=\; \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\,\frac{e^{px}}{(1+e^x)^{p+q}}, \quad -\infty<x<\infty$$

where p and q are the shape parameters. The corresponding characteristic function is given as

$$\phi(t) \;=\; \frac{\Gamma(p+it)\cdot\Gamma(q-it)}{\Gamma(p)\cdot\Gamma(q)}.$$

This distribution has been earlier on considered by George & Ojo [1].

Theorem 2.2. *Let X_1 and X_2 be independent random variables with a common density. Then the random variable $Y = X_1 - X_2$ has the generalized logistic distribution with parameters p and q if X_1 and X_2 each has the generalized Gumbel distribution.*

Proof. Suppose X_1 and X_2 are independent with density function

$$h_1(x_1) \;=\; \frac{1}{\Gamma(q)}e^{-qx_1}exp - e^{-x_1}, \quad -\infty < x_1 < \infty$$

and

$$h_2(x_2) \;=\; \frac{1}{\Gamma(p)}e^{-px_2}exp - e^{-x_2}, \quad -\infty < x_2 < \infty.$$

Then the charateristic function of $X_1 - X_2$ is given as

$$\Phi_{X_1-X_2}(t) \;=\; \Phi_{X_1}(t)\cdot\Phi_{X_2}(-t) = \frac{\Gamma(q-it)\Gamma(p+it)}{\Gamma(q)\cdot\Gamma(p)}$$

by equation (1.4).

Since this is the characteristic function of the generalized logistic random variable, the theorem is proved.

Theorem 2.3. *Let $X_1, ..., X_{2n-1}$ be a random sample of size $2n-1$ from the logistic population. Let U and V be continuously distributed independent random variables. Then*

$$X_{(n)} \overset{L}{=} U - V$$

if U and V each has the generalised Gumbel distribution with parameter n, where $X_(n)$ denotes the logistic sample median and $\overset{L}{=}$ denotes "equality in distribution".

Proof. The density function of $X_{(n)}$, the logistic sample median of a sample of size $2n-1$ is given by

$$g_n(x) \;=\; \frac{(2n-1)!}{(n-1)!(n-1)!}(F(x))^{n-1}(1-F(x))^{n-1}f(x)$$

$$= \frac{\Gamma(2n)}{(\Gamma(n))^2}(f(x))^n = \frac{\Gamma(2n)}{(\Gamma(n))^2} \cdot \frac{e^{nx}}{(1+e^x)^{2n}}, \quad -\infty < x < \infty.$$

The characteristic function of this distribution is readily obtained as

$$\Phi_n(t) = \frac{\Gamma(n+it)\cdot\Gamma(n-it)}{(\Gamma(n))^2}.$$

Since the characteristic function of U-V is

$$\Phi_{u-v}(t) = \frac{\Gamma(n-it)}{\Gamma(n)} \cdot \frac{\Gamma(n+it)}{\Gamma(n)},$$

the theorem is proved.

Theorem 2.4. *Let X_1 and X_2 be independent and identically distributed random variables and let $F(2q, 2p)$ denote an f-random variable with $(2q, 2p)$ degrees of freedom. Then $X_2 - X_1 \stackrel{L}{=} -\log \frac{q}{p} F(2q, 2p)$ if X_1 and X_2 each has the generalized Gumbel distribution with parameters p and q, respectively.*

Proof. The probability density function of an f-random variable with $(2q, 2p)$ degrees of freedom is given as

$$f(w) = \frac{1}{B(p,q)}(\frac{q}{p})^q \frac{w^{q-1}}{(1+\frac{qw}{p})^{p+q}}, \quad w > 0$$

and the characteristic function of $-\log \frac{q}{p} F(2q, 2p)$ is given as

$$\begin{aligned}\Phi_w(t) &= \frac{1}{B(p,q)}(\frac{q}{p})^{q-it} \int_0^\infty \frac{w^{q-it-1}}{(1+\frac{qw}{p})^{p+q}} dw \\ &= \frac{1}{B(p,q)} \int_0^1 u^{p-it-1}(1-u)^{q-it-1} du \\ &= \frac{B(p+it, q-it)}{B(p,q)} = \frac{\Gamma(p+it)\Gamma(q-it)}{\Gamma(p)\cdot\Gamma(q)}.\end{aligned}$$

Since this is the characteristic function of $X_2 - X_1$, the theorem is proved.

Theorem 2.5. *Let X_1 and X_2 be independently and identically distributed random variables and let U be a beta (p, q) random variable. Then*

$$X_1 - X_2 \stackrel{L}{=} \log(\frac{U}{1-U})$$

if X_1 and X_2 each has the generalized Gumbel distribution with parameters p and q.

Proof. If U is a beta (p, q) random variable the characteristic function of $\log \frac{U}{1-U}$ is given by

$$\begin{aligned}\phi(t) &= E\left(\frac{U}{1-U}\right)^{it} = \frac{1}{B(p,q)}\int_0^1 U^{p-it-1}(1-U)^{q-it-1}dU \\ &= \frac{B(p+it, q-it)}{B(p,q)} = \frac{\Gamma(p+it)\Gamma(q-it)}{\Gamma(p)\cdot\Gamma(q)}\end{aligned}$$

which is the characteristic function of $X_1 - X_2$ if X_1 and X_2 each has the generalized Gumbel distribution.

Theorem 2.6. *Let $X_1, X_2, ..., X_{n-1}$ be independently distributed random variables each with density function*

$$f_k(x) = \frac{\sin\ kx}{\pi x}, \quad -\infty < x < \infty, \quad k = 1, ..., n-1$$

and $z_1, z_2, ...$ be independent double exponential random variables with density function

$$f(z) = \frac{1}{2}e^{-|z|}, \quad -\infty < z < \infty,$$

$X_1, X_2, ..., X_{n-1}$ and $z_1, z_2, ...$ being independent. Let U and V be independently and identically distributed random variables with characteristic function ϕ. Then

$$U - V \stackrel{L}{=} \sum_{k=1}^{n-1} X_k + \sum_{j=0}^{\infty} Z_j$$

if U and V each has the generalized Gumbel distribution with parameter n.

Proof. If U and V are independent the characteristic function of $U - V$ is given as

$$\phi(t) = \phi_n(t)\phi_v(-t) = \frac{\Gamma(n-it)\Gamma(n+it)}{(\Gamma(n))^2}$$

this can be expressed as

$$\phi(t) = \prod_{k=n}^{\infty}(1 + \frac{t^2}{k^2})^{-1}$$

(George and Ojo [1]). This can further be written as

$$\phi(t) = \prod_{j=1}^{\infty}(1 + \frac{t^2}{j^2})^{-1} \cdot \prod_{k=1}^{n-1}(1 + \frac{t^2}{k^2}). \tag{2.2}$$

Now $\prod_{j=1}^{\infty}(1+\frac{t^2}{j^2})^{-1}$ is the characteristic function of an infinite sum of double exponential random variables.

Furthermore it can easily be shown by direct inversion of characteristic function that the density function corresponding to the characteristic function $1+\frac{t^2}{k^2}$ is given as

$$f_k(x) = \frac{\sin\ kx}{\pi x}, \quad -\infty < x < \infty.$$

The theorem then follows by equation (2.2). We give a corollary to this theorem

Corollary 2.6. *Under the same condition as in Theorem 2.6, if* X_{1j} *and* X_{2j} *are exponentially distributed random variables with density function* $f_j = je^{-jx},\ x > 0$, *then*

$$U - V \stackrel{L}{=} \sum_{k=1}^{n-1} X_k + \sum_{j=1}^{\infty}(X_{ij} - X_{2j}).$$

Proof. The corollary follows since it is known that

$$Z_j \stackrel{L}{=} (X_{1j} - X_{2j}).$$

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