Kragujevac J. Math. 23 (2001) 59–62.

ONE PROBLEM OF D. S. MITRINOVIĆ AND ITS GENERALIZATION

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(Received February 17, 2001)

Abstract. It is shown that, if y_0 is the solution of the differential equation (1), then y_0' is the solution of the differential equation (1.1).

Besides this, it is shown that general solutions of differential equations (4) and (5) satisfy relation (6), if their coefficients are satisfy by the relations (9) and (10).

D.S. Mitrinović [1] define the following problem:

If y_0 is the solution of the differential equation

$$xy'' + y' + xy = 0$$

then y_0' is the solution of the differential equation

$$x^2y'' + xy' + (x^2 - 1)y = 0.$$

Prove this statement.

This result is indicative because it initiates the idea to find the answer to the following more general question:

If y_0 is the solution of the differential equation

$$a_0y^{(n)} + a_1y^{(n-1)} + \ldots + a_ny = 0, \quad a_0' \neq 0 \tag{1}$$

which differential equation has the solution y_0'?

The simplest way to give the answer to the above equation is to write equation (1) in the form

$$y^{(n)} + (a_1/a_0)y^{(n-1)} + \ldots + (a_n/a_0)y = 0$$

and then to find its derivative. After this operation the following equation is obtained

$$y^{(n+1)} + (a_1/a_0)y^{(n)} + \sum_{k=1}^{n-1} \left[(a_k/a_0)' + a_{k+1}/a_0\right] y^{(n-k)} + (a_n/a_0)'y = 0. \tag{2}$$

It is evident that the differential equation (2) will have the solution y_0', if and only if the condition

$$(a_n/a_0)' = 0 \Rightarrow a_n = a_0 \tag{3}$$

is satisfied, so that, with (2) and (3), the sought differential equation has the form

$$y^{(n)} + (a_1/a_0)y^{(n-1)} + \sum_{k=1}^{n-1} \left[(a_k/a_0)' + a_{k+1}/a_0\right] y^{(n-k-1)} = 0$$

or

$$a_0^2 y^{(n)} + a_0 a_1 y^{(n-1)} + \sum_{k=1}^{n-1} \left(a_k' a_0 - a_k a_0' + a_{k+1} a_0\right) y^{(n-k-1)} = 0. \tag{1.1}$$

Therefore, it is proved the following

Theorem 1. *If y_0 is the solution of the differential equation* (1)*, then y_0' is the solution of the differential equation* (1.1)*.*

However, regarding the above presentation, the answer to the following question may be sought:

Which relation must exist between the coefficients of the equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = 0 \tag{4}$$

and

$$z^{(n)} + b_1(x)z^{(n-1)} + \ldots + b_n(x)z = 0 \tag{5}$$

under the condition that their general solutions Y and Z satisfy relation

$$Z = Y'. \tag{6}$$

In order to answer the above equation, it should be noted that, taking into account (6)

$$Y^{(n)} + a_1(x)Y^{(n-1)} + \ldots + a'_n(x)Y \equiv 0 \tag{4.1}$$

and

$$Y^{(n+1)} + b_1(x)Y^{(n)} + \ldots + b_n(x)Y' \equiv 0. \tag{5.1}$$

Besides this, when (4.1) is differentiated, the following identity is obtained

$$Y^{(n+1)} + \sum_{k=1}^{n-1}(a'_k + a_{k+1})Y^{(n-k)} + a'_n Y \equiv 0$$

and, after subtracting (5.1), the identity

$$(a_1 - b_1)Y^{(n)} + \sum_{k=1}^{n-1}(a'_k + a_{k+1} - b_{k+1})Y^{(n-k)} + a'_n Y \equiv 0. \tag{7}$$

If $Y^{(n)}$ is calculated from (4.1) and value

$$Y^{(n)} \equiv -a_1 Y^{(n-1)} - \ldots - a_n Y$$

is substitued into the identity (7), the following identity is obtained

$$(-a_1^2 + a_1 b_1 + a_2 - b_2 + a'_1)Y^{(n-1)} + \sum_{k=2}^{n-1}(a'_k + a_{k+1} - b_{k+1}$$

$$-a_1 a_n - a_k b_1)Y^{(n-k)} + (a'_n - a_1 a_n + a_n b_1)Y \equiv 0$$

which is satisfied for any x, if

$$\begin{aligned} -a_1^2 + a_1 b_1 + a_2 - b_2 + a'_1 &= 0 \\ \sum_{k=1}^{n-1}(a'_k + a_{k+1} - b_{k+1} - a_1 a_n - a_n b_1) &= 0 \\ a'_n - a_1 a_n + a_n b_1 &= 0. \end{aligned} \tag{8}$$

For system equations (8) follows

$$b_1 = (a_1 a_n - a'_n)/a_n \Leftrightarrow b_1 = a_1 - (\ln|a_n|)' \tag{9}$$

and

$$b_k = [(a_{k-1}/a_k)' + a_k/a_n]a_n, \quad k = 2, 3, \ldots, n. \tag{10}$$

Therefore, from the above follows that it is proven the following

Theorem 2. *The general solutions of differential equations* (4) *and* (5) *are related by* (6), *if and only if their coefficients satisfy the conditions* (9) *and* (10).

It is obvious that the results expressed by Theorem 1 and 2 are of interest for the analytical theory of differential equations, because they provide an extension of the integrable differential equations.

In addition to this statement is the equivalency

$$(a_n/a_0)' = 0 \Leftrightarrow a_n = Ca_0, \quad C = \text{const.}$$

from which follows that, from the integrability of the differential equation (1.1), follows integrability of the differential equation

$$\begin{aligned} a_0^2 y^{(n)} + a_0 a_1 y^{(n-1)} &+ \sum_{k=1}^{n-2} (a_k' a_0 - a_k a_0' + a_{k+1} a_0) y^{(n-k-1)} \\ &+ (a_{n-1}' a_0 - a_{n-1} a_0' + Ca_0^2) y = 0. \end{aligned}$$

References

[1] D.S. Mitrinović, *Zbornik matematičkih problema IV*, Naučna knjiga, Beograd 1972.