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BLOCKED NETWORK OF TANDEM QUEQUES WITH WITHDRAWAL

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Abstract. We consider a network of two queues in tandem with one server in the first queue, and $n \geq 1$ in the second queue. Customers access the system through the first queue in accordance with Poisson input having parameter λ. Holding times are exponentially distributed with independent and identical random variables having rates μ_1 and μ_2 at the first and second queues respectively. A blocking phenomenon is observed in the system with empty buffer between stages, bringing about a withdrawal effect. We obtain the state probabilities and observe that the system has no product form.

INTRODUCTION

The earliest work on queues in tandem is due to [7] with a two-server station in tandem, and its extension to k (> 2) stations in [8] with unlimited waiting spaces between service points. However, when limitations of space are imposed between stages, then the problem of *blocking* is encountered, which imposes some difficulties in the analysis of such systems.

According to [3] and [5], blocking occurs when the flow of units through one queue is momentarily stopped owing to a capacity limitation of the queue ahead. A customer finishing service at the first stage is only allowed to proceed to the second stage if there is a free server in the second stage, otherwise he is blocked and stays in the first queue until there is an exit of at least a customer from the second queue.

There are numerous contributions in the literature on the analysis of tandem network of queues. See for example [14, 9, 12, 13, 11, 1, 2]. For such models in [16], it has been shown that the joint queue length process forms a two-dimensional finite, irreducible Markov process.

Here, we consider a network of two queues in tandem with one server in the first queue, and $n \geq 1$ servers in the second queue. Customers access the system through the first queue in accordance with Poisson input having parameter λ. Service times are independent and identically distributed random variables with rates μ_1 and μ_2 at the first and second queues respectively, with a first-come, first- served discipline. There is no waiting space between the two service stages, i.e. the queue length (L_q) is zero in the second queue. Such a model is referred to as a 3-tuple(1,0,n)-model in [12].

The diagram below describes the model.

$$
\begin{array}{ccccccccccc}
 & & & \mu_1 & & & & & \mu_2 & & \\
\bullet\bullet\bullet & \rightarrow & \lambda & \boxed{\text{Queue 1}} & \rightarrow & \stackrel{(1-w)}{\longrightarrow} & L_q = 0 & \rightarrow & \boxed{\text{Queue 2}} & \rightarrow & \text{Leaves} \\
 & & & \text{1 server} & \downarrow & & \text{No waiting} & & n \geq 1 & & \text{system} \\
 & & & & \bullet & & \text{space} & & \text{servers} & & \downarrow \\
 & & & & w & & & & & & \bullet \\
 & & & & \downarrow & & & & & & \bullet \\
 & & & & \text{withdraws} & & & & & &
\end{array}
$$

A two-stage tandem queue with blocking and withdrawal

THE MODEL

Let at time t, an incoming customer meet m and r customers in the first and second queues respectively, including those being served, where $m \geq 0$ and $0 \leq r \leq n$.

For state i, let $\xi_i(t)$ denote the number of customers at time t. We therefore have,

$$\xi_i(t) = \begin{cases} m, & i = 1 \\ r, & i = 2, \end{cases}$$

Define

$$P_{m,r}(t) = \text{Prob}\{\xi_1(t) = m,\ \xi_2(t) = r\}$$

and

$$P'_{m,n}(t) = \text{Prob}\{\xi_1(t) = m, \text{ and the customer in service is blocked}, \xi_2(t) = n\}$$

Let w denote the probability of withdrawal of a blocked customer who decides to renege from further service. The random process

$$\xi = \{\xi_i(t) : i = 1, 2; 0 \le t < \infty\}$$

assumes values in the strip $\{(m, r); 0 \le m < \infty, 0 \le r \le n\}$ and constitutes a continuous time Markov chain. We note that the definition of $P_{m,n}(t)$ is contained in $P_{m.r}$, since $0 \le r \le n$, and where in this case, the incoming customer is not blocked in the first stage.

Assuming that steady state exists, the global balance equations of the system become

$$\begin{aligned}(\lambda + \mu_1 + r\mu_2)P_{m,r} &= \lambda P_{m-1,r} + \mu_1 P_{m+1,r-1} + (r+1)\mu_2 P_{m,r+1}; \\ &\qquad 0 < m < \infty, 0 < r < n \end{aligned} \tag{1}$$

with the following boundary equations:

$$(\lambda + \mu_1)P_{m,0} = \lambda P_{m-1,0} + \mu_2 P_{m,1}; \quad 0 < m < \infty \tag{2}$$

$$(\lambda + r\mu_2)P_{0,r} = \mu_1 P_{1,r-1} + (r+1)\mu_2 P_{0,r+1} \quad 0 < r < n \tag{3}$$

$$\begin{aligned}(\lambda + \mu_1 + n\mu_2)P_{m,n} &= \lambda P_{m-1,n} + \mu_1 P_{m+1,n-1} + \mu_1 w P_{m+1,n} + n\mu_2 P'_{m+1,n}; \\ &\qquad 0 < m < \infty \end{aligned} \tag{4}$$

$$(\lambda + n\mu_2)P_{0,n} \;=\; \mu_1 P_{1,n-1} + \mu_1 w P_{1,n} + n\mu_2 P'_{1,n} \tag{5}$$

$$(\lambda + n\mu_2)P'_{m,n} \;=\; \lambda P'_{m-1,n} + \mu_1(1-w)P_{m,n} \tag{6}$$

$$\lambda P_{0,0} \;=\; \mu_2 P_{0,1} \tag{7}$$

By Konheim and Reiser [9, 10], the system of equations (1) to (7) is homogeneous and hence always admits a bounded nonnull absolutely summable solution

$$\{P_{m,r}, P'_{m,n}\} = 0, \;\forall m, r, n,$$

which is also needed for the stability of the system. This requires that, $P_{m,r},\; P'_{m,n} \neq 0$ (for some m, r, and n) and

$$\sum_{m,r,n} \mid P_{m,r} + P'_{m,n} \mid < \infty$$

Now define the following generating functions

$$\mathcal{P}_r(z) = \sum_{0 \leq m < \infty} P_{m,r} z^m, \quad 0 \leq r \leq n$$

and

$$\mathcal{P}'_n(z) = \sum_{1 \leq m < \infty} P'_{m,n} z^m$$

where z is a dummy variable satisfying $\mid z \mid \leq 1$, a condition needed for the convergence of the infinite series.

The above generating functions transform the system of equations (1) to (7) to the following:

$$\alpha_r \mathcal{P}_r(z) - \mu_1 z^{-1} \mathcal{P}_{r-1}(z) - (r+1)\mu_2 \mathcal{P}_{r+1}(z) = \mu_1 P_{0,r} - \mu_1 z^{-1} P_{0,r-1};\; 0 < r < n \tag{8}$$

$$\alpha_0 \mathcal{P}_0(z) - \mu_2 \mathcal{P}_1(z) = \mu_1 P_{0,0} \tag{9}$$

$$\begin{aligned}\alpha_n \mathcal{P}_n(z) - \mu_1 z^{-1}\mathcal{P}_{n-1}(z) - n\mu_2 z^{-1}\mathcal{P}'_n(z) &= \mu_1(1-wz^{-1})P_{0,n} \\ &- \mu_1 z^{-1} P_{0,n-1} \end{aligned} \tag{10}$$

$$\alpha'_n \mathcal{P}'_n(z) - \mu_1(1-w)\mathcal{P}(z) = -\mu_1(1-w)P_{0,n} \tag{11}$$

where

$$\alpha_r = \lambda(1-z) + \mu_1(1-wz^{-1}\delta_r) + r\mu_2,$$

$$\delta_r = \begin{cases} 0, & 0 \le r < n \\ 1, & r = n \end{cases}$$

and $\alpha'_n = \lambda(1-z) + n\mu_2$

The linear system of equations (8) to (11) can be expressed in the form

$$A_n(z)\tilde{\mathcal{P}}(z) = B_n(z)\tilde{\mathcal{P}}_0 \tag{12}$$

where $A_n(z)$ and $B_n(z)$ are tridiagonal $(n+2)\times(n+2)$ matrices with

$$\tilde{\mathcal{P}}(z) = (\mathcal{P}_0(z), \mathcal{P}_1(z), ..., \mathcal{P}_n(z), \mathcal{P}'_n(z))'$$

and

$$\tilde{\mathcal{P}}_0 = (P_{0,0}, P_{0,1}, ..., P_{0,n}, 0)'$$

being $(n+2)\times 1$ vectors.

Let (12) be written in the form

$$A_n(z)\tilde{\mathcal{P}}(z) = \mathcal{D}(z) \tag{13}$$

where

$$\mathcal{D}(z) = (d_1(z), d_2(z), ..., d_{n+2}(z))'$$

with

$$d_j(z) = \mu_1(1-wz^{-1}\delta_{j,0})P_{0,j-1} - \mu_1 z^{-1}P_{0,j-2};\ 1 \le j \le n+1$$

and

$$d_{n+2}(z) = -\mu_1(1-w)P_{0,n}.$$

$\delta_{j,0}$ is a Kronecker delta defined by

$$\delta_{j,0} = \begin{cases} 0, & 1 \le j \le n \\ 1, & j = n+1 \end{cases}$$

and $P_{0,k} = 0 \ \forall \ k < 0$

According to [4] and [15], the solution $\tilde{\mathcal{P}}(z) = \mathcal{X}$ of equation (13) is obtained by back substitution as follows:

$$\left.\begin{array}{rcl} x_{n+2} & = & \tilde{d}_{n+2} = \tilde{\mathcal{P}}_{\mathbf{n}}(z) \\ x_i & = & \tilde{d}_i - \tilde{c}_i x_{i+1}, \quad i = n+1, n, ..., 1 \end{array}\right\} \tag{14}$$

where

$$\tilde{c}_j = \frac{-jz\mu_2}{z\alpha_{j-1} + \mu_1 \tilde{c}_{j-1}}, \quad j = 1, 2, ..., n; \ \tilde{c}_0 = 0 \tag{15}$$

$$\tilde{d}_j = \frac{z\mu_1 P_{0,j-1} - \mu_1 P_{0,j-2} + \mu_1 \tilde{d}_{j-1}}{z\alpha_{j-1} - \frac{(j-1)\mu_1\mu_2 T_{j-3}}{T_{j-2}}\delta_{z,j}}, \quad j = 1, 2, ..., n \tag{16}$$

and

$$\alpha_j = \lambda(1-z) + \mu_1(1 - wz^{-1}\delta_j) + j\mu_2;$$

$$\delta_j = \begin{cases} 0, & j = 0, 1, ..., n+1 \\ 1, & j = n+2 \end{cases}$$

with

$$T_j = \alpha_j T_{j-1}\delta_{z,j} - j\mu_1\mu_2 T_{j-2}, \ j \ge 1; \quad T_0 = \alpha_0, \ T_{-1} = 1, \ T_j = 0 \ \text{ for } j \le -2$$

Also,

$$\delta_{z,j} = \begin{cases} z, & \text{for odd } j \\ 1, & \text{for even } j \text{ and zero} \end{cases}$$

with $\tilde{d}_j = 0$ for $j \le 0$ and $P_{0,j} = 0$ for $j < 0$.

It is now easy to see that

$$\tilde{c}_{n+1} = \frac{-n\mu_2\beta(z)}{z(\alpha_n\beta(z) - n\mu_1\mu_2)}$$

where

$$\beta(z) = z\alpha_{n-1} + \mu_1\tilde{c}_{n-1},$$

$$\tilde{d}_{n+1} = \frac{H(.)}{z\lambda(z)(\alpha_n\beta(z) - n\mu_1\mu_2)}$$

and where

$$\begin{aligned} H(.) &= \mu_1\beta(z)\{z\lambda(z)(1 - wz^{-1})P_{0,n} + (z\mu_1 T_{n-2} - \lambda(z))P_{0,n-1} \\ &- \mu_1 T_{n-2}P_{0,n-2} + \mu_1 T_{n-2}\tilde{d}_{n-1}\} \end{aligned}$$

with

$$\lambda(z) = z\alpha_{n-1}T_{n-2} - (n-1)\mu_1\mu_2 T_{n-3}\delta_{z,n}.$$

Finally, we have

$$\tilde{d}_{n+2} = \frac{V(.)}{\lambda(z)[z\alpha'_n\varphi(z) - n\mu_1\mu_2(1-w)\beta(z)]}$$

where

$$\begin{aligned} V(.) &= (1-w)\{\mu_1 z\lambda(z)[\mu_1\beta(z)(1 - wz^{-1}) - \varphi(z)]P_{0,n} \\ &+ \mu_1^2\beta(z)(\mu_1 z T_{n-2} - \lambda(z))P_{0,n-1} - \mu_1^3\beta(z)T_{n-2}P_{0,n-2} + \mu_1^3\beta(z)T_{n-2}\tilde{d}_{n-1}\} \end{aligned}$$

and

$$\varphi(z) = \alpha_n\beta(z) - n\mu_1\mu_2$$

It is now possible to write

$$\mathcal{P}'_n(z) = \frac{(1-w)\sum_{j=0}^{n} D_j(z)P_{0,j}}{D_n(z)}$$

where,

$$D_0(z) = \mu_1^{n+1}(\mu_1 - T_0(z))$$

$$D_j(z) = \mu_1^{n-j+1}[z\mu_1(1 - wz^{-1})T_{j-1}(z) - T_j(z)], \quad j \neq 0, \ j \neq n$$

and

$$D_n(z) = \mid A_n(z) \mid = \alpha'_n T_n(z) - n\mu_1\mu_2(1-w)T_{n-1}(z)$$

We remark that the expression which gave $\mathcal{P}'_n(z)$ is defined for $z = 0$, since z cancels out when this expression is simplified. And by means of successive back substitutions in equation (14), we can express $\mathcal{P}_r(z)$; $0 \leq r < n$, in terms of $\mathcal{P}'_n(z)$ and the boundary values $\{P_{0,r}\}$, from which the state probabilities can subsequently be retrieved.

A SPECIAL CASE

We illustrate the results obtained above for $n = 1$. In this case we have a $(1,0,1)$-model with withdrawal. This presents us with a network of two queues in tandem with one server in each of the queues with unlimited waiting space in front of the first queue, and none between them. Using equations (15) and (16) we obtain the following.

$$\left.\begin{array}{rcl}
\tilde{c}_0 &=& 0 = \tilde{c}_3 \\
\tilde{c}_1 &=& -\mu_2 \alpha_0^{-1} \\
\tilde{c}_2 &=& -\mu_2 \alpha_0 T_1^{-1} \\
\tilde{d}_0 &=& 0 \\
\tilde{d}_1 &=& \mu_1 \alpha_0^{-1} P_{0,0} \\
\tilde{d}_2 &=& \{\mu_1[(\mu_1 - \alpha_0)P_{0,0} + z\alpha_0(1 - wz^{-1})P_{0,1}]\}T_1^{-1}, \text{ and} \\
\tilde{d}_3 &=& \mu_1(1-w)[\mu_1(\mu_1 - \alpha_0)P_{0,0} + (\mu_1 z\alpha_0(1 - wz^{-1}) - T_1)P_{0,1}] \\
& \times & (\Phi(z))^{-1}
\end{array}\right\} \tag{17}$$

where

$$\Phi(z) = \alpha_1' T_1 - \mu_1\mu_2\alpha_0(1-w)$$

and in particular, $\Phi(1) = \mu_1^2 \mu_2 w$.

By substituting corresponding expressions in (17) into (14), we shall obtain

$$\begin{array}{rcl}
\mathcal{P}_1'(z) &=& \tilde{d}_3 \\
\mathcal{P}_1(z) &=& \{\mu_1(\mu_1 - \alpha_0)\alpha_1' P_{0,0} + \mu_1\alpha_0[z\alpha_1'(1 - wz^{-1}) \\
&-& \mu_2(1-w)]P_{0,1}\}\Phi^{-1}(z) \\
\mathcal{P}_0(z) &=& \{\mu_1[z\alpha_1\alpha_1' - \mu_1\mu_2(1-w) - \alpha_1'\mu_2]P_{0,0} \\
&+& \mu_1\mu_2[z\alpha_1'(1 - wz^{-1}) - \mu_2(1-w)]P_{0,1}\}\Phi^{-1}(z)
\end{array}$$

Using the normalizing condition

$$\sum_{0 \le r \le n} \mathcal{P}_r(z=1) + \mathcal{P}_n'(z=1) = 1$$

and (7), we can show that

$$P_{0,0} = \frac{\mu_1\mu_2(\mu_1 + \mu_2) - \lambda(\mu_1^2(1-w) + \mu_1\mu_2 + \mu_2^2)}{\mu_1\mu_2(\mu_1 + \mu_2) + \lambda\mu_1(\mu_1 w + \mu_2)} \tag{18}$$

By definition,

$$P_{k,i} = \frac{1}{k!} d^{(k)} \mathcal{P}_i(z) \mid_{z=0}, \quad i = 0, 1$$

and

$$P_{k,1} = \frac{1}{k!} d^{(k)} \mathcal{P}'_1(z) \mid_{z=0} .$$

We retrieve $\{P_{k,i}\}_k$ and $\{P'_{k,1}\}$ from thier generating functions, noting that $P'_{0,i} = 0 = P'_{k,0}$ $\forall i$ and $k \geq 0$ for obvious reasons. Finally if we set $n = 1$ in this model, we have a network with no waiting space in front of the first queue. The implication of this by [16] and [6], is that arriving customers to the first queue are turned away, either, when the system is blocked or when a service is in progress at station one, even if the second queue is empty, as the system is in sequence. In this case, the system (1) to (7) becomes

$$\left.\begin{array}{rcl} (\mu_1 + \mu_2) P_{1,1} & = & \lambda P_{0,1} \\ \mu_1 P_{1,0} & = & \lambda P_{0,0} + \mu_2 P_{1,1} \\ (\lambda + \mu_2) P_{0,1} & = & \mu_1 P_{1,0} + \mu_1 w P_{1,1} + \mu_2 P'_{1,1} \\ \mu_2 P'_{1,1} & = & \mu_1 (1 - w) P_{1,1} \\ \lambda P_{0,0} & = & \mu_2 P_{0,1} \end{array}\right\} \tag{19}$$

The solution of (19) produces the state probabilities

$$\begin{array}{rcl} P_{0,1} & = & \dfrac{\lambda}{\mu_2} P_{0,0} \\ P_{1,0} & = & \dfrac{\lambda(\lambda + \mu_1 + \mu_2)}{\mu_1(\mu_1 + \mu_2)} P_{0,0} \\ P_{1,1} & = & \dfrac{\lambda^2}{\mu_2(\mu_1 + \mu_2)} P_{0,0} \\ P'_{1,1} & = & \dfrac{\mu_1(1 - w)\lambda^2}{\mu_2^2(\mu_1 + \mu_2)} P_{0,0}, \quad \text{giving} \\ P_{0,0} & = & \dfrac{\mu_1 \mu_2^2 (\mu_1 + \mu_2)}{\mu_1 \mu_2 \{(\mu_1 + \mu_2)(\mu_2 + \lambda) + \lambda^2\} + \lambda\{\mu_2^2(\lambda + \mu_1 + \mu_2) + \lambda \mu_1^2 (1 - w)\}} \end{array} \tag{20}$$

CONCLUSION

In this paper, we have shown that our model generalises the result in the literature. We observe that by setting $w = 0$ in equation (20), we have the results obtained in

[16], when it is assumed that the servers are homogeneous. That is, when $\mu_1 = \mu_2$. We also observe that expressions (18) and (19) do not have a product form, since by [16], the queues are not independent.

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