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# THE NON–CENTRAL G-DISTRIBUTION AND ITS APPLICATION

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ABSTRACT. In this paper, the non-central version of the generalized logistic distribution is introduced. It is used to approximate the non-central $F$ distribution. The approximation is found to compare favourably well with its competitors. Tables are provided for illustrations.

## 1. Introduction

The density function of the logistic random variable can be written in its simplest form as

$$f(x) = \frac{e^x}{(1+e^x)^2}, \quad -\infty < x < \infty.$$

A generalization of this distribution whose density function is given by

$$f(x) = \frac{\Gamma(p+q)e^{px}}{\Gamma_p \Gamma_q (1+e^x)^{p+q}}, \quad -\infty < x < \infty,$$

was considered by George and Ojo (1980). The cumulants of this distribution were obtained and it was demonstrated that the distribution function can be well approximated by the $t$-distribution. Recently, this same distribution was characterized and shown to be related to many distributions in use (Ojo, 1977). In this article, the non-central version of the distribution is introduced as follows.

It has earlier been observed (George and Ojo, 1980) that if $F(\nu_1, \nu_2)$ is an $f$-random variable with $\nu_1$ and $\nu_2$ degrees of freedom, then the random variable

$$X = -\log F(\nu_1, \nu_2) - \log\left(\frac{\nu_1}{\nu_2}\right)$$

has the generalized logistic distribution with $\frac{1}{2}\nu_2$ and $\frac{1}{2}\nu_1$ degrees of freedom. Because of this exact relationship with the $F$-distribution, we now introduce the non-centrality parameter $\lambda$ into the distribution of the generalized logistic random variable. Indeed, given two independent variables $\chi'^2$ distributed as a non-central chi-square with with degree of freedom $\nu_1$ and non-centrality parameter $\lambda$, and $\chi^2$, distributed as a central chi-square with $\nu_2$ degree of freedom, then the negative logarithm of the ratio

$$\frac{\chi'^2}{\nu_1} / \frac{\chi^2}{\nu_2}$$

will have a non-central generalized logistic distribution with $\frac{1}{2}\nu_2$ and $\frac{1}{2}\nu_1$ degrees of freedom and non-centrality parameter $\lambda$. Thus, the random variable

$$G^* \left( \frac{1}{2}\nu_2, \frac{1}{2}\nu_1, \lambda \right) = -\log F^* \left( \frac{1}{2}\nu_2, \frac{1}{2}\nu_1, \lambda \right) - \log \left( \frac{\nu_2}{\nu_1} \right)$$

will be refered to as a non-central $G$ random variable. Furthermore, since the distribution function of the generalized logistic random variable has been successfully approximated by the $t$-distribution we strongly supset that the distribution function of the non-central $G$ random variable can as well be satisfactorily approximated by matching Kurtosis with the $t$-distribution. This can then be used in approximating the non-central $F$ distribution. This approximation is, in fact proposed in section 3 of this paper. It is found to be superior to some earlier approximations and found to compare favourably well with some others. Furthermore, it is simple to evaluate.

## 2. The cumulants of the non-central $G$

The cumulant of $\log \left( \frac{\nu_2}{\nu_1} \right) + \log \left( \frac{1}{2}\chi'^2 \right) - \log \left( \frac{1}{2}\chi^2 \right)$ has been obtained in a paper by Barton et al (1960). These cumulants were obtained by first obtaining the characteristic function in terms of $\nu_1, \nu_2, \lambda$ and $\theta$, taking the logarithm of the characteristics function, differentiating with respect to $\theta$ and then equating $\theta$ to zero. As many cumulants as desired could then be obtained. Essencially, these cumulants, with a slight modification, are the cumulants of the non-central $G$ distribution. For the purpose of this paper, we give the first four cumulants as follows.

$$k_1 = \psi(a_2) - \psi(a_1) - \log \left( \frac{a_2}{a_1} \right) - m_1$$

$$k_2 = \psi'(a_1) + \psi'(a_2) + m_2 - m_1^2$$

$$k_3 = \psi''(a_2) - \psi''(a_1) - m_3 + 3m_2 m_1^2 - 2m_1^3$$

$$k_4 = \psi'''(a_1) + \psi'''(a_2) + m_4 - 4m_1 m_3 + 6m_1^2 m_2 - 3m_1^4 - 3(m_2 - m_1^2)^2$$

where $a_1 = \frac{1}{2}\nu_1, \; a_2 = \frac{1}{2}\nu_2$

$$m_k = k! \sum_{r=k}^{\infty} \frac{(\lambda/2)^r}{r!} \frac{(-1)^{r-k}}{(a_1)^{(r)}} |S_r^k|$$

where $|S_r^k|$ is the Stirling's number of the first kind in modulus, $(a_1)^{(r)} = a_1(a_1 + 1) \ldots (a_1 + r - 1)$, and

$$\psi(x) = \frac{d}{dx} \log \Gamma x, \quad \psi'(x) = \frac{d}{dx} \psi(x).$$

It is noted that when $\lambda = 0$ the cumulants reduce to those of the generalized logistic distribution.

The above cumulants are needed to obtain the approximation that follows.

## 3. Approximation to the non-central $F$

The importance of the distribution function of the non-central $F$ as the power function of the $F$-test is well known and the attendant problem of its tabulation is also well recognized. Because of these reasons, attempts had been made to approximate this distribution and there continues to be efforts in getting more accurate and simpler approximations.

Partnaik (1949) proposed an approximation that requires the use of an incomplete beta table. Severo and Zelen (1960), Laubscher (1960) simultaneously and independetly presented a normal approximation to the distribution. Darton et al (1960) used an Edgeworth series expansion to approximate the same distribution. An approximation that requires the use of an $F$ distribution was due to Tiku (1965). Mudholkar et al (1976) presented Tiku-type approximation by equating moments. Although the approximations due to Mudholkar et al appear to be more accurate than most of the earlier ones, the approximation require a very tedius evaluation of cumulants together with an elaborate computer programming for their implementation.

### 3.1 The proposed approximation.

In what follows, we present a $t$-approximation to the non-central $F$ by matching Kurtosis. Let $\beta_2$ be the index of Kurtosis of the non-central $G$ distribution. By equating this to the coefficient of Kurtosis of the student's $t$ distribution, we have the simple relationship

$$\nu = 4 + 6/\beta_2$$

where $\nu$ is the degree of freedom of the $t$-distribution.

Now let $(G^* - k_1)/\sqrt{k_2}$ and $t_\nu/\sigma_t$ be standardized non-central $G$ and $t$ random variables respectively.

Then,

$$P\{G^* - k_1 \leq x\sqrt{k_2}\} \approx P\{t_\nu \leq x\sigma_t\}.$$

That is

$$P\{-\log F^*(\nu_1, \nu_2, \lambda) - \log(\nu_2/\nu_1) - k_1 \leq x\sqrt{k_2}\} \approx \{t_\nu \leq x\sigma_t\}.$$

That is

$$P\{F^*(\nu_1, \nu_2, \lambda) \geq Z_\alpha\} \approx P\{t_\nu \leq c(k_1 + \log((\nu_2/\nu_1)Z_\alpha)\}$$

or equivalently

$$P\{Z^* \geq z_\alpha\} \approx P\{t_\nu \leq c(k_1 + \log((\nu_2/\nu_1)) + 2Z_\alpha\}$$

where

$$Z^* = \frac{1}{2}\log F^*(\nu_1, \nu_2, \lambda)$$

$$c = \sigma_t/\sqrt{k_2}, \quad \sigma_t = \nu/(\nu - 2)$$

and $k_i$ is the $i^{th}$ cumulant of the non-central $G$ distribution.

## 4. Illustrations

Four tables are provided. Table I gives the relationships between $\beta_2$ and $\nu$ for some selected values of $\nu_1, \nu_2$ and $\lambda$. Table II gives comparisons of the proposed $t$ - approximation with the normal and the series approximations of Barton et al (1960). It is clear from this table that both the series and the $t$-approximations are uniformly better than the normal approximation and that the series and the $t$-approximationsare quite comparable in accuracy. It is also noted that the $k$-approximation in easier to compute than the series expansion. Table III contains values of type II error, $\beta$ for selected values of $\alpha$, the type I error. It compares the proposed approximation with Tiku's series and Patnaik's incomplete beta approximations. Table IV contains the exact and approximate values of the probability integral of $F^*(\nu_1, \nu_2, \lambda)$ for some combinations of $\nu_1, \nu_2$ and some selected values of $\lambda$ and $Z_\alpha$, the percentage point.

The table compares the proposed approximation with those of Patnaik, Severo and Zelen, Tiku and Mudholkar et al. Comparisons in tables III and IV show that the $t$-approximation is superios to those of Patnaik, Severo and Zelen and comparable with those of Tiku and Mudholkar et al and it is simpler to evaluate.

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