*Kragujevac J. Math.* 22 (2000) 41–48.

# ON QUADRATIC LOOPS OF BOL-MOUFANG TYPE

**J. O. Adeniran, E. O. Adeleke, A. A. A. Agboola, J. A. Oguntuase & A. R. T. Solarin**

*Department of Mathematical Sciences, University of Agriculture*
*Abeokuta, Nigeria*

*(Received January 3, 2000)*

ABSTRACT. In this work connections between a class of quadratic differential equations and a class of local analytic Bol-Moufang type loop identities are established.

## 1. INTRODUCTION

The study of "non-associative" (local) Lie group called (local) Lie loops or analytic (local) loops had attracted the attention of authors like Malcev [4], Hofmann and Strambach [3] and Solarin [5] just to mention a few.

Gerber in [2] studied LIP loops and quadratic differential equations. It is the object of this paper to study the relationship between a class of quadratic differential equations and a class of local analytic Bol-Moufang type loop identities.

Before stating our main results, we shall need the following definitions and lemmas.

**Definition 1.1.** [2] *The function $f : \Re^n \times \Re^n \longrightarrow \Re^n$ defines a local analytic loop L(f) with the product $x \circ y$ if and only if $f$ is analytic near the origin $f(x, 0) = x$, $f(0, y) = y$ and $x \circ y = f(x, y)$.*

**Definition 1.2.** [2] *$\mathcal{L}$ is the related algebra of $L(f)$ if and only if multiplication in $\mathcal{L}$ is given by*

$$pq = -g_{11}[p : q] = -f_{xy}(0, 0)[p, q]$$

**Definition 1.3.** [2] *L(f) is a quadratic loop if and only if f is quadratic, in which case we denote the loop by* $L(\mathcal{L})$ *and we have*

$$x \circ y = x + y - xy \tag{1}$$

**Lemma 1.1.** [2] *The quadratic loop* $L(\mathcal{L})$ *is*

*(i)* *associative if and only if* $\mathcal{L}$ *is associative;*

*(ii)* *left-Bol if and only if* $\mathcal{L}$ *is left alternative;*

*(iii)* *left alternative if and only if* $\mathcal{L}$ *is left alternative;*

*(iv)* *LIP if and only if* $\mathcal{L}$ *is left alternative;*

*(v)* *Moufang if and only if* $\mathcal{L}$ *is alternative;*

*(vi)* *Power-associative if and only if* $\mathcal{L}$ *is power associative.*

For the definition of a loop, readers are to consult Bruck [1]. For all identities used Fenyves [3] is to be consulted.

## 2. MAIN RESULTS

**Theorem 2.1.** *The quadratic loop* $L(\mathcal{L})$ *is*

*(i)* *Extra if and only if* $\mathcal{L}$ *is extra;*

*(ii)* *Bol if and only if* $\mathcal{L}$ *is right alternative;*

*(iii)* *C if and only if* $\mathcal{L}$ *is alternative;*

*(iv)* *RC if and only if* $\mathcal{L}$ *is right alternative;*

*(v)* *LC if and only if* $\mathcal{L}$ *is left alternative;*

*(vi)* *LS(RS,S) if and only if* $\mathcal{L}$ *is LS(RS,S);*

*(vii)* *RM if and only if* $\mathcal{L}$ *is right alternative;*

*(viii)* *LM if and only if* $\mathcal{L}$ *is left* alternative.

*Proof.*

(i) The extra identity is

$$(xy.z)x = x(y.zx) \tag{2}$$

Substituting the operation in (1) in the left-hand side of (2), we obtain,

$$\begin{aligned}((x\circ y)\circ z)\circ x &= ((x+y-xy)\circ z)\circ x\\ &= (x+y+z-xy-xz-yz+xy.z)\circ x\\ &= 2x+y+z-xy-xz-yz+xy.z-x^2-yx-zx\\ &+ xy.x+xz.x+yz.x-(xy.z)x \qquad (3)\end{aligned}$$

Also the right-hand side implies

$$\begin{aligned}x\circ(y\circ(z\circ x)) &= x\circ(y\circ(z+x-zx))\\ &= x\circ(y+z+x-zx-yz+y.zx)\\ &= 2x+y+z-zx-yz-yx+y.zx-xy-xz-x^2\\ &+ x.zx+x.yz+x.yx-x(y.zx) \qquad (4)\end{aligned}$$

Comparing (3) and (4) implies

$$-(xy.z)x+yz.x+xz.x+xy.x+xy.z=-x(y.zx)+x.yx+x.yz+x.zx+y.zx$$

and this polynomial identity is equivalent to the three homogenous identities

$$(xy.z)x=x(y.zx);\quad yz.x=y.zx\quad xy.z=x.yz$$

The first of these implies that L is extra, and this in turn implies the rest.

The proofs of $(ii),(iii),\cdots,(viii)$ are similar.

**Definition 2.1.** [2] *L(f) is a first degree loop if and only if $f(x,y)$ is a first degree polynomial in $x$ and we denote this by $L=L(F)$,*

$$x\circ y=F(y)x+y$$

*where $F:\Re^n\longrightarrow\Re^n$ is analytic linear transformation satisfying $F(0)=I$. The related algebra of L(F) is*

$$pq=-F_y(0)[q]_p=R[q]_p.$$

**Theorem 2.2.** *If L(F) is a first degree loop then the following identities hold:*

*(i)* *Extra if*

*(a)* $F(x)F(z)F(y) = F[F(F(x)z + x)y + F(x)z + x]$

*(b)* $F(x)F(z) = F(F(x)z + x)$

*(ii)* *Moufang if*

*(a)* $F(z)F(x)F(y) = F[F(F(z)x + x)y + F(z)x + z]$

*(b)* $F(z)F(x) = F(F(z)x + z)$

*(iii)* *RM if*

$$F(z)F(x)F(x) = F[F(z)F(x) + F(z)x + z]$$

*(iv)* *LM if*

*(a)* $F(z)F(F(x)x + x) = F[(F(z)x + z)x + F(z)x + z]$

*(b)* $F(z)F(x) = F(F(z)x + z)$

*(v)* *C if*

*(a)* $F(z)F(y)F(y) = F[F(F(z)y + z)y + F(z)y + z]$

*(b)* $F(z)F(y) = F(F(z)y + z)$

*(vi)* *LS if*

*(a)* $F(z)F(y)F(x) = F(F(z)y + z)F(x)$

*(b)* $F(z)F(y) = F(F(z)y + z)$

*(vii)* *RS if*

$$F(z)F[F(x)x+x] = F[F(F(x)x+x)z+F(x)x+x]$$

*(viii)* *Bol if*

$$F(y)F(z)F(y) = F[F(y)(F(z)y+z)+y]$$

*(ix)* *Left Bol if*

*(a)* $F(x)F(F(y)z+y) = F[F(F(x)y+x)z+F(x)y+x]$

*(b)* $F(x)F(y) = F(F(x)y+x)$

*(x)* *RC if*

$$F(z)F(z)F(y) = F[F(z)(F(z)y+z)+z]$$

*(xi)* *LC if*

*(a)* $F(x)F[F(y)z+y] = F[F(F(x)y+x+F(x)y+x]$

*(b)* $F(x)F(y) = F(F(x)y+x)$

*(xii)* *S if*

$$F(z)F[F(x)x+x] = F[F(z)(F(x)x+x)+z]$$

*Proof.*

(i) If L(F) is extra then

$$(xy \cdot z)x = x(y \cdot zx)$$

applying the "$\circ$" operation to the left hand side we have

$$\begin{aligned} ((x \circ y) \circ z) \circ x &= ((F(y)x+y) \circ z) \circ x \\ &= (F(z)(F(y)x+y) \circ z) \circ x \\ &= (F(x)[F(z)(F(y)x+y)+z]+x \\ &= F(x)F(z)F(y)x+F(x)F(z)y+F(x)z+z \qquad (5) \end{aligned}$$

Similarly to the right hand side, we have

$$\begin{aligned} x \circ (y \quad \circ \quad & (z \circ x)) = x \circ (y \circ (F(x)z + x)) \\ = \quad & x \circ [F(F(x)z + x)y + F(x)z + x] \\ = \quad & F[F(F(x)z + x)y + F(x)z + x]x + F(F(x)z + x + F(x)z + x \end{aligned} \tag{6}$$

comparing (5) and (6) we obtain

$$F(x)F(z)F(y) = F[F(F(x)z + x)y + F(x)z + x]$$

and

$$F(x)F(z) = F(F(x)z + x).$$

The proof of $(ii), (iii), \cdots, (xii)$ are similar to that of (i) hence they are omitted.

**Theorem 2.3.** *Let (L,.) be a loop. If we define the product*

$$x \circ y = F^{-1}(y)x + y$$

*for all $x, y \in L$, then the following properties hold.*

*(i) Extra if*

*(a)* $F^{-1}(x)F^{-1}(z)F^{-1}(y) = F^{-1}[F^{-1}(x)z + x)y + F^{-1}(x)z + x]$

*(b)* $F^{-1}(x)F^{-1}(z) = F^{-1}(F^{-1}(x)z + x)$

*(ii) Moufang if*

*(a)* $F^{-1}(z)F^{-1}(x)F^{-1}(y) = F^{-1}[F^{-1}(F^{-1}(z)x + z)y + F^{-1}(z)x + z]$

*(b)* $F^{-1}(z)F^{-1}(x) = F^{-1}(F^{-1}(z)x + z)$

*(iii) RM if*

$$F^{-1}(z)F^{-1}(x)F^{-1}(x) = F^{-1}[F^{-1}(z)F^{-1}(x)x + F^{-1}(z)x + z]$$

*(iv) LM if*

$$F^{-1}(z)F^{-1}(F^{-1}(x)x + x) = F^{-1}[F^{-1}(F^{-1}(z)x + z)x + F^{-1}(z)x + z]$$

*(v) C if*

*(a)* $F^{-1}(z)F^{-1}(y) = F^{-1}[F^{-1}(F^{-1}(z)y + z)]$

*(b)* $F^{-1}(z)F^{-1}(y)F^{-1}(y) = F^{-1}[F^{-1}(F^{-1}(z)y + z)y + F^{-1}(z)y + z]$

*(vi) LS if*

*(a)* $F^{-1}(z)F^{-1}(y)F^{-1}(x) = F^{-1}(x)F^{-1}(F^{-1}(z)y+z)$

*(b)* $F^{-1}(z)F^{-1}(y) = F^{-1}[F^{-1}(y)(F^{-1}(z)y+z)+y]$

*(vii)* *Bol if*

$$F^{-1}(y)F^{-1}(z)F^{-1}(y) = F^{-1}[F^{-1}(y)F^{-1}(z)y + F^{-1}(y)z + y]$$

*(viii)* *left Bol if*

*(a)* $F^{-1}(x)F^{-1}(F^{-1}(y)z+y) = F^{-1}[F^{-1}(F^{-1}(x)y+x)z + F^{-1}(x)y + x]$

*(b)* $F^{-1}(x)F^{-1}(y) = F^{-1}(F^{-1}(x)y+x)$

*(ix)* *RC if*

$$F^{-1}(z)F^{-1}(z)F^{-1}(y) = F^{-1}[F^{-1}(z)(F^{-1}(z)y+z)]$$

*(x)* *S if*

$$F^{-1}(y)F^{-1}(z)F^{-1}(y) = F^{-1}[F^{-1}(y)(F^{-1}(z)y+z)+y]$$

*(xi)* *LC if*

*(a)* $F^{-1}(x)F^{-1}(F^{-1}(y)z+y) = F^{-1}[F^{-1}((F^{-1}(x)y+x)z + F^{-1}(x)y + x]$

*(b)* $F^{-1}(x)F^{-1}(y) = F^{-1}(F^{-1}(x)y+x)$

*(xii)* *RS if*

$$F^{-1}(z)F^{-1}(F^{-1}(x)x+x) = F^{-1}[F^{-1}(F^{-1}(x)x+x)z + F^{-1}(x)x + x]$$

*Proof.*

(i) If $L(F)^{-1}$ is extra then

$$(xy \cdot z)x = x(y \cdot zx)$$

applying the "$\circ$" operation to the left hand side we obtain

$$\begin{aligned} ((x \circ y) \circ z) \circ x &= ((F^{-1}(y)x + y) \circ z) \circ x \\ &= (F^{-1}(z)(F^{-1}(y)x+y)+z) \circ x \\ &= F^{-1}(x)[F^{-1}(z)(F^{-1}(y)x+y)+z] + x \\ &= F^{-1}(x)F^{-1}(z)F^{-1}(y)x + F^{-1}(x)F^{-1}(z)y + F^{-1}(x)z + x \quad (7) \end{aligned}$$

Similarly appling the "$\circ$" operation, the right hand side gives

$$
\begin{aligned}
x \quad &\circ \quad (y) \circ (z \circ x)) = x \circ (y \circ (F^{-1}(x)z + x) \\
&= \quad x \circ (F^{-1}(F^{-1}(x)z + x) + F^{-1}(x)z + x) \\
&= \quad F^{-1}[F^{-1}(F^{-1}(x)z + x)y + F^{-1}(x)z + x]x + F^{-1}(F^{-1}(x)z + x)y \\
&+ \quad F^{-1}(x)z + x \qquad (8)
\end{aligned}
$$

Comparing (7) and (8) we have

(a)

$$F^{-1}(x)F^{-1}(z)F^{-1}(y) = F^{-1}[F^{-1}(F^{-1}(x)z + x)y + F^{-1}(x)z + x]$$

$$F^{-1}(x)F^{-1}(z) = F^{-1}(F^{-1}(x)z + x)$$

The proof of $(ii), (iii), \cdots, (xii)$ are similar and hence omitted.

# References


[1] R. H. Bruck; A Survey of Binary Systems, Springer- Verlag, Berlin, 1958, MR 20(1959) # 76

[2] O. Chein et al (Ed); Quasigroups and loops: Theory and Applications, Heldermann Verlag, Berlin 1990.

[3] F. Fenyves;- Extra loops, I & II, Publ. Math. Debreen 15(1968), 235-238, 16(1969), 187-192.

[4] K. H. Hofmann & K. Strambach; Topological and analytical quasi-groups and loops and related structures, In the theory of Quasigroups and loops, Heldermann Verlag, Berlin.

[5] A. Malcev; Analytical loops, Math. Sb. 36(1955), 429-458

[6] Solarin, A.R.T. On the identities of Bol-Moufang type. Koungpook Math. J. Vol.28, No.1 (1988) pp. 51-62