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A NOTE ON DENSITY OF THE ZEROS OF σ-ORTHOGONAL POLYNOMIALS

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Abstract. A new proof of density of the zeros of σ-orthogonal polynomials is presented. Some numerical results for a class of σ-orthogonal polynomials on two symmetric intervals are included.

INTRODUCTION

Let $d\lambda(t)$ be a nonnegative distribution on the compact support $[a,b]$, for which all moments

$$\mu_k = \int_a^b t^k \, d\lambda(t), \quad k = 0, 1, \ldots,$$

exist and are finite, and $\mu_0 > 0$.

Let $\sigma = (s_1, s_2, \ldots, s_n, \ldots)$ be a bounded sequence of nonnegative integers, and denote $(s_1, s_2, \ldots, s_n)$ by σ_n. Let

$$\overline{s} = \max\{s_k \mid k = 1, 2, \ldots\}.$$

Assume that for $\tau_\nu (= \tau_\nu^{(\sigma_n)}) \in (a,b)$, $\nu = 1, \dots, n$, hold

$$\tau_1 < \tau_2 < \cdots < \tau_n.$$

It is known (for more details see the survey paper [5]) that the Chakalov-Popoviciu quadrature formula with multiple nodes,

$$\int_a^b f(t)\, d\lambda(t) = \sum_{\nu=1}^{n} \sum_{h=0}^{2s_\nu} A_{h\nu} f^{(h)}(\tau_\nu) + R(f), \tag{1}$$

has the maximum degree of exactness

$$d_{\max} = 2\left(\sum_{\nu=1}^{n} s_\nu + n\right) - 1$$

if and only if

$$\int_a^b \prod_{\nu=1}^{n} (t-\tau_\nu)^{2s_\nu+1} t^k\, d\lambda(t) = 0, \quad k = 0, 1, \dots, n-1. \tag{2}$$

The proofs of the existence and the uniqueness of (1) which have been obtained recently can be found in [6], [7].

The orthogonality conditions (2) define a sequence of polynomials $\{\pi_{n,\sigma}\}_{n\in\mathcal{N}_0}$ which are called σ-orthogonal polynomials. In the case $s_1 = s_2 = \ldots = s$ the above polynomials reduce to the s-orthogonal polynomials, and (1) is known as the Gauss-Turán quadrature formula.

MAIN RESULT

Recently Shi [8] has stated a property of density of σ-orthogonal polynomials. He obtained this statement as a consequence of the convergence of the corresponding quadrature formula (1) for $f \in C^{2\overline{s}}[a,b]$.

The main result of our paper can be stated in the following form:

Theorem. *Let $d\lambda(t)$ be a nonnegative distribution on the finite segment $[a,b]$, and let $\{\pi_{n,\sigma}(t)\}$ denote the associated orthogonal set of (monic) σ-polynomials. Let*

$[a', b']$ be a subinterval of $[a, b]$ such that $\int_{a'}^{b'} d\lambda(t) > 0$. Then if n is sufficiently large, every polynomial $\pi_{n,\sigma}(t)$ has at last one zero in $[a', b']$.

Proof. Let $\varrho(t)$ be an arbitrary polynomial of degree m, which is not greater than 0 in $[a, b]$, except possibly in $[a', b']$. Assuming that the polynomial $\pi_{k,\sigma}(t)$ ($k = n, n+1, \ldots$) has no zeros $\tau_\nu^{(\sigma_k)}$ in $[a', b']$, and taking $2n - 1 \geq m$ (therefore $2k - 1 \geq m$ ($k = n, n+1, \ldots$)), we obtain

$$\int_a^b \varrho(t) \prod_{\nu=1}^{k} (t - \tau_\nu^{(\sigma_k)})^{2s_\nu}\, d\lambda(t) = \sum_{\nu=1}^{k} \lambda_\nu^{(\sigma_k)} \varrho(\tau_\nu^{(\sigma_k)}) \ (\leq 0), \tag{3}$$

where (3) is the corresponding Gauss quadrature formula subject to the new nonnegative distribution

$$d\mu(t) = d\mu^{(\sigma_k)}(t) = \prod_{\nu=1}^{k} (t - \tau_\nu^{(\sigma_k)})^{2s_\nu}\, d\lambda(t).$$

(A general class implicitly defined polynomials was introduced and studied by Engels (cf. [2, pp. 214–226]).)

It is clear that the condition $\int_{a'}^{b'} d\lambda(t) > 0$ implies $\int_{a'}^{b'} d\mu^{(\sigma_k)}(t) > 0$.

Hence, when we apply the theorem of Weierstrass, it follows that

$$\int_a^b f(t)\, d\mu^{(\sigma_k)}(t) \leq 0,$$

where $f(t)$ is continuous in $[a, b]$ and not greater than 0 in $[a, b]$, except possibly in $[a', b']$. If we define (see Szegö [9, pp. 111–112])

$$f(t) = \begin{cases} 0, & \text{in} \quad a \leq t \leq a' \quad \text{and} \quad b' \leq t \leq b, \\ (t - a')(b' - t), & \text{in} \quad a' \leq t \leq b', \end{cases}$$

we reach a contradiction.

Remark. In the special case we have that the above result holds for the zeros of s-orthogonal polynomials. By a different method it was also proved by Martinelli, Ossicini and Rosati [4].

Let $[a', b'] \subset [a, b]$, with $b' - a' < b - a$, and $\lambda(t)$ is constant on $[a', b']$ ($\lambda(a') = \lambda(b') = \text{const}$). We can prove that $\pi_{n,\sigma}(t)$ ($n \geq 2$) *has at most one zero in* $[a', b']$.

As we know, the n zeros of $\pi_{n,\sigma}(t)$ are distinct, real and all contained in the open interval (a, b) (see [5]).

Let $\pi_{n,\sigma}(t)$ has at least two zeros in $[a', b']$, and τ and T are the minimal and maximal zero of $\pi_{n,\sigma}(t)$ belong to $[a', b']$, respectively. Consider a polynomial of degree $n-2$ defined by

$$\varphi_{n-2}(t) = \frac{\pi_{n,\sigma}(t)}{(t-\tau)(t-T)},$$

for which holds

$$\int_a^b \varphi_{n-2}(t) \prod_{\nu=1}^{n} (t-\tau_\nu)^{2s_\nu+1}\, d\lambda(t) = 0. \tag{4}$$

We have

$$\varphi_{n-2}(t) \prod_{\nu=1}^{n} (t-\tau_\nu)^{2s_\nu+1} = \frac{\prod_{\nu=1}^{n} (t-\tau_\nu)^{2s_\nu+2}}{(t-\tau)(t-T)} \geq 0, \quad \text{for} \quad t \notin (\tau, T),$$

and

$$\begin{aligned} \int_a^b \varphi_{n-2}(t) \prod_{\nu=1}^{n} (t-\tau_\nu)^{2s_\nu+1}\, d\lambda(t) &= \int_a^{a'} \varphi_{n-2}(t) \prod_{\nu=1}^{n} (t-\tau_\nu)^{2s_\nu+1}\, d\lambda(t) \\ &+ \int_{b'}^b \varphi_{n-2}(t) \prod_{\nu=1}^{n} (t-\tau_\nu)^{2s_\nu+1}\, d\lambda(t) > 0, \end{aligned}$$

and this contradicts the condition (4).

SOME NUMERICAL RESULTS

In this section we consider a class of σ-orthogonal polynomials on two symmetric intervals. The theory of polynomials orthogonal on $[-1, 1]$ with respect to the measure $d\lambda(t) = w(t)dt$, where

$$w(x) = \begin{cases} |t+\alpha|(t^2-\xi^2)^p(1-t^2)^q, & t \in [-1, -\xi] \cup [\xi, 1], \\ 0, & \text{elsewhere}, \end{cases} \tag{5}$$

and $0 < \xi < 1$, $p, q > -1$, has been studied by Barkov [1] and Gautschi [3] (when $|t+\alpha|$ was replaced by the symmetric factor $|t|^\gamma$, $\gamma \in \mathbf{R}$).

Table 1: Zeros $\tau_\nu^{(\sigma_n)}$ for $n = 2(1)6$ and $r = 0.8$, 0.4, 0.1, and 0.01

n	$r = 0.8$	$r = 0.4$	$r = 0.1$	$r = 0.01$
	$\xi = 0.11111$	$\xi = 0.42857$	$\xi = 0.81818$	$\xi = 0.98020$
2	−0.612120313218	−0.660305047380	−0.833833863424	−0.970705737842
	0.791538623729	0.809370364068	0.917214970670	0.990186207142
3	−0.866277989768	−0.870064707368	−0.933909373897	−0.991074266969
	−0.141858314662	$\boxed{-0.261739694431}$	$\boxed{-0.673968179280}$	$\boxed{-0.935042701919}$
	0.782829639687	0.797042350068	0.915546320831	0.990167593418
4	−0.942511607904	−0.942347586802	−0.970405718164	−0.996201935187
	−0.595280051440	−0.597351468349	−0.841926044199	−0.982306992400
	$\boxed{0.047822651573}$	$\boxed{0.073329568533}$	$\boxed{0.187853844540}$	$\boxed{0.211943408769}$
	0.857082617275	0.854477211319	0.922462041206	0.990246578198
5	−0.950394843409	−0.950161150659	−0.972884655185	−0.996281089352
	−0.648039232684	−0.644004681231	−0.845156714371	−0.982358957317
	$\boxed{-0.068106927896}$	$\boxed{-0.101880137962}$	$\boxed{-0.317922361302}$	$\boxed{-0.605397923021}$
	0.735898910602	0.740189699069	0.888329106698	0.987049364689
	0.990493563273	0.990585955919	0.995327538187	0.999408079763
6	−0.966074166811	−0.972033302583	−0.988877645473	−0.998693354845
	−0.755777758344	−0.803594791068	−0.926201432719	−0.991561285102
	−0.333780971761	−0.517591055853	−0.839550753398	−0.982352377951
	0.421729771267	0.562644302197	0.850062845879	0.983403047720
	0.803498401912	0.839828653597	0.938151635137	0.992853899335
	0.965688670635	0.971557550110	0.988524020929	0.998642775028

The special case $\alpha = 0$, $p = q = -1/2$, $\xi = (1-r)/(1+r)$ $(0 < r < 1)$ of (5) arises in the study of the diatomic linear chain, where $r = m/M$ has the meaning of a mass ratio, m and M $(m < M)$ being the masses of the the two kinds of particles alternating along the chain (see [10] and [3]). In that case, the coefficients in the three-term recurrence relation for polynomials $\{\pi_n\}$, orthogonal in a usual sense $(s_1 = s_2 = \cdots = 0)$ with respect to $d\lambda(t)$,

$$\pi_{n+1}(t) = t\pi_n(t) - \beta_n\pi_{n-1}(t), \quad \pi_{-1}(t) = 0, \ \pi_0(t) = 1,$$

are known explicitly

$$\beta_0 = \pi, \quad \beta_1 = \frac{1}{2}(1+\xi^2), \quad \beta_2 = \frac{1}{4}\cdot\frac{(1-\xi^2)^2}{1+\xi^2}, \quad \beta_3 = \frac{1}{4}\cdot\frac{1+6\xi^2+\xi^4}{1+\xi^2},$$

$$\beta_{2k} = \frac{1}{16}\cdot\frac{(1-\xi^2)^2}{\beta_{2k-1}}, \quad \beta_{2k+1} = \frac{1}{2}(1+\xi^2) - \beta_{2k} \quad (k=2,3,\ldots).$$

Using the iterative procedure for determining the zeros of σ-orthogonal polynomials given in [6], we illustrate the results from the previous section. We take a σ-sequence, for example, $\sigma = \{2,1,3,4,0,2,\ldots\}$, and calculate the zeros of $\pi_{n,\sigma}(t)$ for some selected values of r, i.e., ξ, and n. The corresponding zeros $\tau_\nu^{(\sigma_n)}$ are presented in Table 1. The boxed zeros belong to the internal interval $[-\xi,\xi]$. Notice that at most one zero of the polynomial $\pi_{n,\sigma}(t)$ can be inside this "hole," which spreads when r decreases. In the other words, at least $n-1$ zeros of of this polynomial is very close to the points ± 1, when $r \to 0$.

In the case of s-orthogonal polynomials with respect to the weight (5) (for $\alpha = 0$), the corresponding zeros are symmetrically distributed around the origin, so that only polynomials of odd degree has one zero in $t = 0$. The all zeros of polynomials of even degree are outside the "hole" $[-\xi,\xi]$. For $s = \max\limits_{1\le k\le n} \sigma_k$, $n = 2(1)6$, and for the same selected values of r as before, these zeros are given in Table 2.

Table 2: Zeros of s-orthogonal polynomials for $n = 2(1)6$ and $r = 0.8$, 0.4, 0.1, and 0.01

n	s	$r = 0.8$	$r = 0.4$	$r = 0.1$	$r = 0.01$
		$\xi = 0.11111$	$\xi = 0.42857$	$\xi = 0.81818$	$\xi = 0.98020$
2	2	∓ 0.711458248604	∓ 0.769309258162	∓ 0.913625056466	∓ 0.990148513614
3	3	∓ 0.865757289350	∓ 0.860704264964	∓ 0.922589947265	∓ 0.990247509655
		0.	0.	0.	0.
4	4	∓ 0.924857488171	∓ 0.938323903257	∓ 0.975493668722	∓ 0.997124645515
		∓ 0.396212450065	∓ 0.550655719996	∓ 0.847250606306	∓ 0.983122881094
5	4	∓ 0.950979456907	∓ 0.952575621577	∓ 0.977369073062	∓ 0.997148921613
		∓ 0.586918631541	∓ 0.608465051352	∓ 0.850010598227	∓ 0.983148119846
		0.	0.	0.	0.
6	4	∓ 0.966353820074	∓ 0.972273876714	∓ 0.988865733374	∓ 0.998685788750
		∓ 0.711458248604	∓ 0.769309258162	∓ 0.913625056466	∓ 0.990148513614
		∓ 0.280189174380	∓ 0.488218166445	∓ 0.831604502730	∓ 0.981536985229

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