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SOME CHARACTERIZATIONS OF THE SPACELIKE, THE TIMELIKE AND THE NULL CURVES ON THE PSEUDOHYPERBOLIC SPACE $\mathbf{H}_0^2$ IN $\mathbf{E}_1^3$

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ABSTRACT. In [4], [5] and [6] the authors have characterized the Lorentzian spherical curves in the Minkowski 3–space E_1^3. In this paper, we shall characterize the curves whose image lies on the pseudohyperbolic space H_0^2 in the same space.

1. Introduction

In the Euclidean 3–space E^3 a spherical unit speed curves and their characterizations are given in [2]. In [4], [5] and [6] the authors have characterized the Lorentzian spherical curves in the Minkowski 3–space E_1^3. In this paper, we shall characterize the spacelike, the timelike and the null (the lightlike) curves whose image lies on the pseudohyperbolic space H_0^2 in the same space.

Recall that the Minkowski 3–space E_1^3 is the Euclidean 3–space E^3 provided with the Lorentzian inner product

$$g(a,b) = -a_1b_1 + a_2b_2 + a_3b_3,$$

where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$.

An arbitrary vector $a = (a_1, a_2, a_3)$ in E_1^3 can have one of three Lorentzian causal characters: it can be *spacelike* if $g(a,a) > 0$ or $a = 0$, *timelike* if $g(a,a) < 0$ and *null (lightlike)* if $g(a,a) = 0$ and $a \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 where s is a pseudo–acrlength parameter, can localy be *spacelike*, *timelike* or *null (lightlike)*, if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null, for every $s \in I \subset R$.

The pseudohyperbolic space H_0^2 of radius $r > 0$ and with center in the origin in the space E_1^3 is defined by

$$H_0^2 = \{p = (p_1, p_2, p_3) \in E_1^3 \mid g(p,p) = -r^2\}$$

and the pseudo–norm of an arbitrary vector $a \in E_1^3$ is given by

$$|| \, a \, || = \sqrt{| \, g(a,a) \, |}.$$

The vectors $v, w \in E_1^3$ are orthogonal if and only if

$$g(v,w) = 0.$$

Denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve $\alpha(s)$. Then T, N and B are the tangent, the principal normal and the binormal vector of the curve $\alpha(s)$ respectively. Depending on the causal character of the curve α, we have the following Frenet formulae:

$$\begin{cases} T' = kN, \quad N' = -kT + \tau B, \quad B' = \tau N \\ g(T,T) = g(N,N) = 1, \quad g(B,B) = -1, \quad g(T,N) = g(T,B) = g(N,B) = 0 \end{cases}$$

if α is a spacelike curve with a spacelike principal normal N,

$$\begin{cases} T' = kN, \quad N' = kT + \tau B, \quad B' = \tau N \\ g(T,T) = g(B,B) = 1, \quad g(N,N) = -1, \quad g(T,N) = g(T,B) = g(N,B) = 0 \end{cases}$$

if α is a spacelike curve with a timelike principal normal N and

$$\begin{cases} T' = kN, \quad N' = \tau N, \quad B' = -kT - \tau B \\ g(T,T) = 1, \quad g(N,N) = g(B,B) = 0, \quad g(T,N) = g(T,B) = 0, \quad g(N,B) = 1 \end{cases}$$

if α is a spacelike curve with a null principal normal N. In this last case, k can take only two values: $k = 0$ when α is a straight line or $k = 1$ in all other cases.

Further,

$$\begin{cases} T' = kN, \quad N' = kT + \tau B, \quad B' = -\tau N \\ g(T,T) = -1, \quad g(N,N) = g(B,B) = 1, \quad g(T,N) = g(T,B) = g(N,B) = 0 \end{cases}$$

if α is a timelike curve and finally,

$$\begin{cases} T' = kN, \quad N' = \tau T - kB, \quad B' = -\tau N \\ g(T,T) = g(B,B) = 0, \quad g(N,N) = 1, \quad g(T,N) = g(N,B) = 0, \quad g(T,B) = 1 \end{cases}$$

if α is a null curve. In this last case, k can take only two values: $k = 0$ when α is a straight null line or $k = 1$ in all other cases. The functions $k = k(s)$ and $\tau = \tau(s)$ are called the curvature and the torsion of α respectively ([1]).

2. Spacelike curves on pseudohyperbolic space H_0^2 in E_1^3

For a spacelike curves whose image lies on H_0^2 in E_1^3 we have the following results. First we shall characterize spacelike curves with a spacelike principal normal.

Theorem 2.1. *Let $\alpha(s)$ be a unit speed spacelike curve with a spacelike principal normal N, whose image lies on a pseudohyperbolic space H_0^2 of radius*

$r \in R^+$ and with center m in E_1^3. Then $\kappa \neq 0$ for every $s \in I \subset R$. If $\tau \neq 0$ for every $s \in I \subset R$, then

$$\left(\frac{1}{\kappa}\right)^2 - \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)^2 = -r^2.$$

Proof. By assumption we have

$$g(\alpha - m, \alpha - m) = -r^2,$$

for every $s \in I \subset R$. Differentiation in s gives

$$g(T, \alpha - m) = 0. \tag{1}$$

By a new differentiation we find that

$$\begin{aligned} &g(T', \alpha - m) + g(T, T) = 0, \\ &\kappa g(N, \alpha - m) = -1. \end{aligned}$$

Thus $\kappa \neq 0$ for every $s \in I \subset R$ and

$$g(N, \alpha - m) = -\frac{1}{\kappa}. \tag{2}$$

Next, assume that $\tau \neq 0$. Then one more differentiation in s gives

$$g(N', \alpha - m) + g(N, T) = -\left(\frac{1}{\kappa}\right)',$$

which using (1) implies that

$$g(B, \alpha - m) = -\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'. \tag{3}$$

Denote the vector $\alpha - m$ by

$$\alpha - m = aT + bN + cB,$$

where $a = a(s)$, $b = b(s)$ and $c = c(s)$ are arbitrary functions. Then the relations (1), (2) and (3) imply that

$$g(T, \alpha - m) = a = 0, \quad g(N, \alpha - m) = b = -\frac{1}{\kappa}, \quad g(B, \alpha - m) = -c = -\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'.$$

Therefore,

$$\alpha - m = -\frac{1}{\kappa}N + \frac{1}{\tau}\left(\frac{1}{\kappa}\right)' B,$$

and hence

$$\left(\frac{1}{\kappa}\right)^2 - \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)^2 = -r^2. \quad \blacksquare$$

Theorem 2.2. *Let $\alpha(s)$ be a unit speed spacelike curve in E_1^3 with a spacelike principal normal N, with $1/\kappa \neq 0$ and $1/\tau \neq 0$ for each s. If*

$$\left(\frac{1}{\kappa}\right)^2 - \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)^2 = -r^2,$$

$r \in R^+$, then image of α lies on a pseudohyperbolic space of radius r in E_1^3.

Proof. Consider the vector

$$m = \alpha + \frac{1}{\kappa}N - \frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'B.$$

We shall prove that $m =$ constant. By differentiation in s we have that

$$\begin{aligned} m' &= \alpha' + \Big(\frac{1}{\kappa}\Big)'N + \frac{1}{\kappa}N' - \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)'B - \frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'B' \\ &= T + \Big(\frac{1}{\kappa}\Big)'N + \frac{1}{\kappa}(-\kappa T + \tau B) - \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)'B - \frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\tau N \\ &= \frac{\tau}{\kappa}B - \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)'B \\ &= \Big(\frac{\tau}{\kappa} - \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)'\Big)B \end{aligned} \tag{1}$$

By differentiation in s of the assumption

$$\Big(\frac{1}{\kappa}\Big)^2 - \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2 = -r^2,$$

we have

$$2\Big(\frac{1}{\kappa}\Big)\Big(\frac{1}{\kappa}\Big)' - 2\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)' = 0,$$

and thus

$$\frac{\tau}{\kappa} - \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)' = 0.$$

Substituting the last relation in the relation (1), we find that $m' = 0$ for each s and therefore $m =$ constant. Since

$$\alpha - m = -\frac{1}{\kappa}N + \frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'B,$$

we have that

$$g(\alpha - m, \alpha - m) = \Big(\frac{1}{\kappa}\Big)^2 - \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2 = -r^2, \quad r \in R^+.$$

Consequently, α lies on the pseudohyperbolic space of radius r and center m in the space E_1^3. ∎

Theorem 2.3. *If $\alpha(s)$ is a unit speed spacelike curve in E_1^3 with a spacelike principal normal N which satisfies $1/\kappa \neq 0$ and $1/\tau \neq 0$ for each $s \in I \subset R$, then $\alpha(s)$ lies on a pseudohyperbolic space if and only if $\frac{\tau}{\kappa} = \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)'$ and $\Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2 > \Big(\frac{1}{\kappa}\Big)^2$.*

Proof. Firstly assume that α is a curve satisfying the mentioned conditions and which lies on a pseudohyperbolic space of radius $r \in R^+$ and center $m = (m_1, m_2, m_3) \in E_1^3$. Then

$$g(\alpha - m, \alpha - m) = -r^2,$$

for each $s \in I \subset R$. Further, by theorem 2.1 we get

$$\alpha - m = -\frac{1}{\kappa}N + \frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)' B,$$

and thus

$$g(\alpha - m, \alpha - m) = \Big(\frac{1}{\kappa}\Big)^2 - \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2.$$

It follows that

$$\Big(\frac{1}{\kappa}\Big)^2 - \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2 = -r^2, \tag{1}$$

so that

$$\Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2 > \Big(\frac{1}{\kappa}\Big)^2.$$

Differentiation of the relation (1) then gives

$$2\frac{1}{\kappa}\Big(\frac{1}{\kappa}\Big)' - 2\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)' = 0,$$

and consequently

$$\frac{\tau}{\kappa} = \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)'.$$

Conversely, assume that $\alpha(s)$ is a unit speed spacelike curve with the spacelike principal normal N, which satisfies $1/\kappa \neq 0$ and $1/\tau \neq 0$ for each $s \in I \subset R$. Further assume that

$$\Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2 > \Big(\frac{1}{\kappa}\Big)^2$$

and that

$$\frac{\tau}{\kappa} = \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)'$$

holds. Then the last equation can be easily transformed into

$$2\frac{1}{\kappa}\Big(\frac{1}{\kappa}\Big)' - 2\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)' = 0.$$

But the last expression is the differential of the equation

$$\Big(\frac{1}{\kappa}\Big)^2 - \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2 = c = \text{constant} < 0,$$

so that we may take $c = -r^2$, $r \in R^+$. Then the theorem 2.2 implies that image of α lies on a pseudohyperbolic space. ■

Theorem 2.4. *A unit speed spacelike curve $\alpha(s)$ with a spacelike principal normal N lies on a pseudohyperbolic space H_0^2 in E_1^3 if and only if $\kappa(s) > 0$ for every s and there is a differentiable function $f(s)$ such that $f\tau = (1/\kappa)'$, $f' = \tau/\kappa$ and $1/\kappa < |f|$.*

Proof. Firstly assume that α satisfies the mentioned conditions and lies on a pseudohyperbolic space of radius r and center m in E_1^3. Then theorem 2.1 implies that $\kappa(s) \neq 0$ for each s and

$$\Big(\frac{1}{\kappa}\Big)^2 - \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2 = -r^2.$$

Hence $k(s) = ||T'(s)|| > 0$. Further by theorem 2.3 we have

$$\frac{\tau}{\kappa} = \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)'.$$

Next, define the differentiable function $f(s)$ by

$$f = \frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'.$$

Then we have $f' = \tau/\kappa$ and

$$f^2 = \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2 > \Big(\frac{1}{\kappa}\Big)^2.$$

Therefore $|f| > |1/\kappa| = 1/\kappa$.

Conversely, assume that $\alpha(s)$ is a unit speed spacelike curve with a spacelike principal normal N which satisfies $\kappa(s) > 0$ for every s. Then $1/\kappa \neq 0$ for every s. Next, assume that there is a differentiable function $f(s)$ such that $f\tau = (1/\kappa)'$, $f' = \tau/\kappa$ and $|f| > 1/\kappa$. Since f is differentiable, it is also continuous, so that $\tau \neq 0$ for every s. Then the assumption implies that

$$\Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2 > \Big(\frac{1}{\kappa}\Big)^2$$

and

$$\Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)' = \frac{\tau}{\kappa}.$$

Therefore the theorem 2.3 implies that α lies on a pseudohyperbolic space in the space E_1^3. ■

Theorem 2.5. *A unit speed spacelike curve $\alpha(s)$ in E_1^3 with a spacelike principal normal N lies on a pseudohyperbolic space H_0^2 if and only if there are constants $A, B \in R^+$ such that*

$$\kappa\Big(A\sinh\Big(\int_0^s \tau(s)\,ds\Big) - B\cosh\Big(\int_0^s \tau(s)\,ds\Big)\Big) = 1.$$

Proof. Firstly assume that $\alpha(s)$ is a unit speed spacelike curve with a spacelike principal normal N which lies on the pseudohyperbolic space H_0^2. Then by the theorem 2.4 $\kappa(s) > 0$ and there is a differentiable function $f(s)$ such that $f\tau = (1/\kappa)', f' = \tau/\kappa$ and $|f| > 1/\kappa$. Next define the C^2 function $\theta(s)$ and the C^1 functions $g(s)$ and $h(s)$ by

$$\theta(s) = \int_0^s \tau(s)\,ds,$$

$$g(s) = -\frac{1}{\kappa}\sinh\theta + f(s)\cosh\theta, \quad h(s) = -\frac{1}{\kappa}\cosh\theta + f(s)\sinh\theta.$$

Then differentiation in s of the functions θ, g and h easily gives

$$\theta'(s) = \tau(s), \quad g'(s) = h'(s) = 0,$$

and therefore

$$g(s) = A = \text{const}, \quad h(s) = B = \text{const}, \qquad (A, B \in R^{+}).$$

Hence we get

$$-\frac{1}{\kappa}\sinh\theta + f(s)\cosh\theta = A, \qquad -\frac{1}{\kappa}\cosh\theta + f(s)\sinh\theta = B.$$

Multiplying the first equation with $\sinh\theta$ and the second with $-\cosh\theta$ and by adding, we find

$$-\frac{1}{\kappa}(\sinh^2\theta - \cosh^2\theta) = A\sinh\theta - B\cosh\theta.$$

Therefore

$$\frac{1}{\kappa} = A\sinh\theta - B\cosh\theta,$$

that is

$$\kappa\Big(A\sinh\big(\int_0^s \tau(s)\,ds\big) - B\cosh\big(\int_0^s \tau(s)\,ds\big)\Big) = 1.$$

Conversely, assume that $\alpha(s)$ is a unit speed spacelike curve with the spacelike principal normal N and that exist constants $A, B \in R^{+}$ such that

$$\kappa\Big(A\sinh\big(\int_0^s \tau(s)\,ds\big) - B\cosh\big(\int_0^s \tau(s)\,ds\big)\Big) = 1, \tag{1}$$

holds for every $s \in I \subset R$. Then obviously $\kappa(s) \neq 0$ for every s and hence $\kappa(s) > 0$. Differentiation in s of the relation (1) gives

$$\Big(\frac{1}{\kappa}\Big)' = \tau\Big(A\cosh\big(\int_0^s \tau(s)\,ds\big) - B\sinh\big(\int_0^s \tau(s)\,ds\big)\Big). \tag{2}$$

Next define the differentiable function $f(s)$ by

$$f(s) = A\cosh\big(\int_0^s \tau(s)\,ds\big) - B\sinh\big(\int_0^s \tau(s)\,ds\big). \tag{3}$$

Then the relations (2) and (3) give $(1/\kappa)' = \tau f$ and using (1) we find that

$$f' = \tau\Big(A\sinh\big(\int_0^s \tau(s)\,ds\big) - B\cosh\big(\int_0^s \tau(s)\,ds\big)\Big) = \frac{\tau}{\kappa}.$$

Since

$$f(s) = A\cosh\theta - B\sinh\theta > A\sinh\theta - B\cosh\theta = \frac{1}{\kappa} > 0,$$

it follows that $|f| > 1/\kappa$. Consequently, the theorem 2.4 implies that α lies on a pseudohyperbolic space in E_1^3. ■

In particular, if the principal normal N is a timelike, we obtain analogous results, which are contained in the theorems 2.6, 2.7, 2.8, 2.9 and 2.10.

Theorem 2.6. *Let $\alpha(s)$ be a unit speed spacelike curve in E_1^3 with a timelike principal normal N, whose image lies on a pseudohyperbolic space H_0^2 of radius $r \in R^+$ and with center m. Then $\kappa \neq 0$ for every $s \in I \subset R$. If $\tau \neq 0$, then*

$$-\Big(\frac{1}{\kappa}\Big)^2 + \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2 = -r^2.$$

Theorem 2.7. *Let $\alpha(s)$ be a unit speed spacelike curve in E_1^3 with a timelike principal normal N, with $1/\kappa \neq 0$ and $1/\tau \neq 0$ for each s. If*

$$-\Big(\frac{1}{\kappa}\Big)^2 + \Big(\frac{1}{\tau}\Big(\frac{1}{\kappa}\Big)'\Big)^2 = -r^2,$$

$r \in R^+$, then image of α lies on a pseudohyperbolic space of radius r in E_1^3.

Theorem 2.8. *If $\alpha(s)$ is a unit speed spacelike curve in E_1^3 with a timelike principal normal N which satisfies $1/\kappa \neq 0$ and $1/\tau \neq 0$ for each $s \in I \subset R$, then $\alpha(s)$ lies on a pseudohyperbolic space if and only if $\dfrac{\tau}{\kappa} = \Big(\dfrac{1}{\tau}\Big(\dfrac{1}{\kappa}\Big)'\Big)'$ and $\Big(\dfrac{1}{\kappa}\Big)^2 > \Big(\dfrac{1}{\tau}\Big(\dfrac{1}{\kappa}\Big)'\Big)^2$.*

Theorem 2.9. *A unit speed spacelike curve $\alpha(s)$ with a timelike principal normal N lies on a pseudohyperbolic space H_0^2 in E_1^3 if and only if $\kappa(s) > 0$ for every s and there is a differentiable function $f(s)$ such that $f\tau = (1/\kappa)'$, $f' = \tau/\kappa$ and $|f| < 1/\kappa$.*

Theorem 2.10. *A unit speed spacelike curve α in E_1^3 with a timelike principal normal N lies on a pseudohyperbolic space H_0^2 if and only if there are constants $A, B \in R^+$, $B > A$, such that*

$$\kappa\Big(- A\sinh\big(\int_0^s \tau(s)ds\big) + B\cosh\big(\int_0^s \tau(s)ds\big)\Big) = 1.$$

The proof of the theorems 2.6, 2.7, 2.8, 2.9 and 2.10 is analogous to the proof of the theorems 2.1, 2.2, 2.3, 2.4 and 2.5. In the sequel, we shall characterize a spacelike curves with a null principal normal.

Theorem 2.11. *Let $\alpha(s)$ be a unit speed spacelike curve with a null principal normal N in E_1^3. Then α lies on a pseudohyperbolic space H_0^2 of radius r and center m if and only if α is a planar curve and we have*

$$\alpha - m = \frac{r^2}{2}N - B, \quad r \in R^+.$$

Proof. Firstly assume that $\alpha(s)$ is a curve which satisfies the mentioned conditions and which lies on a pseudohyperbolic space of radius r and center m. Then it follows that

$$g(\alpha - m, \alpha - m) = -r^2,$$

for every $s \in I \subset R$. By differentiation in s we find that

$$g(T, \alpha - m) = 0. \tag{1}$$

Differentiation in s of the previous equation gives

$$g(T', \alpha - m) + g(T, T) = 0,$$

$$\kappa g(N, \alpha - m) = -1.$$

In this case $\kappa = 1$ for every s, which implies that

$$g(N, \alpha - m) = -1. \tag{2}$$

Next by differentiation in s of the relation (2), we find that

$$\tau g(N, \alpha - m) = 0,$$

which together with the relation (2) gives $\tau = 0$, for every s. Consequently, $\alpha(s)$ is a planar curve.

In the sequel, denote the vector $\alpha - m$ by

$$\alpha - m = aT + bN + cB,$$

where $a = a(s)$, $b = b(s)$ and $c = c(s)$ are arbitrary functions. Then using the relations (1) and (2) we get

$$g(T, \alpha - m) = a = 0, \quad g(N, \alpha - m) = c = -1, \quad g(B, \alpha - m) = b. \tag{3}$$

By differentiation of the equation $g(B, \alpha - m) = b$, we find that

$$g(T, \alpha - m) = -b',$$

which together with the relation (1) gives $b' = 0$. Hence $b = b_0 = \text{constant} \in R$ and therefore $\alpha - m = b_0 N - B$. Since $g(\alpha - m, \alpha - m) = -2b_0 = -r^2$, we find that $b_0 = r^2/2$ and consequently

$$\alpha - m = \frac{r^2}{2} N - B.$$

Conversely, assume that $\alpha(s)$ is a planar unit speed spacelike curve with the null principal normal N, which satisfies the equation $\alpha - m = \dfrac{r^2}{2} N - B$, $r \in R^+$. Then by differentiation in s we find

$$m' = \alpha' - \frac{r^2}{2} N' + B'.$$

In this case we have $k = 1$ and $\tau = 0$ for every s, so that the Frenet formulae read

$$T' = N, \quad N' = 0, \quad B' = -T.$$

Hence it follows that $m' = 0$ and therefore $m = \text{constant}$. Next we easily find that

$$g(\alpha - m, \alpha - m) = -r^2.$$

Thus α lies on the pseudohyperbolic space of radius r and center m in E_1^3. ■

Theorem 2.12. *A unit speed spacelike curve $\alpha(s)$ in E_1^3 with a null principal normal N lies on a pseudohyperbolic space H_0^2 if and only if there are constants $A, B \in R$ such that*

$$A \sinh\Big(\int_0^s \tau(s)\,ds\Big) - B \cosh\Big(\int_0^s \tau(s)\,ds\Big) = 1.$$

Proof. Firstly assume that α lies on a pseudohyperbolic space in E_1^3. Then the theorem 1 implies that $\tau = 0$ for every $s \in I \subset R$. Next define the C^2 function $\theta(s)$ and the C^1 functions $g(s)$ and $h(s)$ by

$$\theta(s) = \int_0^s \tau(s)\,ds, \quad g(s) = -\sinh(\theta(s)), \quad h(s) = -\cosh(\theta(s)).$$

Since $\tau = 0$ for every s, it follows that

$$\theta(s) = c = \text{constant}, \quad g(s) = -\sinh(c) = A, \quad h(s) = -\cosh(c) = B.$$

Therefore, there exist constants $A, B \in R$ such that the equation

$$A\text{sinh}\,\Big(\int_0^s \tau(s)\,ds\Big) - B\text{cosh}\,\Big(\int_0^s \tau(s)\,ds\Big) = 1$$

is satisfied.

Conversely, assume that there are constants $A, B \in R$, such that the torsion $\tau = \tau(s)$ of α satisfies the equation

$$A\text{sinh}\,\Big(\int_0^s \tau(s)\,ds\Big) - B\text{cosh}\,\Big(\int_0^s \tau(s)\,ds\Big) = 1, \tag{1}$$

for every $s \in I \subset R$. By differentiation in s of the previous relation, we get that

$$\tau\Big(A\text{cosh}\,\Big(\int_0^s \tau(s)\,ds\Big) - B\text{sinh}\,\Big(\int_0^s \tau(s)\,ds\Big)\Big) = 0. \tag{2}$$

Next differentiation in s of (2) gives

$$\begin{aligned} &\tau'\Big(A\text{cosh}\Big(\int_0^s \tau(s)\,ds\Big) - B\text{sinh}\Big(\int_0^s \tau(s)\,ds\Big)\Big) + \\ &+ \tau^2\Big(A\text{sinh}\Big(\int_0^s \tau(s)\,ds\Big) - B\text{cosh}\Big(\int_0^s \tau(s)\,ds\Big)\Big) = 0. \end{aligned} \tag{3}$$

Then the relations (1) and (3) imply that

$$\tau'\Big(A\text{cosh}\Big(\int_0^s \tau(s)\,ds\Big) - B\text{sinh}\Big(\int_0^s \tau(s)\,ds\Big)\Big) + \tau^2 = 0.$$

Multiplying the last equation with τ and using the relation (2), we get that $\tau^3 = 0$, which implies $\tau = 0$ for every s.

Next, consider the vector

$$m = \alpha - \frac{r^2}{2}N + B,$$

where $r \in R^+$. Since $\tau = 0$, we easily find that $m' = 0$ and hence $m =$ constant. Finally, we obtain that

$$g(\alpha - m, \alpha - m) = -r^2,$$

and consequently $\alpha(s)$ lies on the pseudohyperbolic space of radius r and center m in E_1^3. ■

3. A timelike and a null curves on pseudohyperbolic space H_0^2

For a timelike and a null curves on H_0^2 in E_1^3 we have the following results.

Theorem 3.1. *There are no timelike unit speed curves which lie on the pseudohyperbolic space H_0^2 of radius $r \in R^+$ and center m in E_1^3.*

Proof. Assume that $\alpha(s)$ is a unit speed timelike curve which lies on the pseudohyperbolic space H_0^2 of radius r and center m in E_1^3. Then we have

$$g(\alpha - m, \alpha - m) = -r^2,$$

$r \in R^+$, for every $s \in I \subset R$. Differentiation in s of the previous equation gives

$$g(T, \alpha - m) = 0.$$

It follows that there exist two mutually orthogonal timelike vectors T and $\alpha - m$ in E_1^3, which is imposible. ■

Theorem 3.2. *There are no null curves which lying on the pseudohyperbolic space H_0^2 of radius $r \in R^+$ and center m in E_1^3.*

Proof. Assume that $\alpha(s)$ is a null curve which lies on the pseudohyperbolic space H_0^2 of radius r and center m in E_1^3. Then we have

$$g(\alpha - m, \alpha - m) = -r^2,$$

$r \in R^+$. By differentiation in s of the previous equation, we find that

$$g(T, \alpha - m) = 0.$$

It follows that a null vector T and a timelike vector $\alpha - m$ are mutually orthogonal vectors in E_1^3, which is imposible. ■

References

[1] G. S. Birman, K. Nomizu, *Trigonometry in Lorentzian geometry*, Amer. Math. Month. **91(9)**, (1984), 543–549.

[2] R. S. Millman, G. D. Parker, "Elements of Differential Geometry", Prentice–Hall Inc., Englewood Cliffs, New Jersey (1977).

[3] B. O'Neill, "Semi–Riemannian Geometry", Academic Press, New York (1983).

[4] U. Pekmen, S. Pasali, *Some characterizations of Lorentzian spherical space–like curves*, Mathematica Moravica **3** (1999), 31–37.

[5] M. Petrović–Torgašev, E. Šućurović, *Some characterizations of Lorentzian spherical spacelike curves with the timelike and the null principal normal*, to appear in Mathematica Moravica.

[6] M. Petrović–Torgašev, E. Šućurović, *Some characterizations of the Lorentzian spherical timelike and null curves*, to appear in Mathematica Moravica.

[7] I. Van de Woestyne, *Minimal surfaces of the 3–dimensional Minkowski space*, World Scientific Pub., Singapore (1990), pp.344–369.