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ON CENTRALLY-EXTENDED GENERALIZED JORDAN $*$-DERIVATIONS IN RINGS

BHARAT BHUSHAN[1], GURNINDER S. SANDHU[2], SHAKIR ALI[3],
AND BRUNO LEONARDO MACEDO FERREIRA[4,5,6]

Abstract. Let R be an associative ring with an involution '$*$'. In this article, we introduce the notions of *centrally-extended generalized Jordan $*$-derivation*, *centrally extended Jordan left $*$-centralizer* and characterize these mappings in involutive prime rings.

1. Introduction

Throughout this study, R is an associative ring with center $Z(R)$. R is called *prime*, if for any $a, b \in R$, $aRb = (0)$ implies either $a = 0$ or $b = 0$; and it is called *semiprime*, if $aRa = (0)$, implies $a = 0$. Clearly, every prime ring is semiprime ring but the converse need not be true, for instance $\mathbb{Z} \times \mathbb{Z}$. The symmetric ring of quotients of R is denoted by Q_s with center C, which is known as the extended centroid of R; clearly $R \subseteq Q_s$ and $Z(R) \subseteq C$. It is well-known that if R is prime then Q_s is prime and C is a field. The central closure of R is denoted by $A(= RC + C)$; for more details of these objects, we refer the reader to [6]. For any $x, y \in R$, the commutator (resp. anti-commutator) of x, y is defined as $[x, y] = xy - yx$ (resp. $x \circ y = xy + yx$). It is established knowledge that R satisfies s_4 (the standard identity in four noncommuting variables), if for all $x_1, x_2, x_3, x_4 \in R$, the equation

$$s_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} (-1)^{\sigma} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)} = 0,$$

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where S_4 is the symmetric group of degree 4 and $(-1)^\sigma$ is the sign of permutation $\sigma \in S_4$. For some interesting equivalent forms of s_4, one can refer [11, Lemma 1].

For any $n \in \mathbb{Z}^+$, R is called an n-torsion free if for any $x \in R$, $nx = 0$ implies $x = 0$. An anti-automorphism $'*'$ of R is called *involution* if $(x^*)^* = x$ for all $x \in R$. A ring equipped with an involution $'*'$ is called $*$-*ring* or *ring with involution* or *involutive ring*. An element x in a $*$-ring R is called *symmetric* if $x^* = x$, and it is called *skew-symmetric* if $x^* = -x$. The set of symmetric and skew symmetric elements of a ring R is denoted by $H(R)$ and $S(R)$, respectively. Moreover, if R is a prime ring endowed with the involution $'*'$, then $'*'$ can be uniquely extended to $Q_s(R)$ (see [15, page 4]).

Let R be a $*$-ring, an additive mapping $d : R \to R$ is called $*$-*derivation* if $d(xy) = d(x)y^* + xd(y)$ for all $x, y \in R$ and it is called *Jordan* $*$-*derivation* if $d(x^2) = d(x)x^* + xd(x)$ for all $x \in R$. These notions are first mentioned in [12]. Note that the mapping $x \mapsto xa - ax^*$, where a is a fixed element of R, is an example of Jordan $*$-derivation, called *inner Jordan* $*$-*derivation*. Moreover, if $a \in Q_s$, then such a map is called X-inner Jordan $*$-derivation. The study of Jordan $*$-derivations has been originated from the problem of representability of quadratic forms by bilinear forms (see [28, 30]). Thereafter, some significant studies have taken place on the structure of Jordan $*$-derivations in rings (see [5, 14, 16, 21]).

In [2], Ali introduced the notion of *generalized* $*$-*derivation*, which is a self-map F of R associated with a $*$-derivation d satisfying $F(xy) = F(x)y^* + xd(y)$ for all $x, y \in R$. In addition to this, a self-map F of R is called a *generalized Jordan* $*$-*derivation* associated with a Jordan $*$-derivation d if $F(x^2) = F(x)x^* + xd(x)$ for all $x \in R$ (see [16], [19]). For any fixed $a, b \in R$, a map $x \mapsto ax^* + xb$ is a basic example of generalized Jordan $*$-derivation, which is called generalized inner Jordan $*$-derivation. Further, if a, b comes from Q_s, then this map is called generalized X-inner Jordan $*$-derivation. There has been an ongoing interest in the investigation of such mappings, for more details we refer the reader to [1, 4, 5, 16, 19] and references therein.

A mapping f of R is called centralizing (resp. commuting) on a subset S of R if $[f(x), x] \in Z(R)$ (resp. $[f(x), x] = 0$) for all $x \in S$. To our best knowledge, Divinsky [17] initiated the study of commuting and centralizing mappings in rings by proving a classical result which states that a simple Artinian ring is commutative if it admits a commuting nontrivial automorphism. Since then, many significant results on commuting and centralizing mappings have been established by Posner [27], Mayne [25], Bell and Martindale [8], Brešar [10]. Moreover, let R be a $*$-ring, a mapping $f : R \to R$ is called $*$-centralizing ($*$-commuting) on a subset S of R if $[f(x), x^*] \in Z(R)$ (resp. $[f(x), x^*] = 0$) for all $x \in S$ (see [3]).

Bell and Daif [7] introduced *centrally-extended derivations* and discussed their existence. Accordingly, a mapping $d : R \to R$ is called *centrally-extended derivation* if $d(x + y) - d(x) - d(y) \in Z(R)$ and $d(xy) - d(x)y - xd(y) \in Z(R)$ for all $x, y \in R$. Motivated by this, El-Deken and Nabiel [31] introduced that a mapping d is called *centrally-extended* $*$-*derivation* if $d(x + y) - d(x) - d(y) \in Z(R)$ and $d(xy) -$

$d(x)y^* - xd(y) \in Z(R)$ for all $x, y \in R$. Furthermore, a mapping F is called *centrally-extended generalized $*$-derivation* associated with a centrally-extended $*$-derivation d if $F(x+y)-F(x)-F(y) \in Z(R)$ and $F(xy)-F(x)y^*-xd(y) \in Z(R)$ for all $x, y \in R$. In a recent paper [9], we introduced and studied the concept of *centrally-extended Jordan $*$-derivation*, which is a mapping $d : R \to R$ such that $d(x + y) - d(x) - d(y) \in Z(R)$ and $d(x \circ y) - d(x)y^* - d(y)x^* - xd(y) - yd(x) \in Z(R)$ for all $x, y \in R$. Nowadays, centrally-extended mappings are getting attention of many researchers, consequently there has been rising literature on these mappings in rings under different settings, for instance, see [7, 9, 18, 26, 31–34].

In this article, we shall introduce centrally-extended generalized Jordan $*$-derivation in rings and discuss their existence in noncommutative prime ring under suitable torsion conditions. We also investigate some specific functional identities involving centrally-extended generalized Jordan $*$-derivations. Precisely, in Section 4 of this article, we prove a structural result on centrally-extended generalized Jordan $*$-derivations which plays a key role in Sections 5 and 6, where we study centralizing and hypercommuting conditions involving centrally-extended generalized Jordan $*$-derivations, respectively.

2. Preliminaries

The following lemmas constitute a set of results that will be instrumental in the development of the paper.

Lemma 2.1 (Brauer's Trick). *Let G be a group and H_1, H_2 be subgroups of G. If $G = H_1 \cup H_2$, then either $G = H_1$ or $G = H_2$.*

Lemma 2.2. *If R is a prime ring, then $Z(R)$ has no proper zero divisor.*

Lemma 2.3. *Let R is a prime for any $a \in Z(R)$. If there exists $b \in R$ such that $ab \in Z(R)$, then either $a = 0$ or $b \in Z(R)$.*

Lemma 2.4. ([1, Proposition 2.3]). *Let R be a 2-torsion free semiprime ring with involution $'*'$. If $f : R \to R$ is an additive map such that $f(x^2) = f(x)x^*$ for all $x \in R$, then there exists $q \in Q_r(R)$ such that $f(x) = qx^*$ for all $x \in R$.*

Lemma 2.5. ([3, Lemma 2.2]). *Let R be a 2-torsion free semiprime ring with involution $'*'$. If an additive mapping f of R into itself such that $[f(x), x^*] \in Z(R)$ for all $x \in R$, then $[f(x), x^*] = 0$ for all $x \in R$.*

Lemma 2.6. ([6, Theorem 6.4.6]). *Let R be a prime ring with extended centroid C, anti-automorphism g and maximal right ring of quotients $Q_{mr}(R) = Q$. If $0 \neq \phi = \phi(x_1, \ldots, x_n, g(x_1), \ldots, g(x_n)) \in Q_C< X \cup g(X) >$ is a g-identity on K ideal of R, then ϕ is a g-identity on $Q_s = Q_s(R)$.*

Lemma 2.7. ([9, Theorem 4.6]). *Let R be a 2-torsion free noncommutative prime ring. If R admits a non-zero centrally-extended Jordan derivation $\mathfrak{d} : R \to R$ such* *that $[\mathfrak{d}(x), x] \in Z(R)$ for all $x \in R$, then either $\mathfrak{d} = 0$ or R is an order in a central simple algebra of dimension at most 4 over its center.*

Lemma 2.8. ([9, Theorem 4.7]). *Let R be a 2-torsion free noncommutative prime ring with involution $*$. If R admits a non-zero centrally-extended Jordan derivation $\mathfrak{d} : R \to R$ such that $[\mathfrak{d}(x), x^*] \in Z(R)$ for all $x \in R$, then either $\mathfrak{d} = 0$ or R is an order in a central simple algebra of dimension at most 4 over its center.*

Lemma 2.9. ([10, Proposition 3.1]). *Let R be a 2-torsion free semiprime ring and U be a Jordan subring of R. If an additive mapping F of R into itself is centralizing on U, then F is commuting on U.*

Lemma 2.10. ([10, Theorem 3.2]). *Let R be a prime ring. If an additive mapping $F : R \to R$ is commuting on R, then there exists $\lambda \in C$ and an additive $\xi : R \to C$, such that $F(x) = \lambda x + \xi(x)$ for all $x \in R$.*

Lemma 2.11. ([19, Theorem 2.2]). *Let R be a 2-torsion free prime ring with involution $'*'$. Let $F : R \to R$ be a generalized Jordan $*$-derivation associated with a Jordan $*$-derivation d. Then, F is of the form $F(x) = qx^* + d(x)$ for all $x \in R$ and some $q \in Q_s(R)$.*

Lemma 2.12. ([20, Theorem 1]). *Let R be a prime ring with involution $'*'$ and center $Z(R)$. If d is a non-zero derivation such that $[d(h), h] \in Z(R)$ for all $h \in H(R)$, then R satisfies s_4.*

Lemma 2.13. ([20, Theorem 3]). *Let R be a prime ring with involution $'*'$ and center $Z(R)$. If n be a fixed natural number such that $x^n \in Z(R)$ for all $x \in H(R)$, then R satisfies s_4 identity.*

Lemma 2.14. ([20, Theorem 6]). *Let R be a prime ring with involution $'*'$ and center $Z(R)$. If d is a non-zero derivation on R such that $d(x)x + xd(x) \in Z(R)$ for all $x \in H(R)$, then R satisfies s_4 identity.*

Lemma 2.15. ([20, Theorem 7]). *Let R be a prime ring with involution $'*'$ and center $Z(R)$. If d is a non-zero derivation on R such that $d(x)x + xd(x) \in Z(R)$ for all $x \in S(R)$, then R satisfies s_4 identity.*

Lemma 2.16. [21, Theorem 1.3]). *Let R be a 2-torsion free noncommutative prime ring with involution $'*'$, then any Jordan $*$-derivation on R is X-inner.*

Lemma 2.17. *Let R be a 2-torsion free prime ring. If $q_1 \in Q_s(R)$ such that $[q_1, h] \in C$ for all $h \in H(R)$, then R satisfy s_4 identity or $q_1 \in C$.*

Proof. Let us consider

$$[q_1, h] \in C, \quad \text{for all } h \in H(R). \tag{2.1}$$

Replacing h by h^2, where $h \in H(R)$, we obtain $[q_1, h]h + h[q_1, h] \in C$, i.e., $d(h)h + hd(h) \in C$ for all $h \in H(R)$, where $d(x) = [q_1, x]$. If $d \neq 0$, we have the result by Lemma 2.14. If $d = 0$, then we conclude $q_1 \in C$, as desired. □

Lemma 2.18. *Let R be a 2-torsion free prime ring with involution. If $[h,k]=0$ for all $h\in H(R)$, $k\in S(R)$, then R satisfy s_4 identity.*

Proof. Suppose that R does not satisfy s_4 identity. By hypothesis, we have $[h,k]=0\in C$ for all $h\in H(R)$ and $k\in S(R)$. In view of Lemma 2.17, it follows that either R satisfies s_4 or $k\in Z(R)$ for all $k\in S(R)$. Under the given hypothesis, we left with $S(R)\subseteq Z(R)$. Clearly $h\circ k\in S(R)$ for all $h\in H(R)$ and $k\in S(R)$, therefore we have $[h\circ k,k]=0$, i.e., $[h,k]k+k[h,k]=0$ for all $h\in H(R)$ and $k\in S(R)$. For a fixed $h\in H(R)$, we have $d(k)k+kd(k)=0$ for all $k\in S(R)$, where $d(x)=[x,h]$ for all $x\in R$. If $d\neq 0$, then we have the result by Lemma 2.15. In case $d=0$, we conclude $H(R)\subseteq Z(R)$ and hence by Lemma 2.13, we have a contradiction. Hence, R must satisfies s_4 identity. $\square$

3. Definitions and Examples

We begin our discussions with the definition of centrally-extended generalized Jordan $*$-derivations of rings with involution.

Definition 3.1. Let R be a ring with involution $'*'$. A mapping $F: R\to R$ is called *centrally-extended generalized Jordan $*$-derivation* associated with an centrally-extended Jordan $*$-derivation d, if

$$F(x+y)-F(x)-F(y)\in Z(R), \tag{A}$$

$$F(x\circ y)-F(x)y^*-F(y)x^*-xd(y)-yd(x)\in Z(R), \tag{B}$$

for all $x,y\in R$.

Remark 3.1. If R is 2-torsion free noncommutative prime ring with involution $'*'$, then for an additive mapping F to be a centrally-extended generalized Jordan $*$-derivation, it is sufficient to satisfy the condition $F(x^2)-F(x)x^*-xd(x)\in Z(R)$ for all $x\in R$.

Example 3.1. We now show the existence of centrally-extended generalized Jordan $*$-derivations in certain rings.

(I) Let $\mathbb{Z}$ be the ring of integers and $R=\left\{\begin{pmatrix} x & y\\ z & t\end{pmatrix}: x,y,z,t\in\mathbb{Z}\right\}$, a noncommutative prime ring. Then the mapping $*: R\to R$ such that $\begin{pmatrix} x & y\\ z & t\end{pmatrix}^*=\begin{pmatrix} t & -y\\ -z & x\end{pmatrix}$, $F: R\to R$ such that $F\begin{pmatrix} x & y\\ z & t\end{pmatrix}=\begin{pmatrix} x-t & y\\ z & 0\end{pmatrix}$ with associated mapping $d: R\to R$ defined as $d\begin{pmatrix} x & y\\ z & t\end{pmatrix}=\begin{pmatrix} x & 0\\ 0 & x\end{pmatrix}$. One can easily verify that F is a centrally-extended generalized Jordan $*$-derivation with associated centrally-extended Jordan $*$-derivation d.

(II) Let R be a ring defined as $R := \mathfrak{R} \times \mathbb{Z}$, where $\mathfrak{R} = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} : x, y, z \in \mathbb{Z}_2 \right\}$. For any $X = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in \mathfrak{R}$, let us define $X^* = \begin{pmatrix} z & y \\ 0 & x \end{pmatrix}$ and hence $(X, k)^* = (X^*, k)$, which is an involution on R. Define $F : R \to R$ such that $F\left(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}, k \right) = \left(\begin{pmatrix} 0 & x \\ 0 & x \end{pmatrix}, 1 \right)$ with associated mapping $d : R \to R$ defined as $d\left(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}, k \right) = \left(\begin{pmatrix} z & y \\ 0 & x \end{pmatrix}, 1 \right)$. We observe that F is a centrally-extended generalized Jordan $*$-derivation with an associated centrally-extended Jordan $*$-derivation.

Remark 3.2. It can be seen from the above example that every centrally-extended generalized Jordan $*$-derivation is a centrally-extended Jordan $*$-derivation, but the converse does not necessarily hold in general.

Definition 3.2. Let R be a ring with involution $'*'$. A mapping $T : R \to R$ is called *centrally-extended Jordan $*$-left centralizer* (resp. *centrally-extended Jordan $*$-right centralizer*) if

(A) $\qquad T(x + y) - T(x) - T(y) \in Z(R)$,

(B) $\qquad T(x^2) - T(x)x^* \in Z(R)$ (resp. $T(x^2) - x^*T(x) \in Z(R)$),

for all $x, y \in R$. Moreover, T is called *centrally-extended Jordan $*$-centralizer* if it is both a centrally-extended Jordan $*$-left centralizer and a centrally-extended Jordan $*$-right centralizer.

Example 3.2. Let R be a ring defined as $R := M_2(\mathbb{R}) \times \mathbb{C}$, where $M_2(\mathbb{R})$ denotes the ring of 2×2 matrices over real numbers. For any $r = (x, z_1), s = (y, z_2) \in R$, we define $r^* = (x^*, \overline{z}_1)$, where x^* is defined as in Example 3.1 (I) and $\overline{z}_1$ is the complex conjugate of z_1. Define a mapping $T : R \to R$ such that $T(x, z) = (0, 1)$ for all $(x, z) \in R$. Then it can be easily verified that T is a centrally-extended Jordan $*$-left centralizer of R.

4. Auxiliary Results

In this section, we shall mainly prove the following theorem which is crucial for the results proved in the subsequent sections.

Theorem 4.1. *Let R be a 2-torsion free noncommutative prime ring with involution $'*'$ and F be a centrally-extended generalized Jordan $*$-derivation of R associated with a centrally-extended Jordan $*$-derivation $\mathfrak{d}$. Then, there exists $q \in Q_s(R)$ such that $F(x) = qx^* + \mathfrak{d}(x)$ for all $x \in R$.*

After proving few more facts in this regard, we shall return to the proof of Theorem 4.1.

Proposition 4.1. *Let R be a 2-torsion free prime ring with involution $'*'$ and $d : R \to R$ a Jordan $*$-derivation. Then, d can be uniquely extended to A, unless R satisfies s_4.*

Proof. Suppose that R does not satisfy s_4 identity. By Lemma 2.16, there exists $a \in Q_s(R)$ such that $d(x) = xa - ax^*$ for all $x \in R$. Let us define $\bar{d}(q) = qa - aq^*$ for all $q \in A$. Clearly, $\bar{d}$ is a well-defined map and it is an extension of d. Now, we claim that the extension of d is unique.

Let D be also an extension of d. Since A is a noncommutative 2-torsion free prime ring, D is X-inner on A, i.e., there exists $b \in A$ such that $D(u) = ub - bu^*$ for all $u \in A$. As D is an extension of d, so we have

$$xb - bx^* = xa - ax^*, \quad \text{for all } x \in R. \tag{4.1}$$

In particular by taking h for x in (4.1), where $h \in H(R)$, we obtain $[h, b - a] = 0$ for all $h \in H(R)$. Using Lemma 2.17, we have $b - a \in C$. That means there exists $c \in C$ such that $b = a + c$. From (4.1), we have $xc - cx^* = 0$ for all $x \in R$. Replacing x by k in the last expression, where $k \in S(R)$, we get $2ck = 0$ for all $k \in S(R)$. If $c \neq 0$, then by using Lemma 2.2, we have $S(R) = (0)$, thence $H(R) = R$; which implies that for any $x, y \in R$, we have $xy = (xy)^* = y^*x^* = yx$, which is a contradiction. In case $c = 0$, we have $a = b$. It proves our claim. □

Proposition 4.2. *Let R be a 2-torsion free prime ring with involution $'*'$ and $F : R \to R$ a generalized Jordan $*$-derivation associated with d a Jordan $*$-derivation. Then, F can be uniquely extended to A, unless R satisfies s_4.*

Proof. By Lemma 2.11, there exists $a \in Q_s(R)$ such that $F(x) = ax^* + d(x)$ for all $x \in R$. We define $\tilde{F}(u) = au^* + d(u)$ for all $u \in A$. By Proposition 4.1, $\tilde{F}$ is a well defined map and also it is an extension of F. Now, we will show that extension of F is unique. Let G be also extension of F. Using Lemma 2.11, there exists $b \in Q_s$ and g a Jordan $*$-derivation on A such that $G(u) = bu^* + g(u)$ for all $u \in A$. Since, G is an extension of F. Therefore

$$ax^* + d(x) = bx^* + g(x), \quad \text{for all } x \in R. \tag{4.2}$$

By Lemma 2.16, there exists $c, q \in Q_s$ such that $d(x) = xc - cx^*$ and $g(x) = xq - qx^*$ for all $x \in R$. From (4.2), we have $(a - c)x^* + xc = (b - q)x^* + xq$ for all $x \in R$. As the preceding equation is a g-identity on R, application of Lemma 2.6 yields $(a - c)x^* + xc = (b - q)x^* + xq$ for all $x \in Q_s(R)$. Replacing x by 1, we obtain $a = b$. From (4.2), we find $d(x) = g(x)$ for all $x \in R$. In fact, Proposition 4.1 gives $d(x) = g(x)$ for all $x \in A$. It completes the proof. □

For the sake of brevity, we omit the proof of the following result, as it follows proceeding along the same lines as the proof of Lemma 4.4 of [9], with insignificant variations.

Proposition 4.3. *Let R be a ring with involution $*$ and with no non-zero central ideal. If F is a centrally-extended generalized Jordan $*$-derivation associated with centrally-extended Jordan $*$-derivation $\mathfrak{d}$ of R, then F is additive.*

Corollary 4.1. *Let R be a noncommutative prime ring with involution $'*'$. If F is a centrally-extended generalized Jordan $*$-derivation of R associated with centrally-extended Jordan $*$-derivation $\mathfrak{d}$, then F is additive.*

Proposition 4.4. *Let R be a 2-torsion free noncommutative prime ring with involution $'*'$ and $T : R \to R$ a centrally-extended Jordan $*$-left centralizer. Then, there exists $q \in Q_s(R)$ such that $T(x) = qx^*$ for all $x \in R$.*

Proof. Note that T is additive by Proposition 4.1. If $Z(R) = (0)$, then $T(x^2) = T(x)x^*$ for all $x \in R$. Thus, we conclude the desired result by Lemma 2.4. In case $Z(R) \neq (0)$, we first claim that there exists $0 \neq z \in Z(R)$ such that $z^* = z$.

Let us suppose that $0 \neq z_c \in Z(R)$. If $z_c^* = z_c$, then we are done. If $z_c^* \neq z_c$, then we take $z_1 = z_c + z_c^*$; and observe that $z_1 = z_1^*$. Therefore, we can say that there exists $0 \neq z \in Z(R)$ such that $z^* = z$.

By the assumption, we have

$$T(x^2) - T(x)x^* \in Z(R), \quad \text{for all } x \in R. \tag{4.3}$$

Polarizing (4.3), we get

$$T(xy + yx) - T(x)y^* - T(y)x^* \in Z(R), \quad \text{for all } x, y \in R. \tag{4.4}$$

Replacing y by z^2, where $z \in H(R) \cap Z(R)$ in (4.4) to get

$$T(xz^2 + z^2x) - T(x)z^2 - T(z^2)x^* \in Z(R), \quad \text{for all } x \in R.$$

It implies

$$T(xz^2 + z^2x) - T(x)z^2 - T(z)zx^* - c_1x^* \in Z(R), \quad \text{for all } x \in R,$$

where c_1 is the corresponding central element. It implies

$$T(4xz^2) - 2T(x)z^2 - 2T(z)zx^* - 2c_1x^* \in Z(R). \tag{4.5}$$

Also, $4xz^2 = z(xz + zx) + (xz + zx)z$. From (4.4), we have

$$T(z(xz + zx) + (xz + zx)z) - T(z)(xz + zx)^* - T(xz + zx)z \in Z(R).$$

Again using (4.4) in the last summand of the above relation, we get

$$T(4xz^2) - T(z)(x^*z + zx^*) - (T(x)z + T(z)x^* + c_2)z \in Z(R), \tag{4.6}$$

where c_2 is the corresponding central element. Comparing (4.5) and (4.6) to obtain

$$T(x)z^2 - T(z)x^*z - c_1x^* \in Z(R).$$

It implies

$$([T(x)z, x^*] - [T(z)x^*, x^*])z = 0.$$

Application of Lemma 2.2 yields

$$[T(x)z - T(z)x^*, x^*] = 0. \tag{4.7}$$

Define $\mathfrak{G}(x) = T(x)z - T(z)x^*$ for all $x \in R$. From (4.7), we find $\mathfrak{G}$ is an additive $*$-commuting map. Applying involution in (4.7) and using Lemma 2.10 in order to get $\mathfrak{G}(x)^* = \lambda x + \sigma(x)$ for all $x \in R$, for some $\lambda \in C$ and $\sigma : R \to C$. It implies $\mathfrak{G}(x){=}T(x)z - T(z)x^* = \lambda^* x^* + \sigma(x)^*$. Therefore, $T(x)z = (T(z) + \lambda^*)x^* + \sigma(x)^*$ for all $x \in R$. It can also be written as $T(x) = z^{-1}(T(z) + \lambda^*)x^* + \sigma(x)^* z^{-1}$. Thus, $T(x) = qx^* + \sigma'(x)$ for all $x \in R$ where $q = z^{-1}(T(z) + \lambda^*)$ and $\sigma'(x) = z^{-1}\sigma(x)^*$. From (4.3), we have $q(x^2)^* + \sigma'(x^2) - (qx^* + \sigma'(x))x^* \in Z(R)$ for all $x \in R$. It implies

$$x^*\sigma'(x) \in C, \qquad \text{for all } x \in R.. \tag{4.8}$$

For any fixed $x \in R$, using Lemma 2.3 in (4.8), we have either $x \in Z(R)$ or $\sigma'(x) = 0$. As σ' is additive function, by application of Lemma 2.1, we find either $x \in Z(R)$ for all $x \in R$ or $\sigma'(x) = 0$ for all $x \in R$. Since R is noncommutative, we have $\sigma' = 0$. Thus, $T(x) = qx^*$ for all $x \in R$. □

Proof of Theorem 4.1. Let $T(x) = (F - \mathfrak{d})(x)$ for all $x \in R$. Then for any $x \in R$, $T(x^2) = (F - \mathfrak{d})(x^2) = F(x)x^* + x\mathfrak{d}(x) + c_1 - \mathfrak{d}(x)x^* - x\mathfrak{d}(x) - c_2$ where c_1 and c_2 are corresponding central element. It turns out to be $T(x^2) - T(x)x^* = c_1 + c_2 \in Z(R)$ for all $x \in R$. Therefore T is a centrally-extended Jordan $*$-left centralizer. Invoking Proposition 4.4, we get $T(x) = qx^*$ for all $x \in R$, where $q \in Q_s(R)$. Thus $F(x) = qx^* + \mathfrak{d}(x)$ for all $x \in R$. □

5. Centralizing Conditions

An astonishing result of Posner [27] states that a prime ring R is commutative if it possesses a non-zero derivation d which is centralizing on R (i.e. $[d(x), x] \in Z(R)$). Proceeding this investigation, Mayne [24, 25] studied automorphisms and derivations which are centralizing on appropriate subsets of a prime ring. Later on, Bell and Martindale [8] examined centralizing mappings of semiprime rings. Since then several authors have extended these results in different directions. In 2014, Ali and Dar [3] introduced the notion of $*$-centralizing mappings in prime rings with involution. Motivated by these studies, in this section, we are intended to describe the structure of centralizing and $*$-centralizing centrally-extended generalized Jordan $*$-derivations of a prime ring with involution $*$.

Theorem 5.1. *Let R be a 2-torsion free prime ring with involution $'*'$ and $F : R \to R$ a centrally-extended generalized Jordan $*$-derivation associated with centrally-extended Jordan $*$-derivation $\mathfrak{d}$. If $[F(x), x] \in Z(R)$ for all $x \in R$, then there exists $\lambda \in C$ such that $F(x) = \lambda x$ for all $x \in R$, unless R satisfies s_4.*

Proof. Suppose that R does not satisfies s_4. By hypothesis, we have

$$[F(x), x] \in Z(R), \qquad \text{for all } x \in R.$$

If $Z(R) = (0)$, then from Lemma 2.11, we find that there exists $a \in Q_s(R)$ such that

$$F(x) = ax^* + \mathfrak{d}(x), \qquad \text{for all } x \in R. \tag{5.1}$$

Since F is a additive and centralizing map, by using Lemma 2.9 and Lemma 2.10, there exists $\lambda \in C$ and a map $\sigma : R \to C$ such that

$$F(x) = \lambda x + \sigma(x), \quad \text{for all } x \in R. \tag{5.2}$$

In view of Proposition 4.2, it follows from equations (5.1) and (5.2) that

$$\lambda x + \sigma(x) = ax^* + \mathfrak{d}(x), \quad \text{for all } x \in A. \tag{5.3}$$

By Lemma 2.16 and Proposition 4.1, there exists $b \in Q_s(R)$ such that $\mathfrak{d}(x) = xb - xb^*$ for all $x \in A$. Therefore, from (5.3), we have

$$\lambda x + \sigma(x) = ax^* + xb - bx^*, \quad \text{for all } x \in A. \tag{5.4}$$

Taking 1 instead of x in (5.4), we find

$$a \in C. \tag{5.5}$$

Replacing x by h in (5.4), we obtain

$$\lambda h + \sigma(h) = ah + d(h), \quad \text{for all } h \in H(A), \tag{5.6}$$

where $d(x) = [x, b]$. Since $a \in C$, from (5.6), we see that $[d(h), h] = 0$ for all $h \in H(A)$. If $d \neq 0$, then Lemma 2.12 leads us to a contradiction.

On the other hand, let $d = 0$, i.e., $b \in C$. Moreover, from (5.6), we get $(\lambda - a)h \in C$. Since $\lambda - a \in C$, in view of Lemma 2.3, we have either $\lambda = a$ or $H(R) \subseteq Z(R)$. In the latter case, we have a contradiction by Lemma 2.13, so we left with $\lambda = a$. With this, from (5.4) we obtain

$$2\lambda k - 2bk \in C, \quad \text{for all } k \in S(R). \tag{5.7}$$

As $\lambda, b \in C$, it implies either $S(R) \subseteq Z(R)$ or $\lambda = b$. In the former case, we have the desired result by Lemma 2.18 and in the latter case, (5.1) yields $F(x) = \lambda x^* + x\lambda - \lambda x^*$ for all $x \in R$, i.e., $F(x) = \lambda x$ for all $x \in R$, as desired.

In case $Z(R) \neq \{0\}$, we have $0 \neq h_c \in Z(R)$ such that $h_c^* = h_c$. By the assumption $[F(x), x] \in Z(R)$ for all $x \in R$. With the aid of Lemma 2.9 and Proposition 4.3, we have

$$[F(x), x] = 0, \quad \text{for all } x \in R. \tag{5.8}$$

From Lemma 2.10 and (5.8), we have

$$F(x) = \lambda x + \sigma(x), \quad \text{for all } x \in R, \tag{5.9}$$

for some $\lambda \in C$ and a map $\sigma : R \to C$. Application of Proposition 4.1 in (5.9) yields

$$qx^* + \mathfrak{d}(x) = \lambda x + \sigma(x), \quad \text{for all } x \in R, \tag{5.10}$$

for some $q \in Q_s(R)$. From (B), we find

$$F(x \circ h_c) - F(x)h_c - F(h_c)x^* - x\mathfrak{d}(h_c) - h_c\mathfrak{d}(x) \in Z(R), \quad \text{for all } x \in R. \tag{5.11}$$

Using (5.9) in (5.11), we find

$$\lambda(x \circ h_c) - \lambda x h_c - \lambda h_c x^* - \sigma(h_c)x^* - x\mathfrak{d}(h_c) - h_c\mathfrak{d}(x) \in C, \quad \text{for all } x \in R.$$

It implies

$$\lambda(x-x^*)h_c-\sigma(h_c)x^*-x\mathfrak{d}(h_c)-h_c\mathfrak{d}(x)\in C, \quad \text{for all } x\in R. \tag{5.12}$$

Replacing x by h_c in (5.12), we conclude

$$\mathfrak{d}(h_c)\in Z(R). \tag{5.13}$$

Replacing x by h in (5.12), where $h\in H(R)$, we obtain

$$-\sigma(h_c)h-h\mathfrak{d}(h_c)-h_c\mathfrak{d}(h)\in C. \tag{5.14}$$

It implies

$$\mathfrak{d}(h_c)[h,x]+\sigma(h_c)[h,x]+h_c[\mathfrak{d}(h),x]=0, \quad \text{for all } h\in H(R), x\in R. \tag{5.15}$$

Replacing h by h^2 in (5.15), we find

$$(\mathfrak{d}(h_c)+\sigma(h_c))([h,x]h+h[h,x])+h_c[\mathfrak{d}(h)h+h\mathfrak{d}(h),x]=0, \quad \text{for all } h\in H(R), x\in R. \tag{5.16}$$

Using (5.13) and (5.15) in (5.16), we obtain

$$h_c(\mathfrak{d}(h)[h,x]+[h,x]\mathfrak{d}(h))=0, \quad \text{for all } h\in H(R), x\in R.$$

It implies that

$$\mathfrak{d}(h)[h,x]+[h,x]\mathfrak{d}(h)=0, \quad \text{for all } h\in H(R), x\in R. \tag{5.17}$$

Polarizing h in (5.17), we find

$$\mathfrak{d}(h_1)[h,x]+\mathfrak{d}(h)[h_1,x]+[h,x]\mathfrak{d}(h_1)+[h_1,x]\mathfrak{d}(h)=0, \quad \text{for all } h,h_1\in H(R), x\in R. \tag{5.18}$$

In particular, replacing h_1 by h_c in (5.18) and thereby using (5.13), we conclude

$$2\mathfrak{d}(h_c)[h,x]=0, \quad \text{for all } h\in H(R), x\in R. \tag{5.19}$$

If $\mathfrak{d}(h_c)\neq 0$, then Lemma 2.2 in (5.19) implies $H(R)\subseteq Z(R)$. With the aid of Lemma 2.13, we arrive at a contradiction.

In case $\mathfrak{d}(h_c)=0$, we obtain from (5.12) that

$$2\lambda kh_c+\sigma(h_c)k-h_c\mathfrak{d}(k)\in C, \quad \text{for all } k\in S(R).$$

It implies $h_c[\mathfrak{d}(k),k]=0$, for all $k\in S(R)$. From the fact $h_c\neq 0$ and Lemma 2.2, it follows that

$$[\mathfrak{d}(k),k]=0, \quad \text{for all } k\in S(R). \tag{5.20}$$

By using $\mathfrak{d}(h_c)=0$ and replacing h by k^2 in (5.14), where $k\in S(R)$, we have

$$-\sigma(h_c)k^2-h_c\mathfrak{d}(k^2)\in C.$$

It implies

$$-\sigma(h_c)k^2+h_c[\mathfrak{d}(k),k]\in C, \quad \text{for all } k\in S(R).$$

From (5.20), we find

$$\sigma(h_c)k^2\in C, \quad \text{for all } k\in S(R). \tag{5.21}$$

If $\sigma(h_c) \neq 0$, then Lemma 2.3 in (5.21) implies $k^2 \in Z(R)$ for all $k \in S(R)$. For any fixed $x \in R$, we have $[k,x]k + k[k,x] = d(k)k + kd(k) = 0$ for all $k \in S(R)$, where $d(y) = [y,x]$ for all $y \in R$. If $d \neq 0$, then Lemma 2.15 yields a contradiction. Now, if $d = 0$, then we have $[x,y] = 0$ for all $x, y \in R$, hence R is commutative, which is a contradiction.

In case $\sigma(h_c) = 0$, using fact $\mathfrak{d}(h_c) = 0$, from (5.14), we have $h_c\mathfrak{d}(h) \in Z(R)$ for all $h \in H(R)$. Since $h_c \neq 0$, we have $\mathfrak{d}(h) \in Z(R)$ for all $h \in H(R)$ by Lemma 2.2. For any fixed $h \in H(R)$, using Lemma 2.2 in (5.17) we find that for each $h \in H(R)$, either $\mathfrak{d}(h) = 0$ or $h \in Z(R)$. Using Lemma 2.1, we have either $\mathfrak{d}(h) = 0$ for all $h \in H(R)$ or $H(R) \subseteq Z(R)$. In latter case, we have a contradiction by Lemma 2.13. In case $\mathfrak{d}(h) = 0$ for all $h \in H(R)$, we have

$$F(h^2) - F(h)h \in C, \quad \text{for all } h \in H(R). \tag{5.22}$$

Using (5.9) in (5.22), we conclude

$$\sigma(h)h \in C, \quad \text{for all } h \in H(R). \tag{5.23}$$

For any fixed $h \in H(R)$, using Lemma 2.3 in (5.23), we have either $\sigma(h) = 0$ or $h \in Z(R)$. Aid of Lemma 2.1, we have either $\sigma(h) = 0$ for all $h \in H(R)$ or $H(R) \subseteq Z(R)$. In the latter case, we have the desired result by using Lemma 2.13. In case $\sigma(h) = 0$, replacing x by h, where $h \in H(R)$ in (5.10), we get

$$qh = \lambda h \text{ for all } h \in H(R). \tag{5.24}$$

Replacing h by h_c in (5.24) and thereby using Lemma 2.2, we conclude $q = \lambda$. Replacing x by k in (5.10), where $k \in S(R)$, we find

$$\mathfrak{d}(k) = 2\lambda k + \sigma(k), \quad \text{for all } k \in S(R). \tag{5.25}$$

Using (B), we have

$$F(h \circ k) + F(h)k - F(k)h - h\mathfrak{d}(k) \in Z(R), \quad \text{for all } k \in S(R),\ h \in H(R). \tag{5.26}$$

Using (5.9) and (5.25) in (5.26), we conclude

$$\lambda(h \circ k) + \lambda hk - \lambda kh - \sigma(k)h - h(2\lambda k + \sigma(k)) \in C, \quad \text{for all } k \in S(R), h \in H(R).$$

It implies

$$\sigma(k)h \in C, \quad \text{for all } k \in S(R), h \in H(R). \tag{5.27}$$

If there exists $k \in S(R)$ such that $\sigma(k) \neq 0$, then using Lemma 2.3 in (5.27), we find $H(R) \subseteq Z(R)$. With the aid of Lemma 2.13, we get a contradiction. In case $\sigma(k) = 0$ for all $k \in S(R)$. From (5.9) and fact $\sigma(k) = \sigma(h) = 0$, we find $F(x) = F(h + k) = F(h) + F(k) = \lambda h + \lambda k = \lambda(h + k) = \lambda x$ for all $x \in R$. $\square$

Theorem 5.2. *Let R be a 2-torsion free prime ring with involution $'*'$ and $F : R \to R$ a centrally-extended generalized Jordan $*$-derivation with an associated centrally-extended Jordan $*$-derivation $\mathfrak{d}$. If $[F(x), x^*] \in Z(R)$ for all $x \in R$, then there exists $\lambda \in C$ such that $F(x) = \lambda x^*$ for all $x \in R$, unless R satisfies s_4.*

Proof. Suppose that R does not satisfies s_4. If $Z(R) = (0)$, then by Lemma 2.11, we find

$$F(x) = ax^* + \mathfrak{d}(x), \quad \text{for all } x \in R, \tag{5.28}$$

for some $a \in Q_s(R)$. From Lemma 2.5 and assumption, we find $[F(x), x^*] = 0$ for all $x \in R$. Applying involution, we obtain $[F(x)^*, x] = 0$ for all $x \in R$. By Lemma 2.10, we have

$$F(x)^* = \lambda x + \sigma(x), \quad \text{for all } x \in R.$$

It implies

$$F(x) = \lambda^* x^* + \sigma(x)^*, \quad \text{for all } x \in R. \tag{5.29}$$

By Proposition 4.2, Eq. (5.28) and (5.29), we conclude

$$\lambda^* x^* + \sigma(x)^* = ax^* + \mathfrak{d}(x), \quad \text{for all } x \in A. \tag{5.30}$$

Replacing x by 1 in (5.30), we obtain

$$a \in C. \tag{5.31}$$

Using (5.31) in (5.30), we obtain

$$[\mathfrak{d}(x), x^*] = 0, \quad \text{for all } x \in R.$$

Application of Lemma 2.8 implies $\mathfrak{d} = 0$ or R satisfy s_4 identity. For non trivial solution, we have $\mathfrak{d} = 0$. Thus using it in (5.28), we obtain $F(x) = ax^*$ for all $x \in R$, where $a \in C$ as desired.

Let $Z(R) \neq \{0\}$. Then there exists $0 \neq h_c \in Z(R)$ such that $h_c^* = h_c$. By the assumption, we have $[F(x), x^*] \in Z(R)$ for all $x \in R$. With the aid of Lemma 2.5 and Lemma 4.3, we have

$$[F(x), x^*] = 0, \quad \text{for all } x \in R. \tag{5.32}$$

Applying involution both sides in (5.32), we find

$$[F(x)^*, x] = 0, \quad \text{for all } x \in R. \tag{5.33}$$

Application of Lemma 2.10 in (5.33) implies

$$F(x)^* = \lambda x + \sigma(x), \quad \text{for all } x \in R, \tag{5.34}$$

for some $\lambda \in C$ and a mapping $\sigma : R \to C$. Applying involution both sides in (5.34), we get

$$F(x) = \lambda^* x^* + \sigma(x)^*, \quad \text{for all } x \in R. \tag{5.35}$$

From (B), we have

$$F(x \circ y) - F(x)y^* - F(y)x^* - x\mathfrak{d}(y) - y\mathfrak{d}(x) \in Z(R), \quad \text{for all } x, y \in R. \tag{5.36}$$

Using (5.35) in (5.36), we find

$$\lambda^*(x \circ y)^* - \lambda^*(x)^* y^* - \sigma(x)^* y^* - \lambda^* y^* x^* - \sigma(y)^* x^* - x\mathfrak{d}(y) - y\mathfrak{d}(x) \in C, .$$

for all $x, y \in R$. It implies

$$-\sigma(x)^*y^* - \sigma(y)^*x^* - x\mathfrak{d}(y) - y\mathfrak{d}(x) \in C, \quad \text{for all } x, y \in R. \tag{5.37}$$

Replacing x and y by h_c in (5.37), we find $h_c\mathfrak{d}(h_c) \in Z(R)$. It forces

$$\mathfrak{d}(h_c) \in Z(R). \tag{5.38}$$

Replacing y by h_c in (5.37) and using (5.38), we find

$$-\sigma(h_c)^*x^* - x\mathfrak{d}(h_c) - h_c\mathfrak{d}(x) \in Z(R), \quad \text{for all } x \in R. \tag{5.39}$$

Replacing x by h in (5.39), where $h \in H(R)$, we have

$$-\sigma(h_c)^*h - h\mathfrak{d}(h_c) - h_c\mathfrak{d}(h) \in C.$$

Commuting with x and using (5.38), we obtain

$$\mathfrak{d}(h_c)[h, x] + \sigma(h_c)^*[h, x] + h_c[\mathfrak{d}(h), x] = 0, \quad \text{for all } x \in R, h \in H(R). \tag{5.40}$$

Replacing h by h^2 in (5.40) and using it, we conclude

$$\mathfrak{d}(h)[h, x] + [h, x]\mathfrak{d}(h) = 0. \tag{5.41}$$

Polarizing (5.41), we find

$$\mathfrak{d}(h_1)[h, x] + \mathfrak{d}(h)[h_1, x] + [h, x]\mathfrak{d}(h_1) + [h_1, x]\mathfrak{d}(h) = 0, \quad \text{for all } h, h_1 \in H(R). \tag{5.42}$$

In particular, replacing h_1 by h_c in (5.42) to obtain

$$2\mathfrak{d}(h_c)[h, x] = 0, \quad \text{for all } h \in H(R). \tag{5.43}$$

If $\mathfrak{d}(h_c) \neq 0$, then using Lemma 2.2 and (5.38) in (5.43), we find $H(R) \subseteq Z(R)$. With the aid of Lemma 2.13, we have contradiction.

In case $\mathfrak{d}(h_c) = 0$, from (5.39), we find $[\mathfrak{d}(x), x^*] = 0$ for all $x \in R$. For non trivial solution, Lemma 2.8 yields $\mathfrak{d} = 0$. Using it in (5.37), we get

$$\sigma(y)^*[x^*, y^*] = 0, \quad \text{for all } x, y \in R.$$

For any fixed $y \in R$, we have either $\sigma(y) = 0$ or $[y^*, R] = 0$. Application of Lemma 2.1 implies that either $\sigma = 0$ or $[y, R] = (0)$ for all $y \in R$. But R is noncommutative, so we left with $\sigma = 0$. Thus, from (5.35) we have $F(x) = ax^*$, where $a = \lambda^* \in C$. It completes the proof. $\square$

6. Hypercommuting Conditions

A pair of mappings f and g satisfying the condition $f(x)x - xg(x) = 0$ (resp. $f(x)x^* - x^*g(x) = 0$) on an appropriate subset K of a ring (resp. ring with involution) R is called hypercommuting (resp. $*$-hypercommuting). Obviously, it is a more general concept than that of commuting ($*$-commuting) mappings. In this section, we study a pair (F, G) of centrally-extended generalized Jordan $*$-derivations which is hypercommuting or $*$-hypercommuting.

Theorem 6.1. *Let R be a 2-torsion free prime ring with involution $'*'$ and $F, G : R \to R$ are centrally-extended generalized Jordan $*$-derivations with an associated centrally-extended Jordan $*$-derivations $\mathfrak{d}, \mathfrak{g}$, respectively. If $F(x)x - xG(x) = 0$ for all $x \in R$, then there exits $\lambda \in C$ such that $F(x) = G(x) = \lambda x$ for all $x \in R$, unless R satisfies s_4.*

Proof. Assume that R does not satisfy s_4. By the hypothesis, we have

$$F(x)x - xG(x) = 0, \quad \text{for all } x \in R.$$

Suppose that $Z(R) = (0)$. Clearly in this case F and G becomes generalized Jordan $*$-derivations. Thus with the aid of Lemma 2.11, we have $F(x) = ax^* + \mathfrak{d}(x)$, $G(x) = bx^* + \mathfrak{g}(x)$ for all $x \in R$, where $a, b \in Q_s(R)$. Using it in our hypothesis, we find

$$(ax^* + \mathfrak{d}(x))x - x(bx^* + \mathfrak{g}(x)) = 0, \quad \text{for all } x \in R.$$

The fact of Lemma 2.16 yields $\mathfrak{d}(x) = xc - cx^*$, $\mathfrak{g}(x) = xd - dx^*$ for all $x \in R$ for some $c, d \in Q_s(R)$. In this view it follows that R satisfies the functional identity

$$(ax^* + xc - cx^*)x = x(bx^* + xd - dx^*), \quad \text{for all } x \in R. \tag{6.1}$$

Application of Lemma 2.6 in (6.1) yields

$$(ax^* + xc - cx^*)x = x(bx^* + xd - dx^*), \quad \text{for all } x \in A. \tag{6.2}$$

Replacing x by 1 in (6.2), we find $a = b$. Polarizing (6.2), we obtain $(ax^* + xc - cx^*)y + (ay^* + yc - cy^*)x = y(bx^* + xd - dx^*) + x(by^* + yd - dy^*)$ for all $x, y \in A$. Replacing y by 1, we get $ax^* + xc - cx^* + ax = bx^* + xd - dx^* + xb$ for all $x, y \in A$. By using the fact $a = b$ in preceding equation, we find

$$xc - cx^* - xd + dx^* + [a, x] = 0, \quad \text{for all } x \in A. \tag{6.3}$$

Replacing x by h in (6.3), where $h \in H(R)$, to obtain

$$-[c, h] + [d, h] + [a, h] = 0. \tag{6.4}$$

It implies $[d - c + a, h] = 0$. With the aid of Lemma 2.17, we have $d - c + a \in C$. Replacing x by k in (6.3), where $k \in S(R)$ and thereby using the fact $d - c + a \in C$, we find $k \circ (c - d) + [c - d, k] = 0$ for all $k \in S(R)$. It implies

$$(c - d)k = 0, \quad \text{for all } k \in S(R). \tag{6.5}$$

Replacing k by $h \circ k$ in (6.5), where $h \in H(R)$, $k \in S(R)$ and using (6.5), we find

$$(c - d)hk = 0, \quad \text{for all } h \in H(R), k \in S(R). \tag{6.6}$$

From (6.5) and (6.6), we get

$$(c - d)[k, h] = 0, \quad \text{for all } k \in S(R), h \in H(R). \tag{6.7}$$

Replacing k by $k \circ h_1$ in (6.7), we obtain

$$(c - d)h_1[k, h] + (c - d)[k, h]h_1 + (c - d)k[h, h_1] + (c - d)[h, h_1]k = 0. \tag{6.8}$$

Using (6.5) and (6.7) in (6.8), we find

$$(c-d)h_1[k,h]=0, \quad \text{for all } k\in S(R), h, h_1\in H(R). \tag{6.9}$$

Equation (6.5) also implies

$$(c-d)k_1[k,h]=0, \quad \text{for all } k, k_1\in S(R), h\in H(R). \tag{6.10}$$

From (6.9) and (6.10), we have

$$\begin{aligned} 2(c-d)x[k,h] &= (c-d)(2x)[k,h] \\ &= (c-d)(h_1+k_1)[k,h] \\ &= (c-d)h_1[k,h]+(c-d)k_1[k,h] \\ &= 0, \quad \text{for all } x\in R, h\in H(R), k\in S(R). \end{aligned}$$

Thus, we have $(c-d)R[k,h]=(0)$ for all $k\in S(R)$ and $h\in H(R)$. In case $c\neq d$, primeness of R implies $[H(R),S(R)]=(0)$. Lemma 2.18 leads us to the contradiction. In case $c=d$, we find $\mathfrak{d}=\mathfrak{g}$. Using the fact $a=b$, we conclude $F(x)=G(x)$ for all $x\in R$. Thus, we have the desired result by Theorem 5.1.

Let $Z(R)\neq(0)$. Then there exists $0\neq h_c\in Z(R)$ such that $h_c^*=h_c$. By the assumption, we have

$$F(x)x-xG(x)=0, \quad \text{for all } x\in R. \tag{6.11}$$

Replacing x by h_c in (6.11), we find

$$F(h_c)-G(h_c)=0. \tag{6.12}$$

Proposition 4.1 yields

$$F(x)=q_1x^*+\mathfrak{d}(x), \quad \text{for all } x\in R, \tag{6.13}$$

$$G(x)=q_2x^*+\mathfrak{g}(x), \quad \text{for all } x\in R, \tag{6.14}$$

for some $q_1,q_2\in Q_s(R)$. Using (6.13) and (6.14) in (6.12), we obtain

$$(q_1-q_2)(h_c)+(\mathfrak{d}-\mathfrak{g})(h_c)=0. \tag{6.15}$$

Polarizing (6.11) to obtain

$$F(x)y-yG(x)+F(y)x-xG(y)=0, \quad \text{for all } x,y\in R. \tag{6.16}$$

Replacing y by h_c in (6.16) to obtain

$$(F(x)-G(x))h_c+F(h_c)x-xG(h_c)=0, \quad \text{for all } x\in R. \tag{6.17}$$

Using (6.12) in (6.17), we have

$$(F(x)-G(x))h_c+[F(h_c),x]=0, \quad \text{for all } x\in R. \tag{6.18}$$

Replacing h_c by h_c^2 in (6.18) and using Lemma 2.2, we conclude

$$[\mathfrak{d}(h_c),x]=0, \quad \text{for all } x\in R. \tag{6.19}$$

It implies $\mathfrak{d}(h_c) \in Z(R)$. In the same way, we compute $\mathfrak{g}(h_c) \in Z(R)$. Using (6.13) and (6.19) in (6.18), we obtain

$$(F(x) - G(x) + [q_1, x])h_c = 0, \quad \text{for all } x \in R.$$

It implies

$$F(x) - G(x) + [q_1, x] = 0, \quad \text{for all } x \in R. \tag{6.20}$$

Since F and G are centrally-extended generalized Jordan $*$-derivation, replacing x by $h \circ h_c$ in (6.20) to obtain

$$\begin{aligned} &(F(h) - G(h))h_c + (F(h_c) - G(h_c))h + h_c(\mathfrak{d}(h) - \mathfrak{g}(h)) \\ &+ h(\mathfrak{d}(h_c) - \mathfrak{g}(h_c)) + 2[q, h]h_c \in Z(R), \quad \text{for all } h \in H(R). \end{aligned} \tag{6.21}$$

Using (6.12) and (6.20) in (6.21), we see that

$$(\mathfrak{d}(h) - \mathfrak{g}(h))h_c + [q, h]h_c + h(\mathfrak{d}(h_c) - \mathfrak{g}(h_c)) \in Z(R), \quad \text{for all } h \in H(R). \tag{6.22}$$

Replacing h by h^2 in (6.22), we obtain

$$\begin{aligned} &(\mathfrak{d}(h) - \mathfrak{g}(h))hh_c + h(\mathfrak{d}(h) - \mathfrak{g}(h))h_c + [q_1, h]hh_c + h[q_1, h]h_c \\ &+ h^2(\mathfrak{d}(h_c) - \mathfrak{g}(h_c)) \in Z(R), \quad \text{for all } h \in H(R). \end{aligned} \tag{6.23}$$

From (6.22) and (6.23), we conclude

$$h^2(\mathfrak{d}(h_c) - \mathfrak{g}(h_c)) \in Z(R). \tag{6.24}$$

If $\mathfrak{d}(h_c) - \mathfrak{g}(h_c) \neq 0$, then using Lemma 2.3 in (6.24), we have $h^2 \in Z(R)$ for all $h \in H(R)$. Thus, we have the result by Lemma 2.13. If $\mathfrak{d}(h_c) - \mathfrak{g}(h_c) = 0$, then from (6.15), we obtain

$$(q_1 - q_2)h_c = 0. \tag{6.25}$$

Using Lemma 2.2 in (6.25), we have $q_1 = q_2$. With the aid of (6.13), (6.14) and the fact $q_1 = q_2$ in (6.20), we find

$$\mathfrak{d}(x) - \mathfrak{g}(x) + [q_1, x] = 0, \quad \text{for all } x \in R. \tag{6.26}$$

Replacing x by $h \circ k$ where $h \in H(R)$, $k \in S(R)$ in (6.26), we obtain

$$\begin{aligned} &(\mathfrak{d}(h) - \mathfrak{g}(h))(-k) + k(\mathfrak{d}(h) - \mathfrak{g}(h)) + (\mathfrak{d}(k) - \mathfrak{g}(k))h + h(\mathfrak{d}(k) - \mathfrak{g}(k)) \\ &+ [q_1, h]k + k[q_1, h] + [q_1, k]h + h[q_1, k] \in Z(R). \end{aligned} \tag{6.27}$$

Using (6.26) in (6.27) to conclude

$$2[q_1, h]k \in Z(R), \quad \text{for all } h \in H(R), k \in S(R). \tag{6.28}$$

We now split the proof into the following two parts.

Case 1. If mapping induced on centroid is non identity map, then there exists $0 \neq z \in C$ such that $z^* \neq z$. Replacing h by $x + x^*$ and k by $y - y^*$ in (6.28), where $x, y \in R$ in order to obtain

$$2[q_1, x + x^*](y - y^*) \in Z(R), \quad \text{for all } x, y \in R. \tag{6.29}$$

With the aid of Lemma 2.6 in (6.29), we have

$$[q_1, x + x^*](y - y^*) \in Z(R), \quad \text{for all } x, y \in Q_s(R). \tag{6.30}$$

Replace y by z in (6.30), we have

$$[q_1, x + x^*](z - z^*) \in C, \quad \text{for all } x \in Q_s(R). \tag{6.31}$$

Using Lemma 2.3 in (6.31), we have

$$[q_1, x + x^*] \in Z(R), \quad \text{for all } x \in Q_s(R). \tag{6.32}$$

Replacing x by h, where $h \in H(R)$ in (6.32), we obtain $[q_1, h] \in Z(R)$. Using Lemma 2.17, we conclude $q_1 \in Z(R)$. From (6.20), we have $F(x) = G(x)$ for all $x \in R$. Hence, by Theorem 5.1, we get the desired result.

Case 2. If mapping induced on centroid is an identity map, then $c^* = c$ for all $c \in C$. From (6.28), we have

$$([q_1, h]k)^* = [q_1, h]k, \quad \text{for all } h \in H(R), k \in S(R).$$

It implies

$$[q_1, h]k - k[{q_1}^*, h] = 0, \quad \text{for all } h \in H(R), k \in S(R). \tag{6.33}$$

Replacing k by $k \circ h_1$ in (6.33), where $k \in S(R)$, $h_1 \in H(R)$, we obtain

$$[q_1, h]h_1k + [q_1, h]kh_1 - kh_1[{q_1}^*, h] - h_1k[{q_1}^*, h] = 0. \tag{6.34}$$

Using (6.28) and (6.33) in (6.34) to conclude

$$[q_1, h]h_1k - kh_1[{q_1}^*, h] = 0, \quad \text{for all } h, h_1 \in H(R), k \in S(R). \tag{6.35}$$

Replacing h_1 by k_1^2 in (6.35), where $k_1 \in S(R)$, we find

$$([q_1, h]k_1)k_1k - kk_1(k_1[{q_1}^*, h]) = 0.$$

It implies

$$([q_1, h]k_1)k_1k - kk_1([q_1, h]k_1)^* = 0, \quad \text{for all } h \in H(R), k, k_1 \in S(R). \tag{6.36}$$

Using (6.28) and (6.33) in (6.36), we obtain

$$[q_1, h]k_1[k_1, k] = 0. \tag{6.37}$$

For fixed $k_1 \in S(R)$, from the fact $[q_1, h]k_1 \in C$, using Lemma 2.2 in (6.37), we have either $[q_1, h]k_1 = 0$ for all $h \in H(R)$ or $[k_1, k] = 0$ for all $k \in S(R)$. Invoking Lemma 2.1 yields that either $[q_1, h]k_1 = 0$ for all $h \in H(R)$, $k_1 \in S(R)$ or $[k_1, k] = 0$ for all $k, k_1 \in S(R)$. In the latter case, replace k_1 by $h \circ k$ to obtain $[h, k]k + k[h, k] = 0$ for all $h \in H(R)$, $k \in S(R)$. For any fixed $h \in H(R)$, we obtain $d(k)k + kd(k) = 0$, where $d(x) = [h, x]$ for all $k \in S(R)$. With the aid of Lemma 2.15, we find either R satisfies s_4 identity or $d = 0$. In case $d = 0$, we have $H(R) \subseteq Z(R)$. Application of Lemma 2.13 gives a contradiction. If $[q_1, h]k_1 = 0$, then using the similar arguments as in (6.5), we find $[q_1, H(R)] = (0)$. With the aid of Lemma 2.17, we have $q_1 \in C$. From (6.20), we have $F(x) = G(x)$ for all $x \in R$. Thus, we get the desired conclusion from Theorem 5.1. □

Theorem 6.2. *Let R be a 2-torsion free prime ring with involution $'*'$ and $F, G : R \to R$ are centrally-extended generalized Jordan $*$-derivations associated with centrally-extended Jordan $*$-derivations $\mathfrak{d}, \mathfrak{g}$, respectively. If $F(x)x^* - x^*G(x) = 0$ for all $x \in R$, then there exists $\lambda \in C$ such that $F(x) = G(x) = \lambda x^*$, unless R satisfies s_4.*

Proof. Suppose that R does not satisfies s_4. By the hypothesis, we have $F(x)x^* - x^*G(x) = 0$ for all $x \in R$.

If $Z(R) = (0)$, then application of Lemma 2.11 yields $F(x) = ax^* + \mathfrak{d}(x)$, $G(x) = bx^* + \mathfrak{g}(x)$ for all $x \in R$, where $a, b \in Q_s(R)$. Using it in our hypothesis, we find

$$(ax^* + \mathfrak{d}(x))x^* - x^*(bx^* + \mathfrak{g}(x)) = 0, \quad \text{for all } x \in R. \tag{6.38}$$

With the aid of Lemma 2.16, we conclude $\mathfrak{d}(x) = xc - cx^*$, $\mathfrak{g}(x) = xd - dx^*$ for all $x \in R$ for some $c, d \in U(R)$. Using it in (6.38), we find

$$(ax^* + xc - cx^*)x^* = x^*(bx^* + xd - dx^*), \quad \text{for all } x \in R. \tag{6.39}$$

Using Lemma 2.6 in (6.39), we conclude

$$(ax^* + xc - cx^*)x^* = x^*(bx^* + xd - dx^*), \quad \text{for all } x \in A. \tag{6.40}$$

Replacing x by 1 in (6.40), we get $a = b$. Polarizing (6.40), we obtain

$$\begin{aligned}&(ax^* + xc - cx^*)y^* + (ay^* + yc - cy^*)x^* \\ =&y^*(bx^* + xd - dx^*) + x^*(by^* + yd - dy^*), \quad \text{for all } x, y \in R.\end{aligned} \tag{6.41}$$

Replacing y by 1 in (6.41), we find

$$ax^* + xc - cx^* + ax^* = bx^* + xd - dx^* + x^*b, \quad \text{for all } x, y \in R. \tag{6.42}$$

Using the fact $a = b$ in (6.42), we conclude

$$xc - cx^* - xd + dx^* + [a, x^*] = 0, \quad \text{for all } x \in R. \tag{6.43}$$

Replacing x by h in (6.43), where $h \in H(R)$, we obtain

$$-[c, h] + [d, h] + [a, h] = 0. \tag{6.44}$$

Since (6.44) is the same as (6.4), using similar arguments, we can reach our conclusion.

Let $Z(R) \neq (0)$. Then there exists $0 \neq h_c \in Z(R)$ such that $h_c^* = h_c$. By the assumption, we have

$$F(x)x^* - x^*G(x) = 0, \quad \text{for all } x \in R. \tag{6.45}$$

Replacing x by h_c in (6.45), we obtain

$$F(h_c) - G(h_c) = 0. \tag{6.46}$$

Invoking Proposition 4.1, we have

$$F(x) = q_1x^* + \mathfrak{d}(x), \quad \text{for all } x \in R, \tag{6.47}$$

$$G(x) = q_2x^* + \mathfrak{g}(x), \quad \text{for all } x \in R, \tag{6.48}$$

for some $q_1, q_2 \in Q_s(R)$. Using (6.47) and (6.48) in (6.46), we obtain

$$(q_1 - q_2)(h_c) + (\mathfrak{d} - \mathfrak{g})(h_c) = 0. \tag{6.49}$$

Polarizing (6.45) to get

$$F(x)y^* - y^*G(x) + F(y)x^* - x^*G(y) = 0, \quad \text{for all } x, y \in R. \tag{6.50}$$

Replacing y by h_c in (6.50), we see that

$$(F(x) - G(x))h_c + F(h_c)x^* - x^*G(h_c) = 0, \quad \text{for all } x \in R. \tag{6.51}$$

Using (6.46) in (6.51), we have

$$(F(x) - G(x))h_c + [F(h_c), x^*] = 0, \quad \text{for all } x \in R. \tag{6.52}$$

Replacing h_c by h_c^2 in (6.52), we obtain

$$[\mathfrak{d}(h_c), x^*] = 0, \quad \text{for all } x \in R. \tag{6.53}$$

It implies $\mathfrak{d}(h_c) \in Z(R)$. In the same way, one can easily observe that $\mathfrak{g}(h_c) \in Z(R)$. With the aid of (6.47), (6.53) in (6.52), we have

$$(F(x) - G(x) + [q_1, x^*])h_c = 0, \quad \text{for all } x \in R. \tag{6.54}$$

It yields

$$F(x) - G(x) + [q_1, x^*] = 0, \quad \text{for all } x \in R. \tag{6.55}$$

Replacing x by $h \circ h_c$ in (6.55) where $h \in H(R)$, we obtain

$$\begin{aligned}&(F(h) - G(h))h_c + (F(h_c) - G(h_c))h + h_c(\mathfrak{d}(h) - \mathfrak{g}(h))\\ &+ h(\mathfrak{d}(h_c) - \mathfrak{g}(h_c)) + 2[q, h]h_c \in Z(R), \quad \text{for all } h \in H(R).\end{aligned} \tag{6.56}$$

Using (6.46) and (6.55) in (6.56), we conclude

$$(\mathfrak{d}(h) - \mathfrak{g}(h))h_c + [q, h]h_c + h(\mathfrak{d}(h_c) - \mathfrak{g}(h_c)) \in Z(R), \quad \text{for all } h \in H(R). \tag{6.57}$$

Replacing h by h^2 in (6.57), we obtain

$$\begin{aligned}&(\mathfrak{d}(h) - \mathfrak{g}(h))hh_c + h(\mathfrak{d}(h) - \mathfrak{g}(h))h_c + [q_1, h]hh_c + h[q_1, h]h_c\\ &+ h^2(\mathfrak{d}(h_c) - \mathfrak{g}(h_c)) \in Z(R), \quad \text{for all } h \in H(R).\end{aligned} \tag{6.58}$$

From (6.57) and (6.58), we conclude

$$h^2(\mathfrak{d}(h_c) - \mathfrak{g}(h_c)) \in Z(R). \tag{6.59}$$

If $\mathfrak{d}(h_c) - \mathfrak{g}(h_c) \neq 0$, then using Lemma 2.3 in (6.59), we have $h^2 \in Z(R)$ for all $h \in H(R)$. Thus, a contradiction follows from Lemma 2.13. In case $\mathfrak{d}(h_c) - \mathfrak{g}(h_c) = 0$, from (6.49), we obtain

$$(q_1 - q_2)h_c = 0,$$

and it implies $q_1 = q_2$. Now from (6.47), (6.48) and (6.55) we find

$$\mathfrak{d}(x) - \mathfrak{g}(x) + [q_1, x^*] = 0, \quad \text{for all } x \in R. \tag{6.60}$$

Replacing x by $h \circ k$ in (6.60), where $h \in H(R),\ k \in S(R)$, we find

$$\begin{aligned}&(\mathfrak{d}(h) - \mathfrak{g}(h))(-k) + k(\mathfrak{d}(h) - \mathfrak{g}(h)) + (\mathfrak{d}(k) - \mathfrak{g}(k))h + h(\mathfrak{d}(k)\\ &- \mathfrak{g}(k)) - [q_1, h]k - k[q_1, h] - [q_1, k]h - h[q_1, k] \in Z(R).\end{aligned} \tag{6.61}$$

Using (6.60) in (6.61), we conclude

$$-2k[q_1, h] \in Z(R), \qquad \text{for all } h \in H(R), k \in S(R). \tag{6.62}$$

As (6.62) is same as (6.28), similar arguments are taking us to the desired conclusion. □

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[1]Department of Mathematics,
Akal University,
Talwandi Sabo-151302, India.
Email address: bharat_rs18@pbi.ac.in
ORCID iD: https://orcid.org/0000-0001-6127-0731

[2]Department of Mathematics,
Patel Memorial National College,
Rajpura-140401, India
Email address: gurninder_rs@pbi.ac.in
ORCID iD: https://orcid.org/0000-0001-8618-6325

[3]Department of Mathematics,
Aligarh Muslim University,
Aligarh-202002, India
Email address: asmith@example.com
ORCID iD: https://orcid.org/0000-0001-5162-7522

[4]Federal University of Technology-Paraná-Brazil,
[5]Federal University of ABC,
[6]University of São Paulo, Brazil
Email address: brunoferreira@utfpr.edu.br
ORCID iD: https://orcid.org/0000-0003-1621-8197