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AN ALGEBRAIC CONDITION FOR QUALITATIVE STABILITY OF FIRST ORDER LINEAR AUTONOMOUS ORDINARY DIFFERENTIAL SYSTEMS

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Abstract. A necessary condition for qualitative stability of the solutions of first order linear autonomous ordinary differential systems is given.

1. INTRODUCTION

In an earlier work, Collins [2] derived some necessary and sufficient algebraic conditions for the origin of a homogeneous polynomial planar vector field of arbitrary degree to be a centre, an unstable focus or a stable focus. Also, Sleeman et al [10] investigated the minimum number of limit cycles which a certain trigonometric autonomous ordinary differential system has. In this paper, the author established a condition for two first order linear autonomous ordinary differential systems to be qualitatively stable. We prove the following:

Theorem. *Let*

$$A' = AX \tag{1}$$

$$Y' = BY \tag{2}$$

be two linear first order one-dimensional autonomous ordinary differential systems which are qualitatively equivalent. Suppose δ_1, δ_2 *are the eigenvalues of the coefficient matrix* A *of* (1) *and* ρ_1, ρ_2 *the eigenvalues of the coefficient matrix* B *of* (2) *such that* $\det(A) = a_{11}a_{22} - a_{12}a_{21}$, $\det(B) = b_{11}b_{22} - b_{12}b_{21}$, $\delta_1 < \delta_2 < 0$ *and* $\rho_1 < \rho_2 < 0$. *Then a necessary condition for the solutions of* (1) *and* (2) *to be qualitatively stable is the existence of two real non-singular matrices,* M *and* N *defined by* $\det(M) = m_{11}m_{22} - m_{12}m_{21}$ *and* $\det(N) = n_{11}n_{22} - n_{12}n_{21}$ *such that*

(i)

$$\frac{tr(A) + \lambda}{tr(A) - \lambda} - \frac{\det(M)}{m_{12}m_{21}} = 1 \tag{R1}$$

(ii)

$$\frac{tr(B) + \lambda}{tr(B) - \lambda} - \frac{\det(N)}{n_{12}n_{21}} = 1 \tag{R2}$$

where $\delta_1 = \rho_1 = \lambda_1$, $\delta_2 = \rho_2 = \lambda_2$, $\lambda_1 < \lambda_2$, $\lambda = \lambda_1 - \lambda_2$, $m_{12}m_{21} \neq 0$ *and* $n_{12}n_{21} \neq 0$.

2. QUALITATIVE CLASSES

Let

$$X_i' = f_i(X) \tag{3}$$

$$Y_j' = g_j(Y) \tag{4}$$

be two systems of first order autonomous ordinary differential equations. Then the systems are said to be qualitatively equivalent if there exists a continuous bijection which maps the phase portrait of (3) into that of (4) in such a way that the orientation of their trajectories is maintained [8, 9]. A relation ρ between the two qualitatively equivalent systems (3) and (4) is thus an equivalence relation [4]. If S represents the set of all first order autonomous ordinary differential systems, then the qualitative classes are the disjoint equivalence classes into which S is partitioned by ρ. In general, there are ten qualitative classes for linear systems. Each of these classes is characterized by a unique phase portrait called an algebraic type. The ten algebraic types are further grouped into four qualitative (or topological) types according to the following distinct qualitative behaviour viz (*a*) Stable behaviour (*b*) Unstable behaviour (*c*) Centre (*d*) Saddle. The four algebraic types which exhibit stable behaviour are the node, improper node, focus (or spiral) and star.

3. JORDAN CANONICAL FORM

There is a connection between qualitative equivalence of differential systems and similarity (or equivalence) of matrices in the sense that if two differential systems are qualitatively equivalent, then their coefficient matrices are similar [6]. Now, two matrices A and B are similar if there exists a non-singular matrix P such that $B = P^{-1}AP$. Consider the two systems (1) and (2) which belong to the same qualitative class. The solutions of these systems are related by the equation

$$X = PY, \tag{5}$$

where $B = P^{-1}AP$.

Now consider the canonical system

$$Z' = JZ \tag{6}$$

for (1) and (2) where J is the Jordan canonical form or matrix. Using the fact that similarity of matrices is an equivalence relation on the set of $n \times n$ real matrices [7], we have

$$J = M^{-1}AM = N^{-1}BN \tag{7}$$

where M and N are non-singular matrices. Hence, all qualitatively equivalent systems have the same Jordan canonical form. In particular, since similar matrices have the same eigenvalues, then all differential systems which belong to the same qualitative class necessarily have the same eigenvalues. It thus follows that the solutions of (1) and (2) are obtained by solving (6) and using the following relationships:

$$X = MZ \tag{8}$$

$$Y = NZ. \tag{9}$$

4. PROOF OF THEOREM

Since $\delta_1 < \delta_2 < 0$ and $\rho_1 < \rho_2 < 0$, then the phase portraits of (1) and (2) are stable nodes. Therefore,

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where $\lambda_1 = \delta_1 = \rho_1$, $\lambda_2 = \delta_2 = \rho_2$. We note that since A and B are similar, then there exist non-singular matrices M and N such that (7) is satisfied. Using (7) , we have

$$m_{22}m_{11}(a_{11} - \lambda_1) + m_{22}m_{21}a_{12} - m_{12}m_{21}(a_{22} + \lambda_1) - m_{12}m_{11}a_{21} = 0 \tag{10}$$

$$n_{22}n_{11}(b_{11} - \lambda_1) + n_{22}n_{21}b_{12} - n_{12}n_{21}(b_{22} + \lambda_1) - n_{12}n_{11}b_{21} = 0 \tag{11}$$

$$-m_{12}m_{21}(a_{11} - \lambda_2) + m_{11}m_{22}(a_{22} - \lambda_2) - m_{21}m_{22}a_{12} + m_{11}m_{12}a_{21} = 0 \tag{12}$$

$$-n_{12}n_{21}(b_{11} - \lambda_2) + n_{11}n_{22}(b_{22} - \lambda_2) - n_{21}n_{22}b_{12} + n_{11}n_{12}b_{21} = 0. \tag{13}$$

Adding (10) and (12), and (11) and (13), we have

$$m_{11}m_{22}[(a_{11} - \lambda_1) + (a_{22} - \lambda_2)] - m_{12}m_{21}[(a_{22} + \lambda_1) + (a_{11} - \lambda_2)] = 0 \tag{14}$$

$$n_{11}n_{22}[(b_{11} - \lambda_1) + (b_{22} - \lambda_2)] - n_{12}n_{21}[(b_{22} + \lambda_1) + (b_{11} - \lambda_2)] = 0 \tag{15}$$

from which the final results follow.

5. DISCUSSION

As can be seen from (8) and (9), the critical issue in arriving at the solutions of (1) and (2) is the ability to construct the non-singular matrices M and N. Our theorem enables one to construct these matrices. For example, consider the following two differential systems which are qualitatively stable viz

$$X_1^{'} = -X_1$$

$$X_2^{'} = -2X_2 \tag{16}$$

and

$$X_1^{'} = X_1 - 2X_2$$

$$X_2^{'} = 3X_1 - 4X_2. \tag{17}$$

Here,

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix},$$

$tr(A) = tr(B) = -3$, $\delta_1 = \rho_1 = -2$ and $\delta_2 = \rho_2 = -1$. From $(R1)$ and $(R2)$,

$$m_{11}m_{22} = 2m_{12}m_{21} \tag{18}$$

$$n_{11}n_{22} = 2n_{12}n_{21}. \tag{19}$$

When $m_{12}m_{21} \neq 0$, then (18) is satisfied by, for instance, $m_{11} = 2$, $m_{22} = 1$, $m_{12} = m_{21} = 1$. Hence,

$$M = N = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

satisfies $(R1)$ and $(R2)$. We note that $(R1)$ and $(R2)$ are also satisfied by the following matrices amongst others :

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}.$$

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