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ON POSTIAN EQUATIONS

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Abstract. In this paper we give a survey on solving postian equations. We also give a reproductive general solution of postian equation in n unknowns (Theorem 13).

General solution of an equation was known in various fields of mathematics. The general solutions were very extensively studied in boolean algebras. Löwenheim [9], [10] gave a great contribution to the research of general and reproductive general solutions of boolean equations. Various form of general and reproductive general solutions were developed by Rudeanu [14], [15], [16]. Banković [1], [2] gave some methods for solving boolean equations.

A generalization of boolean algebras are Post algebras. The axioms and fundamental theorems can be found in Epstein's paper [8]. Serfati [17], [18] developed some methods for solving postian equations. A contribution to the solving postian equations was given by Banković [3], [4], [5], [6].

Let P be an r-Post algebra with underlying chain $C = \{0 = e_0 < e_1 < \cdots < e_{r-1} = 1\}$ where r is an integer and $r \geq 2$. Let $x \vee y$ and $x \cdot y$ denote supremum and infimum of the elements x and y, respectively.

Theorem 1 [2]. *Every element* $x \in P$ *has the unique representation in the form* $x = \bigvee_{i=0}^{r-1} e_i x^i$, *where* x^i *(boolean elements, called postian components of* x*) satisfy the orthonormality condition*

$$\bigvee_{i=0}^{r-1} x^i = 1 \quad and \quad i \neq j \Rightarrow x^i \cdot x^j = 0. \tag{1}$$

The pseudo-complement x^* of an element $x \in P$ is defined as

$$x^* = max\{y \mid x \cdot y = 0 \wedge y \in P\}.$$

If $z \in P$ and there exists an element $\overline{z} \in P$ satisfying the conditions $z \vee \overline{z} = 1$ and $z \cdot \overline{z} = 0$ then $\overline{z}$ is called the complement of z.

One can prove

$$x^0 = x^* \quad and \quad (\forall i \in \{0, 1, \ldots, r-1\})\, (x^i)^0 = \overline{x^i}.$$

Theorem 2 [2]. *For each* $i \in \{0, 1, \ldots, r-1\}$

$$(x^i)^{r-1} = x^i, \qquad (x^i)^j = 0 \ (0 < j < r-1), \qquad (x^i)^0 = \bigvee_{\substack{k=0 \\ k \neq i}}^{r-1} x^k = \overline{x^i}.$$

Let $X = (x_1, \ldots, x_n)$ and $T = (t_1, \ldots, t_n)$.

Definition 1. Let $f, g_1, \ldots, g_n : P^n \to P$ be postian polynomials and $G = (g_1, \ldots, g_n)$. Formula $X = G(T)$ represents the general solution of the consistent postian equation $f(X) = 0$ if and only if

$$(\forall X \in P^n) f(G(X)) = 0 \wedge (\forall X \in P^n)(f(x) = 0 \Rightarrow (\exists T \in P^n) X = G(T)).$$

Definition 2. Let $f, h_1, \ldots, h_n : P^n \to P$ be postian polynomials and $H = (h_1, \ldots, h_n)$. Formula $X = H(T)$ represents the reproductive general solution of the consistent postian equation $f(X) = 0$ if and only if

$$(\forall X \in P^n) f(H(X)) = 0 \wedge (\forall X \in P^n)(f(X) = 0 \Rightarrow X = H(T)).$$

Theorem 3 [18]. *If f is a postian polynomial with variable x, then*

$$f(x) = \bigvee_{i=0}^{r-1} z_i x^i.$$

Theorem 4 [18]. *Let $f(x) = \bigvee_{i=0}^{r-1} z_i x^i$ be a postian polynomial, where $z_i = f(e_i)$ $(i = 0, 1, \ldots, r-1)$. The equation $f(x) = 0$ is consistent if and only if $z_0 z_1 \cdots z_{r-1} = 0$. In that case the solution of $f(x) = 0$ is*

$$x = \bigvee_{i=0}^{r-1} z_i^* \, e_i.$$

Theorem 5 [18]. *Let $f(x) = \bigvee_{i=0}^{r-1} z_i x^i$ be a postian polynomial, where $z_i = f(e_i)$ $(i = 0, 1, \ldots, r-1)$. If p is a particular solution of the equation $f(x) = 0$ then the formula*

$$x = f^*(t) \cdot t \vee \overline{f^*}(t) \cdot p$$

represents the general reproductive solution of $f(x) = 0$.

Theorem 6 [3]. *Let $f(x) = \bigvee_{i=0}^{r-1} z_i x^i$ be a Postian polynomial, where $z_i = f(e_i)$ $(i = 0, 1, \ldots, r-1)$. If the equation $f(x) = 0$ is consistent $(z_0 z_1 \cdots z_{r-1} = 0)$ then the formula $x = \phi(t)$ represents the reproductive general solution of $f(x) = 0$ if*

$$\phi(t) = \bigvee_{k=0}^{r-1} (z_k^* e_k \vee \overline{z_k^*} z_{i_{k,1}}^* e_{i_{k,1}} \vee \cdots \vee \overline{z_k^*} \; \overline{z_{i_{k,1}}^*} \; \cdots \overline{z_{i_{k,r-2}}^*} z_{i_{k,r-1}}^* e_{i_{k,r-1}}) t^k$$

and

$$(\forall k \in \{0, 1, \ldots, r-1\}) \quad \{k, i_{k,1}, \ldots, i_{k,r-1}\} = \{0, 1, \ldots, r-1\}.$$

Theorem 7 [6]. *Let $f(x) = \bigvee_{i=0}^{r-1} z_i x^i$ be a Postian polynomial, where $z_i = f(e_i)$ $(i = 0, 1, \ldots, r-1)$. If the equation $f(x) = 0$ is consistent $(z_0 z_1 \cdots z_{r-1} = 0)$ then* *the formula $x = \phi(t)$ represents the general solution of $f(x) = 0$ if*

$$\phi(t) = \bigvee_{k=0}^{r-1} (z^*_{i_{k,0}} e_{i_{k,0}} \vee \overline{z^*_{i_{k,0}}} z^*_{i_{k,1}} e_{i_{k,1}} \vee \cdots \vee \overline{z^*_{i_{k,0}}}\ \overline{z^*_{i_{k,1}}}\ \cdots \overline{z^*_{i_{k,r-2}}} z^*_{i_{k,r-1}} e_{i_{k,r-1}}) t^k$$

and

$$(\forall k \in \{0, 1, \ldots, r-1\}) \quad \{i_{k,0}, i_{k,1}, \ldots, i_{k,r-1}\} = \{0, 1, \ldots, r-1\},$$

and

$$\{i_{0,0}, i_{1,0}, \ldots, i_{r-1,0}\} = \{0, 1, \ldots, r-1\}.$$

Comment. One can prove that $z^*_{i_{r-1}}$ can be omitted from the previous formulas.

POSTIAN EQUATIONS IN n UNKNOWNS

Let $R = \{0, 1, \ldots, r-1\}$.

Theorem 8 [8]. *If f is a postian polynomial with the variables $x_1, \ldots, x_n$, then*

$$f(x_1, \ldots, x_n) = \bigvee_{(i_1, \ldots, i_n) \in R^n} f(e_{i_1}, \ldots, e_{i_n}) x^{i_1} \cdots x^{i_n}.$$

Theorem 9 [18]. *Let f be the postian polynomial with the variables $x_1, \ldots, x_n$. The equation $f(x_1, \ldots, x_n) = 0$ is constent if and only if*

$$\prod_{(i_1, \ldots, i_n) \in R^n} f(e_{i_1}, \ldots, e_{i_n}) = 0.$$

Theorem 10 [5]. *Let $f(X) = 0$ be a consistent postian equation in n unknows. The formula $X = G(T)$, where $G = (g_1, \ldots, g_n)$, represents the general solution of $f(X) = 0$ if and only if*

$$(\forall X \in P^n) f^*(X) = (\prod_{B \in R^n} \bigvee_{k=1}^{n} \bigvee_{i=0}^{r-1} (G_k(B))^i \overline{(x_k)^i})^*.$$

Theorem 11 [5]. *Let $f(X)=0$ be a consistent postian equation in n unknows. The formula $X=G(T)$, where $G=(g_1,\ldots,g_n)$, represents the reproductive general solution of $f(X)=0$ if and only if*

$$(\forall X \in P^n) f(g(X))=0 \wedge f^*(X)=(\bigvee_{k=1}^{n}\bigvee_{i=0}^{r-1}(G_k(X))^i\overline{(x_k)^i}\,)^*.$$

Theorem 12 [4]. *Let $G=(g_1,\ldots,g_n)$ and $H=(h_1,\ldots,h_n)$, where $g_1,\ldots,g_n$, $h_1,\ldots,h_n$ are postian polynomials in n variables. If f is a postian polynomial and formula $X=G(T)$ represents the general solution of equation $f(X)=0$, then formula $X=H(T)$ represents the reproductive, general solution of $f(X)=0$ if and only if there exist postian polynomials $p_1,\ldots,p_n$ in n variables such that*

$$H(T)=f^*(T)\cdot T \vee \overline{f^*}(T)\cdot H(P(T))$$

where $P(T)=(p_1,\ldots,p_n))$.

Theorem 13. *Let $f,p_1,\ldots,p_n$ be postian polynomials in n variables. If $P=(p_1,\ldots,p_n)$ is the particular solution of $f(X)=0$, then the formula*

$$X=f^*(T)\cdot T \vee \overline{f^*}(T)\cdot P$$

represents the reproductive general solution of $f(X)=0$.

Proof. The given formula satisfies equation $f(X)=0$ by the proof of Theorem 12. The reproductivity follows from

$$f(X)=0 \Rightarrow f^*(X)\cdot X \vee \overline{f^*}(X)\cdot P = 0^*\cdot X \vee \overline{0^*}\cdot P = X.$$

Theorem 13 describes all reproductive general solutions of the given postian equation, if a general solution of that equation is known. It would be interesting to describe the all general solutions of a given postian equation, if a general solution of that equation is known.

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