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WEIGHTED HARDY'S INEQUALITIES WITH MIXED NORM II

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Abstract. We obtain in this paper conditions on the nonnegative weight functions $u(x)$ and $v(x)$ which ensure an inequality of the form

$$\left(\int_{-\infty}^{\infty} \mid v(x)f(x) \mid^{(1-\frac{1}{r})p} dx\right)^{\frac{1}{(1-\frac{1}{r})p}} \leq C\left(\int_{-\infty}^{\infty} \mid u(x)(Tf)(x) \mid^{(1-\frac{1}{s})q} dx\right)^{\frac{1}{(1-\frac{1}{s})q}},$$

where T is either I or I^*, and C is a constant depending on (k,p,q,r,s) but independent of f.

INTRODUCTION

Let $k(x,y) \geq 0$ be defined on $\triangle = \{(x,y) \in \Re^2 : y < x\}$ and define the operator T and its dual T^* by

$$(If)(x) = \int_{-\infty}^{x} k(x,y)f(y)dy, \quad (I^*f)(x) = \int_{x}^{\infty} k(y,x)f(y)dy. \tag{1}$$

Let $u(x), v(x)$ and $f(x)$ denote the nonnegative extended real valued measurable functions on $(0,\infty)$. We shall give conditions on the nonnegative weight functions u(x) and v(x) in terms of the kernel $k(x,y)$ and the nonnegative real numbers p,q,r and s which guarantees an inequality of the form

$$\left(\int_{-\infty}^{\infty} \mid v(x)f(x) \mid^{(1-\frac{1}{r})p} dx\right)^{\frac{1}{(1-\frac{1}{r})p}} \leq C\left(\int_{-\infty}^{\infty} \mid u(x)(Tf)(x) \mid^{(1-\frac{1}{s})q} dx\right)^{\frac{1}{(1-\frac{1}{s})q}} \quad (2)$$

where T is either I or I^*, and C is constant independent of f.
The purpose of this paper is to generalize some of the results obtained in in [1, 2]. Throughout this paper, we shall let p' denote the conjugate index of p and is defined by $\frac{1}{p}+\frac{1}{p'}=1-\frac{1}{r}, \quad r>1$. The conjugate q' of q is defined by $\frac{1}{q}+\frac{1}{q'}=1-\frac{1}{s}, \quad s>1$. Also C is a constant which may be different at difference occurences.

THE MAIN RESULT

We shall need the following Definitions and Lemmas in the proof of our main results.

Definition 1. Let $k(x,y)\geq 0, \quad (x,y)\in\triangle$. Let $p,q<1, \ p\neq 0, \ q\neq 0$ and $\beta=0$ or 1. Then for any real number a we define K and J by

$$\begin{aligned} K_\beta(a) &= \left(\int_{-\infty}^{a} k(a,z)^{(1-1/s)\beta q} u(z)^{(1-1/s)q} dz\right)^{\frac{1}{(1-1/s)q}} \\ &\times \left(\int_{-\infty}^{a} K(a,z)^{(1-1/r)(1-\beta)p'} v(z)^{-(1-1/r)p'} dz\right)^{\frac{1}{(1-1/r)p'}}, \end{aligned} \quad (3)$$

$$\begin{aligned} J_\beta(a) &= \left(\int_{a}^{\infty} k(z,a)^{(1-1/s)\beta q} v(z)^{(1-1/s)q} dz\right)^{\frac{1}{(1-1/s)q}} \\ &\times \left(\int_{a}^{\infty} K(z,a)^{(1-1/r)(1-\beta)p'} v(z)^{-(1-1/r)p'} dz\right)^{\frac{1}{(1-1/r)p'}}. \end{aligned} \quad (4)$$

Definition 2. A function f is said to be T-admissible, respectively T^* admissible if $(Tf)(x)$, respectively $(T^*f)(x)$, is finite for $0<x<\infty$.

Lemma 1. *Let $g(x,y)\geq 0$.Suppose $1\leq p\leq\infty$ and $b>-\infty$, then*

$$\left(\int_{b}^{\infty}\left[\int_{b}^{x} g(x,y)dy\right]^{p} dx\right)^{1/p} \leq \int_{b}^{\infty}\left(\int_{y}^{\infty} g(x,y)^{p} dx\right)^{1/p} dy \quad (5)$$

and

$$\left(\int_b^\infty \left[\int_x^\infty g(x,y)dy\right]^p dx\right)^{1/p} \;\le\; \int_b^\infty \left(\int_b^y g(x,y)^p dx\right)^{1/p} dy. \tag{6}$$

If $p<1$, *inequalities (5) and (6) are reversed.*

Proof. See [3, Theorem 202, p. 148].

Lemma 2. *Let* $k(x,y)\ge 0$, $(x,y)\in\triangle$. *Let* $h(y)$ *be defined by*

$$h(y) \;=\; \left(\int_y^\infty v(z)^{-(1-1/r)p'}dz\right)^{\frac{1}{(1-1/r)^2pp'}}. \tag{7}$$

Suppose $0<(1-1/s)q\le(1-1/r)p<1$, $0<q\le p<1$ and $J_1(a)$ is nondecreasing. Then

$$J_1(y)\left(\int_y^\infty k(z,y)^{(1-1/s)q}u(z)^{(1-1/s)q}dz\right)^{-\frac{1}{(1-1/s)q}} \;=\; h(y)^{(1-1/r)p} \tag{8}$$

Proof. By (4) we have
$J_1(y)\left(\int_y^\infty k(z,y)^{(1-1/s)q}u(z)^{(1-1/s)q}dz\right)^{-\frac{1}{(1-1/s)q}}$

$$\begin{aligned}
&= \left(\int_y^\infty k(z,y)^{(1-1/s)q}v(z)^{(1-1/s)q}dz\right)^{\frac{1}{(1-1/s)q}}\left(\int_y^\infty v(z)^{-(1-1/r)p'}dz\right)^{\frac{1}{(1-1/r)p'}}\\
&\times \left(\int_y^\infty k(z,y)^{(1-1/s)q}u(z)^{(1-1/s)q}dz\right)^{-\frac{1}{(1-1/s)q}}\\
&= \left(\int_y^\infty v(z)^{-(1-1/r)p'}dz\right)^{\frac{1}{(1-1/r)p'}}\\
&= h(x)^{(1-1/r)p}.
\end{aligned}$$

This completes the proof of the Lemma.

Lemma 3. *Let* $y\in\Re$, $h(y)$ *as in Lemma 2. If* $0<(1-1/s)q\le(1-1/r)p<1$, *and* $0<q\le p<1$, *then*
$\left(\int_{-\infty}^x v(y)^{-(1-1/r)p'}\left(\int_y^\infty v(z)^{-(1-1/r)p'}dz\right)^{-\frac{1}{(1-1/r)p}}dy\right)^{\frac{1}{(1-1/r)p'}}$

$$= \;(1-1/r)^{\frac{1}{(1-1/r)p'}}\,(-p')^{\frac{1}{(1-1/r)p'}}\,h(x)^{\frac{p}{p'}}. \tag{9}$$

Proof. By Lemma 2, we have

$$\begin{aligned}
&\int_{-\infty}^{x} v(y)^{-(1-1/r)p'} \left(\int_{y}^{\infty} v(z)^{-(1-1/r)p'} dz\right)^{-\frac{1}{(1-1/r)p}} dy \\
&= (1-1/r)\, p' \left(\int_{-\infty}^{x} v(z)^{-(1-1/r)p'} dz\right)^{\frac{1}{(1-1/r)p'}} \\
&= (1-1/r)\,(-p') \left(\int_{x}^{\infty} v(z)^{-(1-1/r)p'} dz\right)^{\frac{1}{(1-1/r)p'}} \\
&= (1-1/r)\,(-p') h(x)^{(1-1/r)p}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left(\int_{-\infty}^{x} v(y)^{-(1-1/r)p'} \left(\int_{y}^{\infty} v(z)^{-(1-1/r)p'} dz\right)^{-\frac{1}{(1-1/r)p}} dy\right)^{\frac{1}{(1-1/r)p'}} \\
&\qquad = (1-1/r)^{\frac{1}{(1-1/r)p'}} (-p')^{\frac{1}{(1-1/r)p'}} h(x)^{\frac{p}{p'}}
\end{aligned}$$

and the proof is complete.

Lemma 4. *Let T be the integral operator defined in (1) and let $k(x,y) \geq 0,\ (x,y) \in \triangle$. Suppose $0 < (1-1/s)q \leq (1-1/r)p < 1$ and $1 < q \leq p < 1$. Then*

$$\begin{aligned}
&[(Tf)(x)]^{(1-1/s)} \geq (1-1/r)^{\frac{(1-1/s)q}{(1-1/r)p'}} (-p')^{\frac{(1-1/s)q}{(1-1/r)p'}} \\
\times &\left(\int_{-\infty}^{x} k(x,y)^{(1-1/r)p} [f(y)v(y)h(y)]^{(1-1/r)p} dy\right)^{\frac{(1-1/s)q}{(1-1/r)p}} h(x)^{\frac{(1-1/s)pq}{p'}}.
\end{aligned} \tag{10}$$

Proof. Using the definition of T we have,

$$\begin{aligned}
(Tf)(x) &= \int_{-\infty}^{x} k(x,y)f(y)dy \\
&= \int_{-\infty}^{x} k(x,y)f(y)v(y)h(y)v(y)^{-1}h(y)^{-1}dy.
\end{aligned}$$

By Holder's inequality, we have

$$(Tf)(x) \geq \left(\int_{-\infty}^{x} k(x,y)^{(1-1/r)p} [f(y)v(y)h(y)]^{(1-1/r)p} dy\right)^{\frac{1}{(1-1/r)p}}$$

$$\times \quad \left(\int_{-\infty}^{x} v(y)^{-(1-1/r)p'} h(y)^{-(1-1/r)p'} dy\right)^{\frac{1}{(1-1/r)p'}}$$

$$= \quad \left(\int_{-\infty}^{x} k(x,y)^{(1-1/r)p} \left[f(y)v(y)h(y)\right]^{(1-1/r)p} dy\right)^{\frac{1}{(1-1/r)p}}$$

$$\times \quad \left(\int_{-\infty}^{x} v(y)^{-(1-1/r)p'} \left(\int_{y}^{\infty} v(z)^{-(1-1/r)p'} dz\right)^{-\frac{1}{(1-1/r)p}} dy\right)^{\frac{1}{(1-1/r)p'}}.$$

By Lemma 3, we have

$$(Tf)(x) \geq (1-1/r)^{\frac{1}{(1-1/r)p'}}$$

$$\times \quad (-p')^{\frac{1}{(1-1/r)p'}} \left(\int_{-\infty}^{x} k(x,y)^{(1-1/r)p} \left[f(y)v(y)h(y)\right]^{(1-1/r)p} dy\right)^{\frac{1}{(1-1/r)p}} h(x)^{\frac{p}{p'}}.$$

Hence

$$[(Tf)(x)]^{(1-1/s)} \geq (1-1/r)^{\frac{(1-1/s)q}{(1-1/r)p'}} (-p')^{\frac{(1-1/s)q}{(1-1/r)p'}}$$

$$\times \quad \left(\int_{-\infty}^{x} k(x,y)^{(1-1/r)p} \left[f(y)v(y)h(y)\right]^{(1-1/r)p} dy\right)^{\frac{(1-1/s)q}{(1-1/r)p}} h(x)^{\frac{(1-1/s)pq}{p'}}.$$

This completes the proof of the Lemma.

Theorem 1. *Let* $k(x,y) \geq 0, \quad (x,y) \in \triangle$ *and* $k(x,y)$ *is nondecreasing in* y. *Let* (u,v) *satisfy* $\inf J_1(a) \equiv B > 0$ *with* $J_1(a)$ *either bounded or nonincreasing. Suppose* $0 < (1-1/s)q \leq (1-1/r)p < 1, \quad 0 < q \leq p < 1$ *and* $f \geq 0$. *Then*

$$\left(\int_{-\infty}^{\infty} \mid v(x)f(x) \mid^{(1-\frac{1}{r})p} dx\right)^{\frac{1}{(1-\frac{1}{r})p}} \quad \leq \quad C\left(\int_{-\infty}^{\infty} \mid u(x)(Tf)(x) \mid^{(1-\frac{1}{s})q} dx\right)^{\frac{1}{(1-\frac{1}{s})q}}, \tag{11}$$

for every T*-admissible* f *and some positive constant* C *where*

$$C^{-1} \quad = \quad (1-1/r)^{\frac{1}{(1-1/r)p'}} (-p')^{\frac{1}{(1-1/r)p'}} (1-1/r)^{\frac{1}{(1-1/s)q}} p^{\frac{1}{(1-1/s)q}} B. \tag{12}$$

Proof. Denote by N the integral on the right hand side of (11) and assume that it is finite and T-admissible. Let $J_1(a)$ be nonincreasing, then we have

$$N \quad = \quad \left(\int_{-\infty}^{\infty} \mid u(x)(Tf)(x) \mid^{(1-1/s)q} dx\right)^{\frac{1}{(1-1/s)q}}$$

$$= \quad \left(\int_{-\infty}^{\infty} u(x)^{(1-1/s)q} \mid (Tf)(x) \mid^{(1-1/s)q} dx\right)^{\frac{1}{(1-1/s)q}}.$$

By Lemma 4, we have

$$N \geq (1-1/r)^{\frac{1}{(1-1/r)p'}} (-p')^{\frac{1}{(1-1/r)p'}} \left(\int_{-\infty}^{x} u(x)^{(1-1/s)q} \right.$$
$$\left. \times \left\{ \int_{-\infty}^{x} k(x,y)^{(1-1/r)p} [f(y)v(y)h(y)]^{(1-1/r)p} \, dy \right\}^{\frac{(1-1/s)q}{(1-1/r)p}} h(x)^{\frac{(1-1/s)pq}{p'}} dx \right)^{\frac{1}{(1-1/s)q}}$$

Hence

$$N^{(1-1/r)p} \geq (1-1/r)^{\frac{p}{p'}} (-p')^{\frac{p}{p'}} \left(\int_{-\infty}^{x} u(x)^{(1-1/s)q} \right.$$
$$\left. \times \left\{ \int_{-\infty}^{x} k(x,y)^{(1-1/r)p} [f(y)v(y)h(y)]^{(1-1/r)p} \, dy \right\}^{\frac{(1-1/s)q}{(1-1/r)p}} h(x)^{\frac{(1-1/s)pq}{p'}} dx \right)^{\frac{(1-1/r)p}{(1-1/s)q}} .$$

By Minkowskii's integral inequality (5), we obtain

$$\begin{aligned} N^{(1-1/r)p} &\geq (1-1/r)^{\frac{p}{p'}} (-p')^{\frac{p}{p'}} \int_{-\infty}^{\infty} \left(\int_{y}^{\infty} k(x,y)^{(1-1/s)q} u(x)^{(1-1/s)q} \right. \\ &\times \left. h(x)^{\frac{(1-1/s)pq}{(1-1/r)p'}} dx \right)^{\frac{(1-1/r)p}{(1-1/s)q}} [f(y)v(y)h(y)]^{(1-1/r)p} \, dy \\ &= (1-1/r)^{\frac{p}{p'}} (-p')^{\frac{p}{p'}} \int_{-\infty}^{\infty} \left(\int_{y}^{\infty} k(x,y)^{(1-1/s)q} u(x)^{(1-1/s)q} \right. \\ &\times \left. \left\{ \int_{y}^{\infty} v(z)^{-(1-1/r)p'} dz \right\}^{\frac{(1-1/s)q}{(1-1/r)^3 (p')^2}} dx \right)^{\frac{(1-1/r)p}{(1-1/s)q}} [f(y)v(y)h(y)]^{(1-1/r)p} \, dy. \end{aligned}$$

From Lemma 2, we have

$$\begin{aligned} &\left(\int_{y}^{\infty} v(z)^{(1-1/r)p'} dz \right)^{\frac{(1-1/s)q}{(1-1/r)^3 (p')^2}} \\ &= J_1(x)^{\frac{(1-1/s)q}{(1-1/r)^2 p'}} \left(\int_{y}^{\infty} k(z,y)^{(1-1/s)q} u(z)^{(1-1/s)q} dz \right)^{-\frac{1}{(1-1/r)p'}} . \end{aligned}$$

Hence

$$\begin{aligned} N^{(1-1/r)p} &\geq (1-1/r)^{\frac{p}{p'}} (-p')^{\frac{p}{p'}} \int_{-\infty}^{\infty} \int_{y}^{\infty} k(x,y)^{(1-1/s)q} u(x)^{(1-1/s)q} \left\{ J_1(x)^{\frac{(1-1/s)q}{(1-1/r)^2 p'}} \right. \\ &\times \left. \left(\int_{y}^{\infty} k(z,y)^{(1-1/s)q} u(z)^{(1-1/s)q} dz \right)^{-\frac{1}{(1-1/r)p'}} dx \right\}^{\frac{(1-1/r)p}{(1-1/s)q}} \\ &\times [f(y)v(y)h(y)]^{(1-1/r)p} \, dy. \end{aligned}$$

Now by Lemma 4, we obtain

$$\begin{aligned}
N^{(1-1/r)p} &\geq (1-1/r)^{\frac{p}{p'}} (-p')^{\frac{p}{p'}} (1-1/r)^{\frac{(1-1/r)p}{(1-1/s)q}} p^{\frac{(1-1/r)p}{(1-1/s)q}} \int_{-\infty}^{\infty} [f(y)v(y)]^{(1-1/r)p} \\
&\times J_1(y)^{\frac{(1-1/r)p}{(1-1/s)q}} J_1(y) \left(\int_y^{\infty} k(z,y)^{-(1-1/s)q} u(z)^{(1-1/s)q} dz \right)^{-\frac{1}{(1-1/s)q}} \\
&\times \left(\int_y^{\infty} k(z,y)^{(1-1/s)q} u(z)^{(1-1/s)q} dz \right)^{\frac{1}{(1-1/s)q}} dy \\
&= (1-1/r)^{\frac{p}{p'}} (-p')^{\frac{p}{p'}} (1-1/r)^{\frac{(1-1/r)p}{(1-1/s)q}} p^{\frac{(1-1/r)p}{(1-1/s)q}} \int_{-\infty}^{\infty} [f(y)v(y)]^{(1-1/r)p} \\
&\quad \times J_1(y)^{\left[1+\frac{(1-1/r)p}{(1-1/s)q}\right]} dy.
\end{aligned}$$

Since $J_1(y)$ does not vary with y, then we can take it out of the integral sign. Hence

$$\begin{aligned}
N^{(1-1/r)p} &\geq (1-1/r)^{\frac{p}{p'}} (-p')^{\frac{p}{p'}} (1-1/r)^{\frac{(1-1/r)p}{(1-1/s)q}} p^{\frac{(1-1/r)p}{(1-1/s)q}} B^{(1-1/r)p} \\
&\times \int_{-\infty}^{\infty} [f(y)v(y)]^{(1-1/r)p} dy
\end{aligned}$$

where

$$B^{(1-1/r)p} = J_1(y)^{\frac{(1-1/r)p}{(1-1/s)q}}.$$

Hence

$$\begin{aligned}
N &\geq (1-1/r)^{\frac{1}{(1-1/r)p'}} (-p')^{\frac{1}{(1-1/r)p'}} (1-1/r)^{\frac{1}{(1-1/s)q}} p^{\frac{1}{(1-1/s)q}} B \\
&\times \left(\int_{-\infty}^{\infty} [f(y)v(y)]^{(1-1/r)p} dy \right)^{\frac{1}{(1-1/r)p}} \\
&= C^{-1} \left(\int_{-\infty}^{\infty} [f(y)v(y)]^{(1-1/r)p} dy \right)^{\frac{1}{(1-1/r)p}}.
\end{aligned}$$

From this we obtain

$$\left(\int_{-\infty}^{\infty} | v(x)f(x) |^{(1-\frac{1}{r})p} dx \right)^{\frac{1}{(1-\frac{1}{r})p}} \leq C \left(\int_{-\infty}^{\infty} | u(x)(Tf)(x) |^{(1-\frac{1}{s})q} dx \right)^{\frac{1}{(1-\frac{1}{s})q}},$$

where C^{-1} is defined above and the proof is complete.

Remark 1. *In the limit $r \to \infty$ and $s \to \infty$ in Theorem 1, we obtain a result which is more general than Theorem 1 obtained by Beesack and Heinig [2].*

Remark 2. In Theorem 1 if we let $r \to \infty$ and $s \to \infty$ then we shall obtain Theorem 3.1 obtained by Andersen and Heinig [1].

Theorem 2. *Let $k(x,y)\geq 0$ be defined in $\triangle$ and $0<q\leq p<1$. If $k(x,y)$ is nondecreasing in y and (u,v) satisfy* $\inf K_1(a)\equiv B>0$ *with $K_1(a)$ either bounded or either bounded above or nondecreasing. Suppose* $0<(1-1/s)q\leq(1-1/r)p<1$. *Then*

$$\left(\int_{-\infty}^{\infty}\mid v(x)f(x)\mid^{(1-\frac{1}{r})p}dx\right)^{\frac{1}{(1-\frac{1}{r})p}} \leq C\left(\int_{-\infty}^{\infty}\mid u(x)(T^{\star}f)(x)\mid^{(1-\frac{1}{s})q}dx\right)^{\frac{1}{(1-\frac{1}{s})q}}, \tag{13}$$

for every $T^{\star}$-admissible f and some positive constant C where C^{-1} is defined above.

Proof. The proof is similar to that of Theorem 1 except that we define h by

$$\begin{aligned} h(y) &= \left(\int_{-\infty}^{y} v(z)^{-(1-1/r)p'}dz\right)^{\frac{1}{(1-1/r)^2pp'}} \\ &= K_1(y)^{\frac{1}{(1-1/r)p}}\left(\int_{-\infty}^{y} k(y,z)^{(1-1/s)q}u(z)^{-(1-1/s)q}dz\right)^{-\frac{1}{(1-1/r)(1-1/s)pq}}. \end{aligned}$$

The remaining part of the proof follows from Theorem 1.

Corollary 1. *Let $k(x,y)\geq 0$ be defined in $\triangle$. Suppose $q\leq p<0$ and* $(1-\frac{1}{s})q\leq(1-\frac{1}{r})p<0$.

(a) If $k(x,y)$ is nonincreasing in x and (u,v) satisfies $\inf K_1(a)\equiv B>0$ *with $K_1(a)$ either bounded or nonincreasing, then (11) holds for every T-admissible f.* *(b) If $k(x,y)$ is nondecreasing in y and (u,v) satisfies* $\inf J_1(a)\equiv B>0$ *with $J_1(a)$ either bounded or nondecreasing, then (13) holds for every T^{*}-admissible f.*

Proof. The proofs of (a) and (b) follow directly from the proofs of Theorem 1 and Theorem 2 since Corollary 1 is just the dual of Theorem 1 while Corollary 2 is the dual of Theorem 2.

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