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AUTOMORPHISMS AND PSEUDO–AUTOMORPHISMS OF EXTRA ALGEBRAS

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Abstract. This paper is devoted to an investigation of some of the mappings of extra algebras. It is proved that the inner mappings of any extra algebra are indeed pseudo-automorphisms. Each pseudo-automorphism of an extra algebra induces an automorphism of the nucleus. The pseudo-automorphisms are also proved to be semi-automorphisms. Every element of the nucleus determines an inner automorphism of the algebra.

1. INTRODUCTION

Previous investigations carried out on non-associative algebras include those of Albert [1], Paige [6], Jacobson [5], Schater [7], Solarin [8], Solarin and Chiboka [9]. Bruck and Kleinfeld [3] have contributed to the study of division ring. In a previous investigation by Chiboka [4], it was proved that an extra algebra is an alternative division algebra containing the identity element. This property lends itself quite generously in the present paper, hence leading to the interesting results thus obtained. For the definition of a loop, the reader may consult Bruck [2].

Definition 1.1. [7] *Let* $(G, \cdot)$ *be a loop written multiplicatively, and* F *an arbitrary field. Define multiplication in the Vector space,* A*, of all formal sums of a finite number of elements in* G*, with coefficients in* F*, by the use of both distributive laws and the definition of multiplication in* G*. The resulting algebra,* $A(G)$*, over* F *is a [linear] non-asociative algebra which is associative if and only if* $G, \cdot)$ *is a group.*

Definition 1.2. *An extra algebra,* E*, over a field,* F*, is a division algebra in which*

$$(xy \cdot z) = x(y \cdot zx), \quad \text{for all} \quad x, y, z \in E. \tag{1}$$

Let x *be any element of an algebra* A *over a field* F*. The linear transformations,* R_x *and* L_x*, of* A *determined by* x *are defined by*

$$aR_x = ax \quad \text{and} \quad aL_x = xa$$

for all a *in* A*, and are called the right multiplication and left multiplication of* A*, respectively. If* A *is division algebra, then for* $x \neq 0$ *in* A, R_x *and* L_x *have inverses* R_x^{-1} *and* L_x^{-1}*, respectively.*

Definition 1.3. [6] *Let* A *be an algebra. Define the set,* $M(A)$*, to be the associative algebra generated by the right and left multiplication of* A*. It is called the multiplication algebra of* A*. The elements of* $M(A)$ *are of the form*

$$S = \sum S_1 S_2 \cdots S_n,$$

where S_i *is either a right or a left multiplication of* A*.*

Definition 1.4. *Let* A *be an algebra, and* $N(A)$ *the subalgebra of* $M(A)$ *consisting of those elements,* T *of* $M(A)$*, such that* $eT = e$*, an element of the centre of* A*,* $N(A)$ *shall be called the inner multiplication algebra of* A*.*

Definition 1.5. [1] *A linear transformation* H *on an algebra* A *is an automorphism of* A *if and only if* H *is non-singular and such that either (and hence both) of the following conditions holds*

$$R_{xH} = H^{-1} R_x H, \quad L_{xH} = H^{-1} L_x H$$

for all $x \in A$.

Definition 1.6. *Let A be an algebra. The automorphisms of A that are contained in $N(A)$, or that leave the centre of A elementwise invariant, are called the inner automorphisms of A.*

Definition 1.7. *A linear non-singular transformation P of an algebra A will be called a pseudo-automorphism of A if there exists at least one element c of A such that*

$$(ab)P \cdot c = aP \cdot (bP \cdot c)$$

for all $a, b \in A$. The element c will be called a companion of P.

Definition 1.8. *Let A be an algebra such that $ab \cdot a = a \cdot ba$ for all $a, b \in A$. A linear non-singular transformation S of A is a semi-automorhism if*

$$(aba)S = aS \cdot bS \cdot aS$$

for all $a, b \in A$.

The left nucleus N_λ, middle nucleus N_μ, right nucleus N_e, nucleus N, centrum C, and centre Z of an algebra A are defined as follows:

$$\begin{aligned}
N_\lambda &= \{x \in A|\ x \cdot yz = xy \cdot z \quad \text{for all} \quad y, z \in A\}\\
N_\mu &= \{y \in A|\ x \cdot yz = xy \cdot z \quad \text{for all} \quad x, z \in A\}\\
N_e &= \{z \in A|\ x \cdot yz = xy \cdot z \quad \text{for all} \quad x, y \in A\}\\
N &= N_\lambda N_\mu N_e\\
C &= \{x \in A|\ xy = yx \quad \text{for all} \quad y \in A\}\\
Z &= NC.
\end{aligned}$$

In an alternative algebra, A, the associator (x, y, z) is an alternative function, and so $N_\lambda = N_\mu = N_e$ in such an algebra.

Lemma 1.1. [4] *Let E be an extra algebra. Then* *i)* $xy \cdot xz = x(yx \cdot z)$,

ii) $yx \cdot zx = (y \cdot xz)x$,

iii) x^2 *is in the nucleus of* E,

for all $x, y, z \in E$.

The Moufang identities

$$xy \cdot zx = (x \cdot yz)x \tag{2}$$

$$(xy \cdot z)y = x(y \cdot zy) \tag{3}$$

$$x(y \cdot xz) = (xy \cdot x)z \tag{4}$$

for all x, y, z, hold in any alternative division algebra, hence they hold in E.

2. MAIN RESULTS

Theorem 2.1. *Let* A *be an algebra and* N *the nucleus of* A. *Then* N *is an associative sub-algebra of* A.

Proof. Since the associator (x, y, z) is a multilinear function, it suffices to prove that, for $u, v \in N$, uv is also in N, i. e.,

$$(x, y, uv) = (x, uv, y) = (uv, x, y)$$

for all $x, y \in A$.

For any $u, v \in N$, we have

$$\begin{aligned}
(x, y, uv) &= xy \cdot uv - x(y \cdot uv) = (xy \cdot u)v - x(yu \cdot v) = (x \cdot yu)v - (x \cdot yu)v = 0,\\
(x \cdot uv, y) &= (x, uv)y - x(uv \cdot y) = (xu \cdot v)y - x(u \cdot vy)\\
&= xy \cdot vy - xu \cdot vy = 0,\\
(uv, x, y) &= (uv \cdot x)y - uv \cdot xy = (u \cdot vx)y - (v \cdot xy) = u(vx \cdot y) - u(vx \cdot y) = 0
\end{aligned}$$

for all $x, y \in A$.

Theorem 2.2. *Let N be the nucleus of an extra algebra E, then the linear transformation $R_{y^{-1}}^{-1}L_y$ is an inner automorphism of E for all $y \in N$.*

Proof. If yEN, then

$$ya \cdot (y \cdot zy^{-1}) = y \cdot a(y \cdot zy^{-1})$$

for all $a, z \in E$. Let $H = R_{y^{-1}}L_y$, then

$$H^{-1} = L_y^{-1}R_{y^{-1}}^{-1} = L_y^{-1}R_y$$

for $y \in N$.

Now,

$$ya \cdot (y \cdot zy^{-1}) = aL_yR_{y \cdot zy^{-1}} = aL_yR_{ZR_{y^{-1}}L_y}$$

and

$$y \cdot a(y \cdot zy^{-1}) = aR_{y \cdot zy^{-1}}L_y = aR_{zR_{y^{-1}}}L_y.$$

So,

$$aL_yR_{zR_{y^{-1}}L_y} = aR_{zR_{y^{-1}}}L_y$$

gives

$$aL_yR_{zH} = aR_{zH}L_y$$

for all $a, z \in E$, hence

$$L_yR_{zH} = R_{zH}L_y$$

for all $z \in E$, which gives

$$R_{zH} = L_y^{-1}R_{zH}L_y = L_y^{-1}R_{y^{-1}}^{-1}R_{y^{-1}}R_{zH}R_{y^{-1}}^{-1}R_{y^{-1}}L_y = H^{-1}R_{y^{-1}}R_{zH}R_{y^{-1}}^{-1}H.$$

But y and y^{-1} are in N, so, we obtain

$$\begin{aligned} R_{y^{-1}}R_{zH}R_{y^{-1}}^{-1} &= R_{(y^{-1}.zH)\cdot(y^{-1})^{-1}} = R_{y^{-1}(zH\cdot(y^{-1})^{-1})} = R_{y^{-1}(zH\cdot y)} \\ &= R_{y^{-1}(zR_{y^{-1}}L_y\cdot y)} = R_{y^{-1}[(y\cdot zy^{-1})y]} = R_{y^{-1}[y\cdot(zy^{-1}\cdot y)]} \\ &= R_{y^{-1}\cdot yz} = R_z. \end{aligned}$$

So, $R_{zH} = H^{-1}R_zH$ for all $z \in E$.

Similarly, we can prove that

$$L_{zH} = H^{-1}L_zH^{-1}$$

for all $z \in E$.

Consequently, the linear transformation H is an automorphism of E.

If a is in the centre of E, then

$$aR_{y^{-1}}L_y = y \cdot ay^{-1} = y \cdot ay^{-1} = y \cdot y^{-1}a = a.$$

So, $R_{y^{-1}}L_y$ leaves every element of the centre invariant, and, hence, $R_{y^{-1}}L_y$ is an inner automorphism of E.

Remark. Every element of the nucleus of E induces an inner automorphism of E.

Theorem 2.3. *Let P be a pseudo-automorphism of an extra algebra, with companion c. Then*

(i) $eP = e$, where e is the identity element of E;

(ii) $x^{-1}P = xP^{-1}$, for all $x \in E$;

(iii) P is a semi-automorphism of E.

Proof. Since P is a pseudo-automorphism of E, with companion c, we have

$$(xy)P \cdot c = xP \cdot (yP \cdot c)$$

for all $x, y \in E$.

(i) Taking $x = y = e$, we obtain

$$e(eP \cdot c) = eP \cdot (eP \cdot c).$$

Hence $e = eP$.

$$R_{y^{-1}}R_{zH}R_{y^{-1}}^{-1} = R_{(y^{-1} \cdot zH)\cdot(y^{-1})^{-1}} = R_{y^{-1}(zH \cdot y)}$$

(ii) $y = x^{-1}$ gives

$$(xx^{-1})P \cdot c = xP \cdot (x^{-1}P \cdot c)$$

that is,

$$eP \cdot c = e \cdot c = c = xP \cdot (x^{-1}P \cdot c).$$

By (4)

$$x(x^{-1} \cdot xy) = (xx^{-1})(xy) = xy,$$

hence

$$x^{-1} \cdot xy = y \qquad \text{for all } x, y \in E.$$

So,

$$(xP)^{-1} \cdot c = (xP)^{-1}(xP \cdot (x^{-1}P \cdot c)) = x^{-1}P \cdot c.$$

Consequently,

$$x^{-1}P = (xP)^{-1} \qquad \text{for all } x \in E.$$

(iii)

$$(xy \cdot x)P \cdot c = (x \cdot yx)P \cdot c = xP \cdot (yx)P \cdot c = xP \cdot (yP \cdot (xP \cdot c)).$$

Since the Moufang identity (4) holds in E, we have

$$(xy \cdot x)P \cdot c = (xP \cdot yP \cdot xP) \cdot c$$

for all $x, y \in E$.

Hence,

$$(xy \cdot x)P = (x \cdot yx)P = (xP \cdot yP) \cdot xP = xP \cdot (yP \cdot xP).$$

Thus,

$$(xyx)P = xP \cdot yP \cdot xP$$

for all $x, y \in E$, and P is a semi-automorphism of E.

Theorem 2.4. *If P is a pseudo-automorphism of an extra algebra, with companion c, then*

(i) P^{-1} is also a pseudo-automorphism of E and $c^{-1}P^{-1}$ is a companion of P^{-1},

(ii) x^2P is in the nucleus of E.

Proof. Let P be a pseudo-automorphism of E, with companion c. Since P is non-singular, P^{-1} exists.

Now,

$$((xy)P^{-1} \cdot c^{-1}P^{-1})P \cdot c = (xy)P^{-1}P \cdot ((c^{-1}P^{-1})P \cdot c) = xy \cdot c^{-1}c = xy \cdot e = xy$$

for all $x, y \in E$.

On the other hand,

$$\begin{aligned}(xP^{-1} \cdot (yP^{-1} \cdot c^{-1}P^{-1}))P \cdot c &= xP^{-1}P \cdot ((yP^{-1} \cdot c^{-1}P^{-1})P \cdot c) \\ &= x \cdot ((yP^{-1})P \cdot ((c^{-1}P^{-1}P) \cdot c)) = xy\end{aligned}$$

for all $x, y \in E$. Consequently,

$$((xy)P^{-1} \cdot c^{-1}P^{-1})P \cdot c = (xP^{-1} \cdot (yP^{-1} \cdot c^{-1}P^{-1}))P \cdot c$$

gives

$$(xy)P^{-1} \cdot c^{-1}P^{-1} = xP^{-1} \cdot (yP^{-1} \cdot c^{-1}P^{-1})$$

for all $x, y \in E$. Hence, P^{-1} is also a pseudo-automorphism of E and $c^{-1}P^{-1}$ is a companion of P^{-1}.

(ii)

$$(xy)P \cdot c = xP \cdot (y \cdot c)$$

for all $x, y \in E$.

In particular, $y = x$ implies

$$x^2P \cdot c = xP \cdot (xP \cdot c) = (xP)^2 \cdot c.$$

So,

$$x^2P = (xP)^2.$$

Since x^2 is in N for all $x \in E$, we have xP in E implies $(xP)^2$ in N, hence x^2P is in N for all $x \in E$.

Theorem 2.5. *Let P be a pseudo-automorphism of an extra algebra E, with companion c.*

(i) If x is in the nucleus, N, of E, then xP is also in N;

(ii) P is an automorphism of N.

Proof. Using Theorem 2.4 (i), we have

$$(xP\cdot y)P^{-1}\cdot c^{-1}P^{-1} = xPP^{-1}\cdot(yP\cdot c^{-1}P^{-1}) = x\cdot(yP^{-1}\cdot c^{-1}P^{-1}) = (x\cdot yP^{-1})\cdot c^{-1}P^{-1}$$

since $x \in N$. So,

$$(xP \cdot y)P^{-1} = x \cdot yP^{-1}$$

for all $y \in E$.

For any $z \in E$, we have

$$\begin{aligned} ((xP \cdot y) \cdot z)P^{-1} \quad \cdot \quad & c^{-1}P^{-1} = (x \cdot y)P^{-1}(zP^{-1} \cdot c^{-1}P^{-1}) \\ = \quad & (x \cdot yP^{-1}) \cdot (zP^{-1} \cdot c^{-1}P^{-1}) = x \cdot (yP^{-1} \cdot (zP^{-1} \cdot c^{-1}P^{-1})) \\ = \quad & x \cdot ((yz)P^{-1}c^{-1}P^{-1}) = (x \cdot (yz)P^{-1}) \cdot c^{-1}P^{-1} \end{aligned}$$

since $x \in N$. Hence,

$$((xP \cdot y) \cdot z)P^{-1} = x \cdot (yz)P^{-1} = (xP \cdot yz)P^{-1}$$

gives

$$(xP \cdot y)z = xP \cdot yz$$

for all $y, z \in E$. Thus $xP \in N$ for all $x \in N$.

(ii) Let x be in N, then xP is also in N. For any $a \in E$, we have $aR_xP = (ax)P$. So,

$$aR_xP \cdot c = (ax)P \cdot c = aP \cdot (xP \cdot c) = (aP \cdot xP) \cdot c$$

since c is a companion of P, and $xP \in N$.

Hence,

$$aR_xP = aP \cdot xP$$

for all $a \in E$ and all $x \in N$.

On the other hand,

$$aPR_xP_{xp} = aP \cdot xP$$

for all $a \in E$ and all $x \in N$. So $aR_xP = aPR_{xp}$ for all $a \in E$ and all $x \in N$.

Hence,

$$R_xP = PR_{xp} \quad \text{and} \quad P^{-1}R_xP = R_{xp}.$$

Similarly,

$$PL_{xp} = xP \cdot aP \quad \text{and} \quad aL_xP = (xa)P$$

for all $a \in E$ and all $x \in N$.

Now,

$$aL_xP \cdot c = (xa)P \cdot c = xP \cdot (aP \cdot c) = (xP \cdot aP) \cdot c.$$

So,

$$aL_xP = xP \cdot aP = aPL_{xp}$$

for all $a \in E$ and all $x \in N$.

Hence,

$$L_xP = PL_{xp} \quad \text{and} \quad L_{xp} = P^{-1}L_xP$$

for all $x \in N$. Consequently, P is an automorphism of N.

Theorem 2.6. *Every inner mapping of an extra algebra E, is a pseudo-automorphism of E.*

Proof. Since the Moufang identity

$$xy \cdot zx = x(yz \cdot x) \tag{5}$$

is satisfied for all $x, y, z \in E$, we obtain

$$yL_x \cdot zR_x = (yz)R_xL_x \tag{6}$$

for all $x, y, z \in E$.

Using (5) and replacing z with yz, we get

$$x^{-1}y^{-1}(yz \cdot x^{-1}) = x^{-1}((y^{-1} \cdot yz) \cdot x^{-1})) = x^{-1} \cdot zx^{-1}.$$

So,

$$yz \cdot x^{-1} = (yx)(x^{-1} \cdot zx^{-1}),$$

which gives

$$(yz)R_{x^{-1}} = yR_x \cdot zR_{x^{-1}}L_{x^{-1}} \tag{7}$$

for all $x, y, y \in E$

Let H an inner mapping of E, then H is of the form

$$H = H_1 H_2 \cdots H_n,$$

where H_k is either a right or a left multiplication of E. It follows from (6) and (7) that for each $x \in E$, there exist S_k and T_k in $M(E)$ such that

$$(ab)T_k = aH_k \cdot bS_k \tag{8}$$

for all $a, b \in E$. So if

$$S = S_1 S_2 \cdots S_n, \quad \text{and} \quad T = T_1 T_2 \cdots T_n,$$

then

$$(ab) \cdot T = aH \cdot bS \tag{9}$$

for all $a, b \in E$.

Since H is an inner mapping, we have $eH = e$. In (9), $a = e$ gives $bT = bS$ for all $b \in E$, so $T = S$; and $b = e$ gives $aT = aH \cdot eS$ for all $a \in E$.

So,

$$\begin{aligned} &(ab)T = (ab)H \cdot eS, \quad \text{and} \quad (ab)T = aH \cdot bs = aH \cdot bT \\ &\text{imply} \quad (ab)H \cdot eS = aH \cdot bT = aH \cdot (bH \cdot eS) \end{aligned}$$

for all $a, b \in E$. Hence, we have

$$(ab)H \cdot eS = aH \cdot (bH \cdot eS).$$

Consequently, H is a pseudo-automorphism with companion eS.

Lemma 2.1. *Let E be an extra algebra, and N the nucleus of E. The inner mappings*

$$R_{(x,y)} = R_x R_y R_{(xy)^{-1}} \quad \text{and} \quad L_{(x,y)} L_x L_y L_{(yx)^{-1}}$$

leave the nucleus of E elementwise invariant.

Proof. If $a \in N$, then

$$aR_xR_yR_{(xy)^{-1}} = (ax \cdot y)(xy)^{-1} = (a \cdot xy)(xy)^{-1} = a;$$

and

$$aL_xL_yL_{(yx)^{-1}} = (yx)^{-1}(y \cdot xa) = (yx)^{-1}(yx \cdot a)^{-1} = a.$$

Theorem 2.7. *Let E be an extra algebra and N the nucleus of E. If x or y lies in N, then the mappings $R_{(x,y)}$ and $L_{(x,y)}$ are identity automorphism of E.*

Proof. If either x or y is in N, then

$$aR_xR_y = (ax)y = a(xy) = R_{xy}$$

for all $a \in E$, hence

$$aR_{(x,y)} = aR_xR_yR_{(x \cdot y)^{-1}} = aR_{xy}R_{(xy)^{-1}} = (a \cdot xy)(xy)^{-1} = a.$$

Theorem 2.8. *Let E be an extra algebra, and N the nucleus of E. Then, the inner mapping $R_{x^{-1}}L_x$ is an automorphism of N for all $x, y \in E$ if the centre of N is equal to the centre of E.*

Proof. For any $a, b \in N$, we have

$$(ab)R_{x^{-1}}L_x = x(ab \cdot x^{-1})$$

and

$$aR_{x^{-1}}L_x \cdot bR_xL_x = (x \cdot ax^{-1}) \cdot (x \cdot bx^{-1}) = x[(ax^{-1} \cdot x) \cdot bx^{-1}] = x(a \cdot bx^{-1}) = x(ab \cdot x^{-1}).$$

So, $R_{x^{-1}}L_x$ is an automorphism of N. If a is in the centre of E then $aR_{x^{-1}}L_x = xax^{-1} = x \cdot x^{-1}a = a$.

Hence, $R_{x^{-1}}L_x$ is inner if the centre of N coincides with the centre of E.

Corollary 2.1. *Let E be an extra algebra, and N the nucleus of E. Then every inner mapping P of E is an inner automorphism of N.*

Proof. The result follows from Theorems 2.5 and 2.6.

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