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THE NEUBERGER SPECTRA OF NONLINEAR SUPERPOSITION OPERATORS IN THE SPACES OF SEQUENCES

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ABSTRACT. In this paper we consider the nonlinear superposition operator F in l_p spaces of sequences, generated by the function

$$f(s,u) = a(s) + u^n \quad \text{or} \quad f(s,u) = a(s) \cdot u^n$$

First we show that these operators are Fréchet differentiable. Then we find out the Neuberger spectra $\sigma_N(F)$ of these operators. We compare it with some other nonlinear spectra and indicate some possible applications.

1. Introduction and Preliminaries

In the last 50 years there have been presented several ways of defining and studying spectra for nonlinear operators ([2], [8], [9]). One of them was introduced by J.W.Neuberger in 1969.([4]) for the class of continuously Fréchet differentiable operators. The Neuberger spectrum of nonlinear operators shares some properties with the usual spectrum of bounded linear operators, such as: it is always nonempty if the underlying space is complex and it always contains the eigenvalues of an operator which keeps zero fixed. In this paper we are finding out the Neuberger spectrum of some nonlinear superposition operators in l_p spaces of sequences. The superposition operator plays an important role in numerous mathematical investigations. The Neuberger spectrum may be useful in solvability of certain operator equations and eigenvalue problems ([4]). First, let us introduce some preliminary definitions and facts for nonlinear superposition operators in Banach spaces l_p.

Let $f = f(s,u)$ be a function defined on $\mathbb{N} \times \mathbb{R}$ (or $\mathbb{N} \times \mathbb{C}$) with the values in $\mathbb{R}$ (or respectively $\mathbb{C}$). Given a function $x = x(s)$, by applying f, we get another

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function $y = y(s)$ on $\mathbb{N}$ by:

$$y(s) = f(s, x(s)).$$

In this way, the function f generates an operator F :

$$Fx(s) = f(s, x(s)), \tag{1.1}$$

which is usually called superposition operator, Nemytskij operator or composition operator ([3], [1]).

We are going to observe the operator of superposition, defined in the spaces of sequences l_p $(1 \leqslant p \leqslant \infty)$.

Theorem 1.1. (*see* [1]) *Let* $1 \leqslant p, q < \infty$. *Then the following properties are equivalent:*

- *the operator* F *acts from* l_p *to* l_q;
- *there are functions* $a(s) \in l_q$ *and constants* $\delta > 0, n \in \mathbb{N}, b \geqslant 0$, *for which* $|f(s, u)| \leqslant a(s) + b|u|^{\frac{p}{q}}$ $(s \geqslant n, |u| < \delta)$;
- *for any* $\varepsilon > 0$ *there exists a function* $a_\varepsilon \in l_q$ *and constants* $\delta_\varepsilon > 0, n_\varepsilon \in \mathbb{N}, b_\varepsilon \geqslant 0$, *for which* $\|a_\varepsilon(s)\|_q < \varepsilon$ *and*

$$|f(s, u)| \leqslant a_\varepsilon(s) + b_\varepsilon |u|^{\frac{p}{q}} \quad (s \geqslant n_\varepsilon, \ |u| \leqslant \delta_\varepsilon).$$

Theorem 1.2. ([1], [7]) *Let* $1 \leqslant p, q < \infty$ *and let the superposition operator* (1.1), *generated by the function* $f(s, u)$, *act from* l_p *to* l_q. *Then this operator is continuous if and only if each of the functions is continuous for every* $s \in \mathbb{N}$.

In the sequence, X and Y denote Banach spaces and $\mathbb{K}$ is a field of real or complex numbers.

Definition 1.1. ([2], [3])An operator $F : X \to Y$ is called Fréchet differentiable at $x_0 \in X$ if there is an linear bounded operator $L : X \to Y$ such that

$$\lim_{\|h\| \to 0} \frac{1}{\|h\|} \|F(x_0 + h) - F(x_0) - Lh\| = 0 \quad (h \in X). \tag{1.2}$$

In this case this linear operator L is called Fréchet derivative of F at x_0 and denoted by $F'(x_0)$. The value $F'(x_0)x \in Y$ for arbitrary $x \in X$, is called Fréchet derivative of operator F at x_0 along x.

If F is differentiable at each point $x \in X$ and the map $x \mapsto F'(x)$ is continuous, we write $F \in \mathfrak{C}^1(X, Y)$ and call F continuously differentiable.

Theorem 1.3. ([6], [7]) *Let* $1 \leqslant p, q < \infty$ *and the operator* F *generated by the function* $f(s, u)$ *acts from* l_p *into* l_q. *The operator* F *is differentiable at* $x_0 \in l_p$ *if and only if* $f'_u(s, \cdot)$ *is continuous at* x_0 *for almost all* $s \in \mathbb{N}$.

More informations on Fréchet differentiable operators may be found in [10]. For the continuously differentiable operators $F \in \mathfrak{C}^1(X)$, Neuberger introduced the Neuberger resolvent set and spectrum.

DEFINITION 1.2. ([2], [4]) Let an operator $F : X \to X$ admit at each point $x \in X$ a Fréchet derivative $F'(x)$ which depends continuously (in the operator norm) on x. The set

$$\rho_N(F) = \left\{\lambda \in \mathbb{K} : \lambda I - F \text{ is bijective and } (\lambda I - F)^{-1} \in \mathfrak{C}^1(X)\right\}$$

is called Neuberger resolvent set, and the set

$$\sigma_N(F) = \mathbb{K} \diagdown \rho_N(F)$$

is called Neuberger spectrum of F.

REMARK 1.1. A point $\lambda \in \mathbb{K}$ belongs to $\rho_R(F)$ if and only if $\lambda I - F$ is a diffeomorphism on X.

2. Fréchet differentiability

As the Neuberger spectrum deals with Fréchet differentiable operators, in this section we will first investigate differentiability of some superposition operators, according to the Definition 1.1.

I) Find out if operator $F : l_p \to l_q$, generated by the function $f(s,u) = u^2$, is differentiable. For arbitrary $x_0 = (x_1, x_2, ...) \in l_p$ and $h = (h_1, h_2, ...) \in l_p$ we have:

$$\begin{aligned} I &= \lim_{\|h\|_p \to 0} \frac{1}{\|h\|_p} \|F(x_0 + h) - F(x_0) - Lh\|_q = \\ &= \lim_{\|h\|_p \to 0} \frac{1}{\|h\|_p} \left\| \left((x_1 + h_1)^2 - x_1^2, (x_2 + h_2)^2 - x_2^2, ... \right) - Lh \right\|_q = \\ & \lim_{\|h\| \to 0} \frac{1}{\|h\|_p} \left\| \left(2x_1h_1 + h_1^2, 2x_2h_2 + h_2^2, ...\right) - Lh \right\|_q . \end{aligned}$$

If we take linear bounded operator $L : l_p \to l_q$ to be a multiplication operator

$$Lh(s) = a(s)h(s) = 2x_sh_s, \text{ i.e.} \qquad Lh = (2x_1h_1, 2x_2h_2, ...),$$

then

$$\begin{aligned} I &= \lim_{\|h\|_p \to 0} \frac{1}{\|h\|_p} \left\| \left(2x_1h_1 + h_1^2, 2x_2h_2 + h_2^2, ...\right) - (2x_1h_1, 2x_2h_2, ...) \right\|_q \\ &= \lim_{\|h\|_p \to 0} \frac{1}{\|h\|_p} \left\| \left(h_1^2, h_2^2, ...\right) \right\|_q \end{aligned} \tag{2.1}$$

a) In case that operator F acts from l_2 to l_1, then

$$I = \lim_{\|h\|_2 \to 0} \frac{1}{(\sum\limits_{i=1}^{\infty} |h_i|^2)^{\frac{1}{2}}} \cdot \sum_{i=1}^{\infty} \left|h_i^2\right| = \lim_{\|h\|_2 \to 0} \left(\sum_{i=1}^{\infty} \left|h_i^2\right| \right)^{\frac{1}{2}} = \lim_{\|h\|_2 \to 0} \|h\|_2 = 0.$$

b) In case that operator F acts from l_1 to l_1, from (2.1) it follows:

$$\begin{aligned} I &= \lim_{\|h\|_p \to 0} \frac{1}{\|h\|_p} \left\| \left(h_1^2, h_2^2, ...\right) \right\|_q = \lim_{\|h\|_1 \to 0} \frac{1}{\|h\|_1} \left\| \left(h_1^2, h_2^2, ...\right) \right\|_1 \\ &= \lim_{\|h\|_1 \to 0} \frac{1}{\sum\limits_{i=1}^{\infty} |h_i|} \cdot \sum_{i=1}^{\infty} \left|h_i^2\right| \leqslant \lim_{\|h\|_1 \to 0} \frac{1}{\sum\limits_{i=1}^{\infty} |h_i|} \cdot \sum_{i=1}^{\infty} |h_i| \cdot \sum_{i=1}^{\infty} |h_i| \\ &= \lim_{\|h\|_1 \to 0} \sum_{i=1}^{\infty} |h_i| = \lim_{\|h\|_1 \to 0} \|h\|_1 = 0. \end{aligned}$$

c) In case that operator F acts from l_∞ to l_∞, from (2.1) it follows:

$$\begin{aligned} I &= \lim_{\|h\|_p \to 0} \frac{1}{\|h\|_p} \left\| \left(h_1^2, h_2^2, ...\right) \right\|_q = \lim_{\|h\|_\infty \to 0} \frac{1}{\|h\|_\infty} \left\| \left(h_1^2, h_2^2, ...\right) \right\|_\infty \\ &= \lim_{\|h\|_\infty \to 0} \frac{1}{\sup |h_i|} \cdot \sup \left|h_i^2\right| = \lim_{\|h\|_\infty \to 0} \frac{1}{\sup |h_i|} \cdot (\sup |h_i|)^2 \\ &= \lim_{\|h\|_\infty \to 0} \sup |h_i| = \lim_{\|h\|_\infty \to 0} \|h\|_\infty = 0. \end{aligned}$$

Anyway, operator F is differentiable (at every point x_0) and Fréchet derivative of operator F at $x_0 = (x_1, x_2, ...)$ along $h = (h_1, h_2, ...)$, is given with:

$$F'(x_0)\, h = (2x_1 h_1, 2x_2 h_2, ..., 2x_n h_n, ...) .$$

II) Let us see if operator $F : l_p \to l_q$, generated by the function $f(s, u) = u^3$, is differentiable. For arbitrary $x_0 = (x_1, x_2, ...) \in l_p$ and $h = (h_1, h_2, ...) \in l_p$ consider:

$$\begin{aligned} I &= \lim_{\|h\|_p \to 0} \frac{1}{\|h\|_p} \|F(x_0 + h) - F(x_0) - Lh\|_q = \\ &= \lim_{\|h\|_p \to 0} \frac{1}{\|h\|_p} \left\| \Big((x_1 + h_1)^3 - x_1^3, (x_2 + h_2)^3 - x_2^3, ... \Big) - Lh \right\|_q \\ &= \lim_{\|h\|_p \to 0} \frac{1}{\|h\|_p} \left\| \left(3x_1^2 h_1 + 3x_1 h_1^2 + h_1^3, 3x_2^2 h_2 + 3x_2 h_2^2 + h_2^3, ...\right) - Lh \right\|_q . \end{aligned}$$

If we assume that operator $L : l_p \to l_q$ is a linear bounded multiplication operator $Lh = L(h_1, h_2, ...) = \left(3x_1^2 h_1, 3x_2^2 h_2, ...\right)$, then we get

$$\begin{aligned} I = &\lim_{\|h\|_p \to 0} \frac{1}{\|h\|_p} \left\| \left(3x_1^2 h_1 + 3x_1 h_1^2 + h_1^3, 3x_2^2 h_2 + 3x_2 h_2^2 + h_2^3, ...\right) \right. \\ &\quad \left. - \left(3x_1^2 h_1, 3x_2^2 h_2, ...\right) \right\|_q \\ = &\lim_{\|h\|_p \to 0} \frac{1}{\|h\|_p} \left\| \left(3x_1 h_1^2 + h_1^3, 3x_2 h_2^2 + h_2^3, ...\right) \right\|_q \end{aligned} \tag{2.2}$$

If operator F acts from l_3 to l_1, then from (2.2) further we get:

$$\begin{aligned}
I &= \lim_{\|h\|_3\to 0}\frac{1}{\|h\|_3}\left\|\left(3x_1h_1^2+h_1^3,3x_2h_2^2+h_2^3,...\right)\right\|_1=\lim_{\|h\|_3\to 0}\frac{1}{(\sum\limits_{i=1}^{\infty}|h_i|^3)^{\frac{1}{3}}}\cdot\sum_{i=1}^{\infty}\left|3x_ih_i^2+h_i^3\right|\\
&\leqslant \lim_{\|h\|_3\to 0}\frac{1}{(\sum\limits_{i=1}^{\infty}|h_i|^3)^{\frac{1}{3}}}\cdot\left\{\sum_{i=1}^{\infty}\left|3x_ih_i^2\right|+\sum_{i=1}^{\infty}\left|h_i^3\right|\right\}\\
&= \lim_{\|h\|_3\to 0}\frac{1}{(\sum\limits_{i=1}^{\infty}|h_i|^3)^{\frac{1}{3}}}\cdot\left\{\sum_{i=1}^{\infty}\left|3x_ih_i^2\right|\right\}+\left\{\sum_{i=1}^{\infty}\left|h_i^3\right|\right\}^{\frac{2}{3}}\\
&= \lim_{\|h\|_3\to 0}\frac{1}{(\sum\limits_{i=1}^{\infty}|h_i|^3)^{\frac{1}{3}}}\cdot(\sum_{i=1}^{\infty}\left|3x_ih_i^2\right|).
\end{aligned}$$

Here, by applying Hölder inequality ([11]), we get:

$$\begin{aligned}
I &= 3\cdot\lim_{\|h\|_3\to 0}\frac{1}{(\sum\limits_{i=1}^{\infty}|h_i|^3)^{\frac{1}{3}}}\cdot(\sum_{i=1}^{\infty}\left|x_ih_i^2\right|)\leqslant\\
&\leqslant 3\cdot\lim_{\|h\|_3\to 0}\frac{1}{(\sum\limits_{i=1}^{\infty}|h_i|^3)^{\frac{1}{3}}}\cdot\left\{\sum_{i=1}^{\infty}\left|x_i^3\right|\right\}^{\frac{1}{3}}\cdot\left\{\sum_{i=1}^{\infty}\left|h_i^2\right|^{\frac{3}{2}}\right\}^{\frac{2}{3}}=\\
&= 3\left\{\sum_{i=1}^{\infty}\left|x_i^3\right|\right\}^{\frac{1}{3}}\cdot\lim_{\|h\|_3\to 0}\frac{1}{(\sum\limits_{i=1}^{\infty}|h_i|^3)^{\frac{1}{3}}}\cdot\left\{\sum_{i=1}^{\infty}|h_i|^3\right\}^{\frac{2}{3}}=\\
&= 3\left\|x_0\right\|\cdot\lim_{\|h\|_3\to 0}\left\{\sum_{i=1}^{\infty}|h_i|^3\right\}^{\frac{1}{3}}=0.
\end{aligned}$$

Hence, operator F is differentiable (at every point x_0) and its Fréchet derivative of operator F at $x_0=(x_1,x_2,...)$ along $h=(h_1,h_2,...)$, is:

$$F'(x_0)(h_1,h_2,...)=(3x_1^2h_1,3x_2^2h_2,...,3x_n^2h_n,...).$$

Generally,

Proposition 2.1. *Let a superposition operator $F:l_p\to l_q$ be generated by the function $f(s,u)=u^n, n\in\mathbb{N}, 1\leqslant p\leqslant nq\leqslant\infty$. It is a continuously Fréchet differentiable operator and its Fréchet derivative at $x_0\in l_p$ along h is given by:*

$$F'(x_0)(h_1,h_{2,},...)=(nx_1^{n-1}h_1,nx_2^{n-1}h_2,...,...).$$

III) If we have a superposition operator F generated by the function $f(s,u) = \sqrt[n]{u}$ then it is not differentiable at $x_0 = 0$. Indeed, the function $f'_u(s,u) = \frac{1}{n \cdot \sqrt[n]{u^{n-1}}}$ is not continuous in $u = 0$, so from the Theorem 1.3 it follows that this operator F is not (continuously) differentiable.

IV) Let us see if a superposition operator $F : l_1 \to l_1$, generated by $f(s,u) = \frac{1}{s(s+1)} + u^2$, is differentiable

For arbitrary $x_0 = (x_1, x_2, ...) \in l_1$, we have:

$$\begin{aligned} I &= \lim_{\|h\|_1 \to 0} \frac{1}{\|h\|_1} \|F(x_0+h) - F(x_0) - Lh\|_1 = \\ &= \lim_{\|h\|_1 \to 0} \frac{1}{\|h\|_1} \|F(x_1+h_1, x_2+h_2, ...) - F(x_1, x_2, ...) - Lh\|_1 = \end{aligned}$$

$$\begin{aligned} &\lim_{\|h\|_1 \to 0} \frac{1}{\|h\|_1} \left\| \left(\left(\frac{1}{2} + (x_1+h_1)^2, \frac{1}{6} + (x_2+h_2)^2, ... \right) - \left(\frac{1}{2} + x_1^2, \frac{1}{6} + x_2^2, .. \right) \right) - Lh \right\|_1 \\ = &\lim_{\|h\|_1 \to 0} \frac{1}{\|h\|_1} \left\| \left(2x_1h_1 + h_1^2, 2x_2h_2 + h_2^2, ... \right) - Lh \right\|_1 . \end{aligned}$$

If we take the operator L as

$$Lh = L(h_1, h_2, ..) = (2x_1h_1, 2x_2h_2, ...),$$

then I becomes:

$$\begin{aligned} I &= \lim_{\|h\|_1 \to 0} \frac{1}{\|h\|_1} \left\| \left(2x_1h_1 + h_1^2, 2x_2h_2 + h_2^2, ... \right) - (2x_1h_1, 2x_2h_2, ...) \right\|_1 = \\ &= \lim_{\|h\|_1 \to 0} \frac{1}{\|h\|_1} \left\| \left(h_1^2, h_2^2, ... \right) \right\|_1 = \lim_{\|h\|_1 \to 0} \frac{\sum_{i=1}^{\infty} |h_i|^2}{\|h\|_1} \leqslant \\ &\leqslant \lim_{\|h\|_1 \to 0} \frac{\sum_{i=1}^{\infty} |h_i| \cdot \sum_{i=1}^{\infty} |h_i|}{\|h\|_1} = \lim_{\|h\|_1 \to 0} \|h\|_1 = 0. \end{aligned}$$

This means that F is differentiable at every $x_0 = (x_1, x_2, ...) \in l_1$ and its derivative is given with: $F'(x_0)(h_1, h_2, ..) = (2x_1h_1, 2x_2h_2, ...)$.

We can see that Fréchet derivative of this operator is the same one as Fréchet derivative of an operator F generated by the function $f(s,u) = u^2$.

Generally,

PROPOSITION 2.2. *Let a superposition operator $G : l_p \to l_q\,(1 \leqslant p, q \leqslant \infty)$ be generated by the function $g(u)$ and a superposition operator $F : l_p \to l_q$ be generated by the function $f(s,u) = \varphi(s) + g(u)$, $(\varphi \in l_t,\ 1 \leqslant t \leqslant q)$. If operator G is differentiable at $x_0 \in l_p$ then operator F is also differentiable at x_0 and $F'(x_0)h = G'(x_0)h$.*

According to the Theorem 1.1, since operator G acts from l_p to l_q, there are $a \in l_q$ and constants $\delta > 0, n_0 \in \mathbb{N}, b \geqslant 0$ such that:

$$|g(u)| \leqslant a(s) + b \cdot |u|^{\frac{p}{q}}, \ (\forall s \geqslant n_0, |u| < \delta). \qquad (*)$$

Now we have

$$\begin{aligned} |f(s,u)| &= |\varphi(s) + g(u)| \leqslant |\varphi(s)| + |g(u)| \overset{(*)}{\leqslant} \\ |\varphi(s)| + a(s) + b \cdot |u|^{\frac{p}{q}} &= r(s) + b \cdot |u|^{\frac{p}{q}}, \end{aligned}$$

where $r(s) = |\varphi(s)| + a(s)$. As $1 \leqslant t \leqslant q$, it holds $l_t \subseteq l_q$, so $\varphi \in l_q$ (and also $|\varphi| \in l_q$). The sequence r is also from the space l_q since it is a sum of two sequences from l_q. We have shown that there are $r \in l_q$ and constants $\delta > 0, n_0 \in \mathbb{N}, b \geqslant 0$ such that:

$$|f(s,u)| \leqslant r(s) + b \cdot |u|^{\frac{p}{q}},$$

and from the Theorem 1.1, it means that operator F acts from l_p to l_q, indeed.

Example 2.1. Consider the superposition operator F generated by $f(s,u) = \frac{1}{2^s} \cdot u^2$,

$$Fx(s) = \frac{1}{2^s} \cdot x^2(s).$$

If $x = (x_1, x_2, ...) \in l_2$ then $Fx = F(x_1, x_2, ...) = \left(\frac{1}{2^1} \cdot x_1^2, \frac{1}{2^2} \cdot x_2^2, ...\right) \in l_1$. Really,

$$\sum_{s=1}^{\infty} \left| \frac{1}{2^s} \cdot x_s^2 \right| < \sum_{s=1}^{\infty} \left| x_s^2 \right| = \|x\|_2^2 < \infty,$$

so operator F acts from l_2 to l_1 (also it can be shown, by Theorem 1.1, that $F : l_1 \to l_1$). We are interested in differentiability of this operator. If we take arbitrary $x_0 = (x_1, x_2, ...)$ from l_2, then

$$\begin{aligned} I &= \lim_{\|h\|_2 \to 0} \frac{1}{\|h\|_2} \|F(x_0 + h) - F(x_0) - Lh\|_1 = \\ &= \lim_{\|h\|_2 \to 0} \frac{1}{\|h\|_2} \|F(x_1 + h_1, x_2 + h_2, ...) - F(x_1, x_2, ...) - Lh\|_1 = \end{aligned}$$

$$\begin{aligned} &= \lim_{\|h\|_2 \to 0} \frac{1}{\|h\|_2} \left\| \left(\frac{1}{2^1}(x_1 + h_1)^2, \frac{1}{2^2}(x_2 + h_2)^2, ... \right) - \left(\frac{1}{2^1} x_1^2, \frac{1}{2^2} x_2^2, ... \right) - Lh \right\|_1 \\ &= \lim_{\|h\|_2 \to 0} \frac{1}{\|h\|_2} \left\| \left(\frac{1}{2^1}\left(2x_1 h_1 + h_1^2\right), \frac{1}{2^2}\left(2x_2 h_2 + h_2^2\right), ... \right) - Lh \right\|_1. \end{aligned}$$

Now, if we take $L(h_1,h_2,...)=\left(\frac{2x_1}{2^1}h_1,\frac{2x_2}{2^2}h_2,...\right)=\left(\frac{x_1}{2^0}h_1,\frac{x_2}{2^1}h_2,...\right)$ we get

$$\begin{aligned}
I &= \lim_{\|h\|_2\to 0}\frac{1}{\|h\|_2}\left\|\left(\frac{1}{2^1}\left(2x_1h_1+h_1^2\right),\frac{1}{2^2}\left(2x_2h_2+h_2^2\right),...\right)-\left(\frac{x_1}{2^0}h_1,\frac{x_2}{2^1}h_2,...\right)\right\|_1 = \\
&= \lim_{\|h\|_2\to 0}\frac{1}{\|h\|_2}\left\|\left(\frac{h_1^2}{2^1},\frac{h_2^2}{2^2},\frac{h_3^2}{2^3},...\right)\right\|_1 = \lim_{\|h\|_2\to 0}\frac{\sum\limits_{i=1}^{\infty}\left|\frac{1}{2^i}\cdot h_i^2\right|}{\left(\sum\limits_{i=1}^{\infty}|h_i|^2\right)^{\frac{1}{2}}} \leqslant \\
&\leqslant \lim_{\|h\|_2\to 0}\frac{\frac{1}{2}\cdot\sum\limits_{i=1}^{\infty}\left|h_i^2\right|}{\left(\sum\limits_{i=1}^{\infty}|h_i|^2\right)^{\frac{1}{2}}} = \frac{1}{2}\cdot\lim_{\|h\|_2\to 0}\left(\sum_{i=1}^{\infty}|h_i|^2\right)^{\frac{1}{2}} = \frac{1}{2}\cdot\lim_{\|h\|_2\to 0}\|h\|_2 = 0.
\end{aligned}$$

It means this operator is Fréchet differentiable and its derivative at $x_0=(x_1,x_2,...)$ along h is:

$$F'(x_0)(h_1,h_2,...)=\left(\frac{x_1}{2^0}h_1,\frac{x_2}{2^1}h_2,...\right)=\left(\frac{1}{2^1}\cdot 2x_1h_1,\frac{1}{2^2}\cdot 2x_2h_2,...\right).$$

PROPOSITION 2.3. *Let a superposition operator $G: l_p\to l_q\,(1\leqslant p,q\leqslant\infty)$ be generated by the function $g(u)$. If operator G is a Fréchet differentiable operator at point $x_0\in l_p$, then operator $F: l_p\to l_q$, generated by the function $f(s,u)=\varphi(s)\cdot g(u)$, $(\varphi\in l_\infty)$, is also Fréchet differentiable operator at the same point x_0.*

Again, according to the Theorem 1.1, there are $a\in l_q$ and constants $\delta>0, n_0\in\mathbb{N}, b\geqslant 0$ such that $(*)$ holds. Now we have

$$\begin{aligned}
|f(s,u)| &= |\varphi(s)\cdot g(u)|\leqslant|\varphi(s)|\cdot|g(u)|\overset{(*)}{\leqslant} \\
|\varphi(s)|\cdot\left(a(s)+b\cdot|u|^{\frac{p}{q}}\right) &\leqslant |\varphi(s)|\cdot a(s)+|\varphi(s)|\cdot b\cdot|u|^{\frac{p}{q}} \qquad (**)
\end{aligned}$$

Since φ is bounded sequence there exists $\sup\limits_{s\in\mathbb{N}}|\varphi(s)|=C<\infty$. From $(**)$ by denoting $d(s)=C\cdot a(s)$ and $k=C\cdot b$, we get

$$|f(s,u)|\leqslant d(s)+k\cdot|u|^{\frac{p}{q}},\ (\forall s\geqslant n_0,|u|<\delta),$$

with $d\in l_q$, $k\geqslant 0$. This means that really operator F acts from l_p to l_q.

We have:

$$f_u'(s,u)=\varphi(s)\cdot g_u'(u)=\varphi(s)\cdot p(u),$$

which gives us a generator for linear bounded multiplication operator-Fréchet derivative at $x_0=(x_1,x_2,...)$:

$$F'(x_0)(h_1,h_2,...)=(\varphi(1)\cdot p(x_1)h_1,\varphi(2)\cdot p(x_2)h_2,...).$$

It is also known the following Theorem which gives us the necessary and sufficient conditions for the superposition operator (1.1) to be a Fréchet differentiable operator (see [1], [6], [7]):

THEOREM 2.1. *Let $f(s,u)$ be a Carathéodory function and operator F generated by the function $f(s,u)$ acts from l_p to l_q. If operator F is differentiable in $x_0 \in l_p$, then its (Fréchet) derivative in x_0 has the form*

$$F'(x_0)\, h(s) = a(s)\, h(s) \tag{2.3}$$

where $a \in l_q/l_p$ is given by

$$a(s) = \lim_{u\to 0} \frac{f(s, x_0(s) + u) - f(s, x_0(s))}{u} \tag{2.4}$$

If superposition operator G, generated by the function

$$g(s,u) = \begin{cases} \frac{1}{u}\left[f(s, x(s)+u) - f(s, x(s))\right]; & u \neq 0 \\ a(s) & ;\ u = 0, \end{cases}$$

acts from l_p to l_q/l_p, and it is continuous in 0, then F is differentiable in x_0 and formula (2.3) holds.

Here space l_q/l_p is the set of all multipliers $(a(s))$ from l_p to l_q. It is a Banach space of sequences, defined by

$$l_q/l_p = \begin{cases} l_{pq(p-q)^{-1}} & \text{for } p > q \\ l_\infty & \text{for } p \leqslant q. \end{cases} \tag{2.5}$$

3. The Neuberger spectrum

In this section we are going to find out the Neuberger spectrum of some nonlinear superposition operators.

First we will consider the superposition operator F generated by the function $f(s,u) = a(s) + u^n$, $n \in \mathbb{N}$, where $(a(s))_{s\in\mathbb{N}}$ is a sequence from the space l_p $(1 \leqslant p \leqslant \infty)$. Since $a \in l_p \subset l_\infty$, we can see that operator F can act from l_∞ to l_∞ or according to the Theorem 1.1, F can act from l_p to l_p.

a) Case $1 \leqslant p < \infty$

$$|f(s,u)| = |a(s) + u^n| \leqslant |a(s)| + |u^n|.$$

For $|u| < 1$ we have $|u^n| < |u|$, so we get

$$|f(s,u)| \leqslant |a(s)| + |u^n| < d(s) + |u|, \qquad (\triangle)$$

where $d(s) = |a(s)|$. So, there exists $d \in l_p$ and constants $\delta = 1, n_0 = 1, b = 1$ such that $\forall s \geqslant n_0, |u| < \delta$ inequality $(\triangle)$ holds. From the Theorem 1.1 it follows that $F : l_p \to l_p$.

b) Case $p = l_\infty$

For arbitrary $x = (x_1, x_2, ...) \in l_\infty \implies \exists \sup\limits_{s\in\mathbb{N}} x_s = B < \infty$; also $a \in l_\infty \implies \exists \sup\limits_{s\in\mathbb{N}} a(s) = A < \infty$.

$$\begin{aligned} Fx &= (a(1) + x_1^n, a(2) + x_2^n, ...) \\ \sup_{s\in\mathbb{N}} |Fx(s)| &= \sup_{s\in\mathbb{N}} |a(s) + x_s^n| \leqslant \sup_{s\in\mathbb{N}} |a(s)| + \sup_{s\in\mathbb{N}} |x_s^n| = A + B^n < \infty. \end{aligned}$$

We see that $Fx \in l_\infty$, so indeed F acts from l_∞ to l_∞.

From the Proposition 2.1 and the Proposition 2.2, we see it is continuously differentiable operator $\left(F \in \mathfrak{C}^1(l_p)\right)$ with

$$F'(x_0)(h_1, h_2, ..) = \left(nx_1^{n-1}h_1, nx_2^{n-1}h_2, ...\right).$$

We can also write

$$F'(x_0)\, h(s) = b(s)\, h(s),$$

where $b(s) = n \cdot (x_0(s))^{n-1}$ is a multiplier from l_p to l_p. Since $x_0 \in l_p \subset l_\infty$, it is clear that $b \in l_\infty$. Compare with Theorem 2.1, (2.3) and (2.5).

Lemma 3.1. *Let the superposition operator $F : l_p \to l_p$ be generated by the function $f(s,u) = a(s) + u^n$, where n is an even number and $(a(s))_s$ is a sequence from the space l_t $(1 \leqslant t \leqslant p \leqslant \infty)$. Then the Neuberger spectrum of F is $\sigma_N(F) = \mathbb{R}$ (or $\sigma_N(F) = \mathbb{C}$).*

Proof. Denote $a = (a_1, a_2, ...) \in l_p$. For $x = (x_1, x_2, ...)$ we have

$$Fx = F(x_1, x_2, ...) = (a_1 + x_1^n, a_2 + x_2^n, ...).$$

Find out if $\lambda I - F$ is an injective operator, for any real λ. Suppose that

$$(\lambda I - F)x = (\lambda I - F)y,$$

for some $x, y \in l_p$. Then

$$(\lambda x_1 - a_1 - x_1^n, \lambda x_2 - a_2 - x_2^n, ...) = (\lambda y_1 - a_1 - y_1^n, \lambda y_2 - a_2 - y_2^n, ...) \tag{3.1}$$

For $\lambda = 0$ we get

$$\begin{aligned}
(-a_1 - x_1^n, -a_2 - x_2^n, ...) &= (-a_1 - y_1^n, -a_2 - y_2^n, ...) \implies \\
(\forall i &\in N) - a_i - x_i^n = -a_i - y_i^n \implies \\
(\forall i &\in N) \;\; x_i^n = y_i^n.
\end{aligned}$$

Number n is an even number, so it does not have to follow $x_i = y_i, (\forall i \in N)$. This is not injective (nor bijective) mapping so $0 \in \sigma_N(F)$. If $\lambda \neq 0$ then from equality (3.1) we get $(\forall i \in N)$:

$$\begin{aligned}
\lambda x_i - a_i - x_i^n &= \lambda y_i - a_i - y_i^n \\
\lambda x_i - x_i^n &= \lambda y_i - y_i^n \iff \lambda(x_i - y_i) = x_i^n - y_i^n \\
\lambda(x_i - y_i) &= (x_i - y_i)\left(x_i^{n-1} + x_i^{n-2}y_i + ... + x_i y_i^{n-2} + y_i^{n-1}\right) \implies
\end{aligned}$$

$$(x_i = y_i) \;\; \vee \tag{3.2}$$

$$\left(x_i^{n-1} + x_i^{n-2}y_i + ... + x_i y_i^{n-2} + y_i^{n-1} - \lambda = 0\right). \tag{3.3}$$

Hence (3.3) is an odd-degree polynomial equation, there is always at least one real (nontrivial) solution and $\lambda I - F$ is not injective mapping. We proved that $\lambda I - F$ is not bijective mapping for any real λ. Operator F is a continuously differentiable operator (as we see from the Proposition 2.1 and the Proposition 2.2) and $\lambda I - F$ is not bijective for any real λ. Thus, according to the Definition 1.2, **the Neuberger spectrum of this operator F is** $\sigma_N(F) = \mathbb{R}$. In case that sequences were defined in $\mathbb{C}$ we would get the Neuberger spectrum $\sigma_N(F) = \mathbb{C}$. □

Lemma 3.2. *Let the superposition operator $F : l_p \to l_p$ be generated by the function $f(s,u) = a(s) + u^n$, where n is an odd number ($n \geqslant 3$) and $(a(s))_s$ is a sequence from the space l_t ($1 \leqslant t \leqslant p \leqslant \infty$).Then the Neuberger spectrum of F is $\sigma_N(F) = [0,\infty)$ (or $\sigma_N(F) = \mathbb{C}$).*

Proof. Consider a continuous superposition operator F defined in spaces of sequences l_p, by the function $f(s,u) = a(s) + u^n$, where $a \in l_p$ and n is an odd number.

$$Fx = F(x_1, x_2, ...) = (a_1 + x_1^n, a_2 + x_2^n, ...).$$

Consider now the operator

$$(\lambda I - F)(x_1, x_2, ...) = \ (\lambda x_1 - a_1 - x_1^n, \lambda x_2 - a_2 - x_2^n, ...).$$

For $\lambda = 0$, the operator $-F$ is injective, because from $-Fx = -Fy \iff (-a_1 - x_1^n, -a_2 - x_2^n, ...) = (-a_1 - y_1^n, -a_2 - y_2^n, ...)$, we get

$$-a_i - x_i^n = -a_i - y_i^n, \forall i \in \mathbb{N} \implies x_i^n = y_i^n, \forall i \in \mathbb{N} \implies x = y.$$

The operator $-F$ is surjective because for arbitrary $y \in l_q$ there are some $x \in l_p$ such that $-Fx = y$. Really:

$$\begin{array}{rcl} -Fx & = & (-a_1 - x_1^n, -a_2 - x_2^n, ...) = (y_1, y_2, ...) \iff \\ x & = & (\sqrt[n]{-a_1 - y_1}, \sqrt[n]{-a_2 - y_2}, ...). \end{array}$$

Let now $\lambda \neq 0$:

$$(\lambda I - F)(x_1, x_2, ...) = \ (\lambda I - F)(y_1, y_2, ...) \tag{3.4a}$$

$$(\lambda x_1 - a_1 - x_1^n, \lambda x_2 - a_2 - x_2^n, ...) = (\lambda y_1 - a_1 - y_1^n, \lambda y_2 - a_2 - y_2^n, ...)$$

$$\lambda x_i - a_i - x_i^n = \lambda y_i - a_i - y_i^n, \forall i \in \mathbb{N}$$

$$x_i^n - \lambda x_i = y_i^n - \lambda y_i, \forall i \in \mathbb{N}. \tag{3.4b}$$

$$(x_i^n - y_i^n) = \lambda (x_i - y_i), \forall i \in \mathbb{N}. \tag{3.4c}$$

From (3.4c) we get $(x_i = y_i)$ or

$$x_i^{n-1} + \ x_i^{n-2} y_i + ... + \ x_i \ \ y_i^{n-2} + \ \ y_i^{n-1} - \lambda = 0. \tag{3.5}$$

If $\lambda < 0$ then for $(x_i \geqslant 0 \wedge y_i \geqslant 0)$, or $(x_i \leqslant 0 \wedge y_i \leqslant 0)$, we have that

$$\begin{array}{rcl} x_i^{n-1} + \ x_i^{n-2} y_i + ... + \ x_i \ \ y_i^{n-1} + \ \ y_i^{n-1} & \geqslant & 0 \qquad \text{and} \\ x_i^{n-1} + \ x_i^{n-2} y_i + ... + \ x_i \ \ y_i^{n-1} + \ \ y_i^{n-1} - \lambda & > & 0. \end{array}$$

If $\lambda < 0$ and $(x_i \geqslant 0 \wedge y_i \leqslant 0)$ then: a) for $x_i \geqslant -y_i$ we have

$$\begin{array}{rcl} x_i^{n-1} & \geqslant & x_i^{n-2}(-y_i) \\ x_i^{n-3}y_i^2 & \geqslant & x_i^{n-4}(-y_i)^3 \\ & \vdots & \\ \end{array}$$
$$x_i^2 y_i^{n-3} \geqslant x_i (-y_i)^{n-2}$$

From these inequalities by summing we get

$$x_i^{n-1} + x_i^{n-3}y_i^2 + ... + x_i^2 y_i^{n-3} \geqslant -x_i^{n-2}y_i - x_i^{n-4}y_i^3 - ... - x_i y_i^{n-2} \implies$$
$$x_i^{n-1} + x_i^{n-2}y_i + x_i^{n-3}y_i^2 + x_i^{n-4}y_i^3 + ... x_i^2 y_i^{n-3} + x_i y_i^{n-2} \geqslant 0$$

By adding two members $y_i^{n-1} \geqslant 0$ and $-\lambda > 0$, to the left side, we get

$$x_i^{n-1} + \quad x_i^{n-2}y_i + ... + \ x_i \quad y_i^{n-2} + \quad y_i^{n-1} - \lambda > 0.$$

b) for $x_i \leqslant -y_i$, we have

$$\begin{array}{rcl} y_i^{n-1} & \geqslant & x_i(-y_i)^{n-2} \\ x_i^2 y_i^{n-3} & \geqslant & x_i^3(-y_i)^{n-4} \\ & \vdots & \\ \end{array}$$
$$x_i^{n-3}y_i^2 \geqslant x_i^{n-2}(-y_i)$$

From these inequalities by summing we get

$$y_i^{n-1} + x_i^2 y_i^{n-3} + ... + x_i^{n-3}y_i^2 \geqslant -x_i y_i^{n-2} - x_i^3 y^{n-4} - ... - x_i^{n-2}y_i \implies$$
$$y_i^{n-1} + x_i y_i^{n-2} + x_i^2 y_i^{n-3} + x_i^3 y^{n-4} + ... + x_i^{n-3}y_i^2 + x_i^{n-2}y_i \geqslant 0.$$

By adding two members $x_i^{n-1} \geqslant 0$ and $-\lambda > 0$, to the left side, we get

$$x_i^{n-1} + \quad x_i^{n-2}y_i + ... + \ x_i \quad y_i^{n-2} + \quad y_i^{n-1} - \lambda > 0.$$

If $\lambda < 0$ and $(x_i \leqslant 0 \wedge y_i \geqslant 0)$ we can analogously get the same inequality. So any way, from (3.4a) it follows that $x = y$ and $\lambda I - F$ is an injective operator (for $\lambda < 0$). We can see from the equations (3.4b) that this operator $\lambda I - F$ is injective if the operator $\lambda I - G$ (where G is operator generated by the function $g(s,u) = u^n$, (n is odd number)) is injective. Let us find out if the equation $(\lambda I - G)x = 0$ has any nontrivial solutions for $\lambda > 0$.

$$(\lambda I - G)(x_1, x_2, ...) = (0, 0, ...) \tag{3.6}$$

$$\begin{array}{rcl} (\lambda x_1 - x_1^n, \lambda x_2 - x_2^n, ...) & = & (0, 0, ...) \\ \lambda x_i - x_i^n & = & 0, \forall i \in \mathbb{N} \\ x_i\left(\lambda - x_i^{n-1}\right) & = & 0, \forall i \in \mathbb{N} \\ \left(x_i = 0 \ \vee \ x_i^{n-1} = \lambda\right), \forall i & \in & \mathbb{N}. \end{array}$$

If $\lambda < 0$ then there is only trivial solution $x = (0, 0, ...)$. If $\lambda > 0$, then it is possible that $x_i = \pm \sqrt[n-1]{\lambda}$ for some $i \in \mathbb{N}$, so the equation (3.6) has nontrivial solutions also, such as $\left(\sqrt[n-1]{\lambda}, 0, 0, ...\right)$. This implies (since $G0 = 0$) for $\lambda > 0$, that operator $\lambda I - G$ is not injective and also $\lambda I - F$ is not injective. So operator $\lambda I - F$ is not bijective mapping for $\lambda > 0$, hence

$$(0, \infty) \subseteq \sigma_N(F) \tag{3.7}$$

Let us see for $\lambda \neq 0$ and arbitrary $y \in l_q$, whether exists $x \in l_p$ such that $(\lambda I - F)x = y$.

$$\begin{aligned} (\lambda x_1 - a_1 - x_1^n, \lambda x_2 - a_2 - x_2^n, ...) &= (y_1, y_2, ...) \implies \\ \lambda x_i - a_i - x_i^n &= y_i, \forall i \in \mathbb{N} \\ x_i^n - \lambda x_i + a_i + y_i &= 0, \forall i \in \mathbb{N}. \end{aligned}$$

These odd-degree polynomial equations have at least one real solutions x_i for every $y_i \in \mathbb{R}$ and it means that operator $\lambda I - F$ is onto for $\lambda \neq 0$. For $\lambda \leqslant 0$ operator $\lambda I - F$ is bijective and now we research if $(\lambda I - F)^{-1}$ is a continuous operator. For $\lambda = 0$ we have

$$(-F)^{-1}(x_1, x_2, ...) = (\sqrt[n]{-a_1 - x_1}, \sqrt[n]{-a_2 - x_2}, \cdots)$$

and this is continuous mapping. It follows from the Theorem 1.2., because $f(i, u) = \sqrt[n]{-a_i - u}$ are continuous functions $\forall i \in \mathbb{N}$. For $\lambda < 0$:

$$(\lambda I - F)(x_1, x_2, ...) = (y_1, y_2, \cdots)$$

$$(\lambda x_1 - a_1 - x_1^n, \lambda x_2 - a_2 - y_2^n, ...) = (y_1, y_2, \cdots)$$

The function $f(i, u) = \lambda u - a_i - u^n$ is bijective and decreasing (for $\lambda < 0$) and continuous, $\forall i \in \mathbb{N}$, so there exists its inverse $f^{-1}(i, u)$ (which is also bijective, decreasing and continuous function) $\forall i \in \mathbb{N}$ ([5]). Now from the Theorem 1.2 follows that operator $(\lambda I - F)^{-1}$, generated by $f^{-1}(i, u)$, is continuous operator. We proved that for $\lambda \leqslant 0$ the operator $(\lambda I - F)$ is bijective and $(\lambda I - F)^{-1}$ is continuous operator. For $\lambda = 0$ a superposition operator $G = (-F)^{-1}$ is generated by the function $g(s, u) = -\sqrt[n]{a(s) + u}$. The function $g'_u(s, u) = -\frac{1}{n} \cdot (a(s) + u)^{-\frac{n-1}{n}}$ is not continuous in $u = -a(s)$ ($\forall s \in \mathbb{N}$), so from the Theorem 1.3 it follows that operator G can not be continuously differentiable at $x_0 = (-a_1, -a_2, -a_3, ...) \in l_p$. Hence, $0 \notin \rho_N(F) \implies$

$$0 \in \sigma_N(F). \tag{3.8}$$

If $\lambda < 0$ then $\lambda I - F$ is bijective mapping and then we have to find out if $(\lambda I - F)^{-1}$ is a continuously differentiable operator.

a) Case that $n = 3$. We have $(\lambda I - F)^{-1}(x_1, x_2, ...) =$

$$\left(\sqrt[3]{\frac{-(a_1+x_1)}{2} + \sqrt{\triangle_1}} - \sqrt[3]{\frac{a_1+x_1}{2} + \sqrt{\triangle_1}}, \right.$$
$$\left. \sqrt[3]{\frac{-(a_2+x_2)}{2} + \sqrt{\triangle_2}} - \sqrt[3]{\frac{a_2+x_2}{2} + \sqrt{\triangle_2}}, \cdots \right),$$

where $\triangle_i = \left(\frac{a_i+x_i}{2}\right)^2 + \left(\frac{-\lambda}{3}\right)^3$. We have a superposition operator $G = (\lambda I - F)^{-1}$ which is generated by the function:

$$g(s,u) = \sqrt[3]{-\frac{a(s)+u}{2} + \sqrt{\triangle}} - \sqrt[3]{\frac{a(s)+u}{2} + \sqrt{\triangle}},$$

with $\triangle = \left(\frac{a(s)+u}{2}\right)^2 + \left(\frac{-\lambda}{3}\right)^3$. Since $\lambda < 0$, we have $\triangle > 0$ and the expressions under the cubic root are positive $\left(\pm\frac{a(s)+u}{2} + \sqrt{\triangle} > 0\right)$. Then for $g'_u(s,u)$ we get ($\forall u \in \mathbb{R}$,$\forall s \in \mathbb{N}$):

$$g'_u(s,u) = -\frac{3}{2\lambda^2}\left[A\left(1 - \frac{a(s)+u}{2\sqrt{\triangle}}\right) + B\left(1 + \frac{a(s)+u}{2\sqrt{\triangle}}\right)\right],$$

where

$$A = \left(\frac{a(s)+u}{2} + \sqrt{\triangle}\right)^{\frac{2}{3}}; \quad B = \left(-\frac{a(s)+u}{2} + \sqrt{\triangle}\right)^{\frac{2}{3}}; \ \triangle > 0.$$

Consequently, this function $g'_u(s,u)$ is continuous $\forall u \in \mathbb{R}$,$\forall s \in \mathbb{N}$ and according to the Theorem 1.3 the operator $G = (\lambda I - F)^{-1}$ is continuously differentiable. So we get for $\lambda < 0$ that $\lambda \in \rho_N(F)$, which together with (3.7) and (3.8) gives us that **Neuberger spectrum of** F is a set $\sigma_N(F) = [0,\infty)$.

b) Case that n is an odd number and $n > 3$. The superposition operator

$$(\lambda I - F)x = (\lambda x_1 - a_1 - x_1^n, \lambda x_2 - a_2 - y_2^n, ...)$$

is generated by the function

$$f(i,u) = \lambda u - a_i - u^n. \tag{3.9}$$

For fixed $i \in \mathbb{N}$ we can consider the function (3.9) as the function of one variable u, where a_i is a real constant. For $\lambda < 0$ it is bijective, decreasing and continuous function (for every $i \in \mathbb{N}$), so there exists its inverse $f^{-1}(i,u)$ which is also bijective, decreasing and continuous function (for every $i \in \mathbb{N}$). The function (3.9) is convex for $u < 0$ and concave for $u > 0$ and it is a continuously differentiable function, that is, the function $f'_u(i,\cdot)$ is continuous at every u ($f'_u(i,0) = \lambda < 0$). Thus $f^{-1}(i,u)$ is concave for $u < -a_i$ and convex for $u > -a_i$ and it is also continuously differentiable function for every $i \in \mathbb{N}$ ([5]). Indeed, it is clear that $f^{-1}(i,u)$ is differentiable for $u < -a_i$ or $u > -a_i$ and it is also differentiable in $u = -a_i$ with $\left(f^{-1}\right)'_u(i,-a_i) = \frac{1}{\lambda} < 0$. By the Theorem 1.3 it means that operator $(\lambda I - F)^{-1}$ generated by the function $f^{-1}(i,u)$, is a continuously differentiable operator (for $\lambda < 0$). Again we get that

$$(-\infty,0) \subseteq \rho_N(F) \quad . \tag{3.10}$$

Finally, from (3.7), (3.8) and (3.10) we get **the Neuberger spectrum of** F is a set $\sigma_N(F) = [0,\infty)$. □

We can now summarize the Lemma 3.1 and the Lemma 3.2 in the following:

THEOREM 3.1. *Let the superposition operator $F : l_p \to l_p$ be generated by the function $f(s,u) = a(s) + u^n$, where $(a(s))_s$ is a sequence from the space l_t ($1 \leqslant t \leqslant p \leqslant \infty$).Then the Neuberger spectrum of F is:*

$$\sigma_N(F) = \begin{cases} \mathbb{R}, & \text{if } n \text{ is even} \\ [0,\infty), & \text{if } n \text{ is odd and } n \geqslant 3 \end{cases}$$

In case that l_p is a space of sequences over $\mathbb{C}$ then $\sigma_N(F) = \mathbb{C}$.

Now we will consider the superposition operator F generated by the function $f(s,u) = a(s) \cdot u^n$, $n \in \mathbb{N}$, where $(a(s))_s$ is a bounded sequence ($a \in l_\infty$) and $(\exists s \in \mathbb{N})\, a(s) \neq 0$. We can see that operator F can act from l_p to l_p ($1 \leqslant p \leqslant \infty$)

a) Case $1 \leqslant p < \infty$. Since a is a bounded sequence of numbers, then there exists a number

$$b = \sup_{s\in\mathbb{N}} |a_s| < \infty \tag{3.11}$$

such that $\forall s \in \mathbb{N}$, $|a(s)| \leqslant b$. Now we have

$$|f(s,u)| = |a(s) \cdot u^n| = |a(s)| \cdot |u^n| \leqslant b \cdot |u|^n .$$

For $|u| < 1$ we have inequality $|u|^n < |u|$ and

$$|f(s,u)| \leqslant b \cdot |u|^n < b \cdot |u| = b \cdot |u|^{\frac{p}{p}} . \tag{3.12}$$

Inequality (3.12) holds for $|u| < 1$ and $\forall s \in \mathbb{N}$, so from the Theorem 1.1 follows that operator F acts from l_p to l_p.

b) Case $p = \infty$.

$$Fx = F(x_1, x_2, ...) = (a_1 x_1^n, a_2 x_2^n, ...)$$
$$x = (x_1, x_2, ...) \in l_\infty \implies$$

$$\exists \sup_{s\in\mathbb{N}} |x_s| = M \quad \text{and} \sup_{s\in\mathbb{N}} |x_s|^n = M^n. \tag{3.13a}$$

Now, from (3.11) and (3.13a) we get

$$\sup_{s\in\mathbb{N}} |a_s x_s^n| = \sup_{s\in\mathbb{N}} |a_s| \cdot \sup_{s\in\mathbb{N}} |x_s|^n = b \cdot M^n < \infty,$$

which means that operator F acts from l_∞ to l_∞.

From the Propositions 2.1 and 2.3 we see that this operator F is a continuously differentiable operator, $F \in \mathfrak{C}^1(l_p)$.

LEMMA 3.3. *Let the superposition operator $F : l_p \to l_p$ ($1 \leqslant p \leqslant \infty$) be generated by the function $f(s,u) = a(s) \cdot u^n$, where n is an even number and $(a(s))_s$ is a sequence from the space l_∞.Then the Neuberger spectrum of F is $\sigma_N(F) = \mathbb{R}$ (or $\sigma_N(F) = \mathbb{C}$).*

PROOF. Denote $a(s) = a_s$, $s \in \mathbb{N}$. We have

$$\begin{aligned} Fx &= F(x_1, x_2, ...) = (a_1 \cdot x_1^n, a_2 \cdot x_2^n, ...) \text{ and} \\ F0 &= F(0, 0, ...) = (0, 0, ...). \end{aligned}$$

First we have to find when the operator $\lambda I - F$ is bijective. For $\lambda = 0$ we get

$$-Fx = (-a_1 \cdot x_1^n, -a_2 \cdot x_2^n, ...).$$

It follows from $-Fx = -Fy$ that

$$(-a_1 \cdot x_1^n, -a_2 \cdot x_2^n, ...) = (-a_1 \cdot y_1^n, -a_2 \cdot y_2^n, ...) \iff$$

$$-a_s \cdot x_s^n = -a_s \cdot y_s^n, \forall s \in \mathbb{N}. \tag{3.14a}$$

From (3.14a) it does not have to follow that $x_s = y_s$, so the operator $-F$ is not injective. Let us find out for $\lambda \neq 0$ if the equation $(\lambda I - F) x = 0$, has some nontrivial solutions.

$$\begin{aligned} (\lambda I - F) x &= (\lambda x_1 - a_1 x_1^n, \lambda x_2 - a_2 x_2^n, ...) = (0, 0, ...) \\ \lambda x_s - a_s x_s^n &= 0, \forall s \in \mathbb{N}. \end{aligned}$$

If $a_s = 0$ for some s, then follows $x_s = 0$. If $a_s \neq 0$ then we have

$$x_s \left(\lambda - a_s x_s^{n-1}\right) = 0 \implies (x_s = 0) \quad \vee (x_s^{n-1} = \frac{\lambda}{a_s}).$$

Since $n-1$ is an odd number there always exists a real number $x_s = \sqrt[n-1]{\frac{\lambda}{a_s}} \neq 0$. Hence, the operator $\lambda I - F$ is not injective for any real number λ. Thus, it follows from the Definition 1.2, that the Neuberger resolvent set is empty and the Neubereger spectrum of F is $\sigma_N(F) = \mathbb{R}$. □

LEMMA 3.4. *Let the superposition operator $F : l_p \to l_p$ $(1 \leqslant p \leqslant \infty)$ be generated by the function $f(s, u) = a(s) \cdot u^n$, where n is an odd number and $(a(s))_s$ is a sequence from the space l_∞. Then the Neuberger spectrum of F is*

$$\sigma_N(F) = \begin{cases} [0, \infty), & \text{if } a(s) \geqslant 0, \forall s \in \mathbb{N} \\ (-\infty, 0], & \text{if } a(s) \leqslant 0, \forall s \in \mathbb{N} \\ \mathbb{R}, & \text{if } (\exists i, j)(a_i > 0 \ \wedge a_j < 0). \end{cases} \tag{3.15}$$

PROOF. Denote $a(s) = a_s$, $s \in \mathbb{N}$. We have

$$\begin{aligned} Fx &= F(x_1, x_2, ...) = (a_1 \cdot x_1^n, a_2 \cdot x_2^n, ...) \\ (\lambda I - F) x &= (\lambda x_1 - a_1 x_1^n, \lambda x_2 - a_2 x_2^n, ...). \end{aligned}$$

I) Case that $a_s \geqslant 0, \forall s \in \mathbb{N}$. For $\lambda = 0$ we get

$$-Fx = (-a_1 x_1^n, -a_2 x_2^n, ...).$$

From $-Fx = -Fy$ follows

$$(-a_1 x_1^n, -a_2 x_2^n, ...) = (-a_1 y_1^n, -a_2 y_2^n, ...) \iff$$

$$-a_s x_s^n = -a_s y_s^n, \forall s \in \mathbb{N} \tag{3.16a}$$

If $\exists a_s = 0$ then from (3.16a) it does not follow $x_s = y_s$ and it means that operator $-F$ is not injective. If $a_s > 0, \forall s \in \mathbb{N}$, then from (3.16a) (since n is an odd number) it follows that $x_s = y_s, \forall s \in \mathbb{N}$, i.e. $x = y$. It means that operator $-F$ is injective in case that $a_s > 0, \forall s \in \mathbb{N}$. For $\lambda \neq 0$ let us first consider the operator equation $(\lambda I - F)\, x = 0$.

$$\begin{aligned} (\lambda x_1 - a_1 x_1^n, \lambda x_2 - a_2 x_2^n, ...) &= (0, 0, ...) \iff \\ \lambda x_s - a_s x_s^n &= 0, \forall s \in \mathbb{N}. \end{aligned}$$

If $a_s = 0$ for some $s \in \mathbb{N}$, then $x_s = 0$ for those s; otherwise (for $s \in \mathbb{N}$ such that $a_s > 0$) we can write

$$x_s \left(\lambda - a_s x_s^{n-1}\right) = 0$$

From the last equation it follows that $x_s = 0$ or

$$a_s x_s^{n-1} = \lambda. \tag{3.17}$$

If $\lambda > 0$ and $a_s > 0$ then the equation (3.17) has solutions $x_s = \pm \sqrt[n-1]{\frac{\lambda}{a_s}}$. If $\lambda < 0$ and $a_s > 0$ then the equation (3.17) has no real solutions (since $n-1$ is even). It means that the equation $(\lambda I - F)\, x = 0$ has nontrivial solutions for $\lambda > 0$ and $a_s \geqslant 0$ and since $F0 = 0$ it gives us the consequence that $\lambda I - F$ is not injective (for $\lambda > 0$ and $a_s \geqslant 0$). So we get

$$(0, \infty) \subseteq \sigma_N (F). \tag{3.18}$$

If $\lambda < 0$ suppose that $(\lambda I - F)\, x = (\lambda I - F)\, y$.

$$(\lambda x_1 - a_1 x_1^n, \lambda x_2 - a_2 x_2^n, ...) = (\lambda y_1 - a_1 y_1^n, \lambda y_2 - a_2 y_2^n, ...) \iff$$
$$\lambda x_s - a_s x_s^n = \lambda y_s - a_s y_s^n, \forall s \in \mathbb{N} \iff$$
$$\lambda \left(x_s - y_s\right) = a_s \left(x_s^n - y_s^n\right), \forall s \in \mathbb{N} \tag{3.19a}$$

If $a_s = 0$ for some $s \in \mathbb{N}$, then $x_s = y_s$ for those s. Otherwise (for $s \in \mathbb{N}$ such that $a_s > 0$) we get from (3.19a) that $x_s = y_s$ or

$$x_s^{n-1} + \ x_s^{n-2} y_s + ... + \ x_s \ \ y_s^{n-2} + \ \ y_s^{n-1} - \frac{\lambda}{a_s} = 0. \tag{3.20}$$

The equation (3.20) is similar to the equation (3.5). Since $\frac{\lambda}{a_s} < 0$, we have already shown in the proof of the Lemma 3.2 that it implies

$$x_s^{n-1} + \ x_s^{n-2} y_s + ... + \ x_s \ \ y_s^{n-2} + \ \ y_s^{n-1} - \frac{\lambda}{a_s} > 0,$$

so the equation (3.20) has no real solutions. It means that from (3.19a) follows $x_s = y_s, \forall s \in \mathbb{N}$. Hence, $\lambda I - F$ is injective operator for $\lambda < 0$ and $a_s \geqslant 0$. Surjectivity of an operator $\lambda I - F$ for every real λ can be proved on the similar way as in the proof of the Lemma 3.2. Thus, $\lambda I - F$ is bijective operator for $(\lambda = 0 \ \wedge (a_s > 0, \forall s \in \mathbb{N}))$ and $(\lambda < 0 \wedge (a_s \geqslant 0, \forall s \in \mathbb{N}))$. For these cases we are going to investigate if $(\lambda I - F)^{-1}$ is a continuously differentiable operator.

a)$\lambda = 0 \wedge (a_s > 0, \forall s \in \mathbb{N})$

$$\left(-F\right)^{-1} x = \left(\sqrt[n]{\frac{-x_1}{a_1}}, \sqrt[n]{\frac{-x_2}{a_2}}, ... \right)$$

This operator $\left(-F\right)^{-1}$ is generated by the function $f(s,u) = \sqrt[n]{\frac{-u}{a_s}}$. The function $f'_u(s,u) = -\left(n\sqrt[n]{a_s}u^{\frac{n-1}{n}}\right)^{-1}$ is not continuous in $u = 0$, so by the Theorem 1.3 we have that $\left(-F\right)^{-1}$ is not differentiable operator in zero, $\left(-F\right)^{-1} \notin \mathfrak{C}(l_p)$. Thus, $0 \notin \rho_N(F) \implies$

(3.21) $$0 \in \sigma_N(F).$$

b) $\lambda < 0 \ \wedge \ (a_s \geqslant 0, \forall s \in \mathbb{N})$

Operator $\lambda I - F$ is generated by the function $f(s,u) = \lambda u - a_s u^n$ which is continuous, bijective and decreasing function (for every fixed s). It is convex for $u < 0$ and concave for $u > 0$ and it is a continuously differentiable function, that is, the function $f'_u(s,u) = \lambda - a_s n u^{n-1}$ is continuous at every u ($f'_u(s,0) = \lambda < 0$). Thus there exists its inverse $f^{-1}(s,u)$ which is continuous, bijective and decreasing function (for every s) and it is concave for $u < 0$ and convex for $u > 0$ and it is also continuously differentiable function for every $s \in \mathbb{N}$ $\left(\left(f^{-1}\right)'_u(s,0) = \frac{1}{\lambda} < 0\right)$. By the Theorem 1.3 it means that operator $\left(\lambda I - F\right)^{-1}$ generated by the function $f^{-1}(s,u)$, is a continuously differentiable operator for $\lambda < 0$. So we get

(3.22) $$(-\infty, 0) \subseteq \rho_N(F).$$

Finally we get from (3.18), (3.21) and (3.22) that $\sigma_N(F) = [0, \infty)$.

II) Case that $a_s \leqslant 0, \forall s \in \mathbb{N}$

Again from (3.16a) we conclude: operator $-F$ is not injective if $\exists a_s = 0$ and it is injective if $a_s < 0, \forall s \in \mathbb{N}$. For $a_s < 0, \forall s \in \mathbb{N}$, the operator $\left(-F\right)^{-1}$ is generated by the function $f(s,u) = \sqrt[n]{\frac{-u}{a_s}}$. The function $f'_u(s,u) = -\left(n\sqrt[n]{a_s}u^{\frac{n-1}{n}}\right)^{-1}$ is not continuous in $u = 0$, so by the Theorem 1.3 we have that $\left(-F\right)^{-1}$ is not differentiable operator in zero. Thus, we get $0 \in \sigma_N(F)$ (3.21). Analogously as in the previous case I), from the equations (3.17) we conclude that the operator equation $(\lambda I - F)x = 0$ has nontrivial solutions if $\lambda < 0$ (with $x_s = \pm \sqrt[n-1]{\frac{\lambda}{a_s}}$ if $a_s < 0$ and $x_s = 0$ if $a_s = 0$). It means that $\lambda I - F$ is not injective operator for $\lambda < 0$. For $\lambda > 0$ from the equation $(\lambda I - F)x = (\lambda I - F)y$ we again comes up to the conclusion that $x_s = y_s$ if $a_s = 0$ and if $a_s < 0$ then from the equation (3.20) where $\frac{\lambda}{a_s} < 0$, we see that again values $x_s = y_s$. Hence, $\lambda I - F$ is an injective operator for $\lambda > 0$ and $a_s \leqslant 0$. Naturally, operator $\lambda I - F$ is surjective for every real λ. So,

(3.23) $$(-\infty, 0) \subseteq \sigma_N(F).$$

Operator $\lambda I - F$ is bijective for $\lambda > 0$ and $a_s \leqslant 0$ and it is generated by the function $f(s,u) = \lambda u - a_s u^n$ which is continuous, bijective and increasing function (for every fixed s). It is concave for $u < 0$ and convex for $u > 0$ and it is a continuously differentiable function, that is, the function $f'_u(s,u) = \lambda - a_s n u^{n-1}$ is continuous at every u ($f'_u(s,0) = \lambda > 0$). Thus there exists its inverse $f^{-1}(s,u)$ which is continuous, bijective and increasing function (for every s) and it is convex for $u < 0$ and concave for $u > 0$ and it is also continuously differentiable function for every $s \in \mathbb{N}$ $\left(\left(f^{-1}\right)'_u(s,0) = \frac{1}{\lambda} > 0\right)$. By the Theorem 1.3 it means that operator $(\lambda I - F)^{-1}$ generated by the function $f^{-1}(s,u)$, is a continuously differentiable operator for $\lambda > 0$. So

$$(0,\infty) \subseteq \rho_N(F). \tag{3.24}$$

Now, from the (3.21), (3.23) and (3.24) we get $\sigma_N(F) = (-\infty, 0]$.

III) Case that $(\exists i,j)(a_i > 0 \wedge a_j < 0)$

From the above observations in cases I) and II), we can conclude that the Neuberger spectrum in this third case is $\sigma_N(F) = \mathbb{R}$. □

We can summarize the Lemma 3.3 and Lemma 3.4 in the following:

THEOREM 3.2. *Let the superposition operator $F : l_p \to l_p$ $(1 \leqslant p \leqslant \infty)$ be generated by the function $f(s,u) = a(s) \cdot u^n$, where $n \in \mathbb{N}$ and $(a(s))_s$ is a sequence from the space l_∞. Then the Neuberger spectrum of F is*

$$\sigma_N(F) = \begin{cases} [0,\infty), & \text{if } n \text{ is odd and } a(s) \geqslant 0, \forall s \in \mathbb{N} \\ (-\infty,0], & \text{if } n \text{ is odd and } a(s) \leqslant 0, \forall s \in \mathbb{N} \\ \mathbb{R}, & \text{if } n \text{ is even or } n \text{ is odd and } (\exists i,j)(a_i > 0 \ \wedge a_j < 0). \end{cases}$$

4. Some other spectra and discussion

We may compare some other notions of spectrum for above mentioned nonlinear operators. For the class of continuous operators F on a Banach space X Rhodius introduced in 1984. the following notion of a spectrum. A point $\lambda \in \mathbb{K}$ belongs to the Rhodius resolvent set $\rho_R(F)$ if $\lambda I - F$ is bijective and $(\lambda I - F)^{-1}$ is a continuous operator on X. The set $\sigma_R(F) = \mathbb{K} \backslash \rho_R(F)$ is called the Rhodius spectrum of F. The set of all eigenvalues of the operator F is the point spectrum of F, i.e. $\sigma_p(F) = \{\lambda \in \mathbb{K} : Fx = \lambda x \text{ for some } x \neq 0\}$. The point spectrum is an important part of the spectrum of a linear operator and it is also important part of the Rhodius and Neuberger spectrum of nonlinear operator. In case $F0 = 0$ we have that the point spectrum is a subset of the Rhodius, as well as, of the Neuberger spectrum. It is not difficult to find out the Rhodius and point spectra from the previous section in this paper (see also [15]).

1) If the superposition operator $F : l_p \to l_p$ is generated by the function $f(s,u) = a(s) + u^n$, where n is an even number and $(a(s))_s$ is a sequence from the space l_t $(1 \leqslant t \leqslant p \leqslant \infty)$, then:

$$\begin{aligned}
\sigma_R(F) &= \mathbb{R}; \\
\sigma_p(F) &= \begin{cases} \mathbb{R}\setminus\{0\}, \text{ if } \ a(s)=0, \forall s\in\mathbb{N} \\ \mathbb{R}, \text{ if } (a(s)\leqslant 0) \ \wedge (\exists a(s)<0) \\ \left(-\infty,-2\sqrt{\sup a(s)}\right] \cup \left[2\sqrt{\sup a(s)},+\infty\right), \text{ if } \sup a(s)>0, n=2. \end{cases}.
\end{aligned}$$

The Rhodius spectrum is the same as the Neuberger spectrum

$\sigma_R(F)=\sigma_N(F)=\mathbb{R}$.

2) If the superposition operator $F: l_p \to l_p$ is generated by the function $f(s,u)=a(s)+u^n$, where n is an odd number and $(a(s))_s$ is a sequence from the space l_t $(1\leqslant t\leqslant p\leqslant\infty)$, then:

$$\sigma_R(F)=(0,\infty)\,; \ \ \sigma_p(F)=\begin{cases} (0,\infty), \text{ if } \ a(s)=0, \forall s\in\mathbb{N}. \\ \mathbb{R}, \text{ if } \exists a(s)\neq 0. \end{cases}.$$

The Rhodius spectrum is a strict subset of the Neuberger spectrum $\sigma_R(F) = (0,\infty)\subset[0,\infty)=\sigma_N(F)$. In case $\exists a(s)\neq 0$ we do not have $F0=0$ and the point spectrum $\sigma_p(F)=\mathbb{R}$ is not a subset of the Rhodius spectrum $\sigma_R(F)=(0,\infty)$, nor of the Neuberger spectrum $\sigma_N(F)=[0,\infty)$. In case $a(s)=0,\forall s\in\mathbb{N}$, we have that $(0,\infty)=\sigma_p(F)=\sigma_R(F)\subset\sigma_N(F)=[0,\infty)$.

3) If the superposition operator $F: l_p \to l_p$ is generated by the function $f(s,u)=a(s)\cdot u^n$, where n is an even number and $(a(s))_s$ is a sequence from the space l_∞, then:

$$\sigma_R(F)=\mathbb{R}\ ; \ \ \sigma_p(F)=\begin{cases} \mathbb{R}\setminus\{0\}, \text{ if } \ a(s)\neq 0, \forall s\in\mathbb{N} \\ \mathbb{R}, \text{ if } \exists a(s)=0. \end{cases}.$$

We have that $\sigma_R(F)=\sigma_N(F)=\mathbb{R}$. If there is some $a(s)=0$ then $\sigma_p(F)=\sigma_R(F)=\sigma_N(F)=\mathbb{R}$; if $a(s)\neq 0,\forall s\in\mathbb{N}$ then $\mathbb{R}\setminus\{0\}=\sigma_p(F)\subset\sigma_R(F)=\sigma_N(F)=\mathbb{R}$.

4) If the superposition operator $F: l_p \to l_p$ is generated by the function $f(s,u)=a(s)\cdot u^n$, where n is an odd number and $(a(s))_s$ is a sequence from the space l_∞, then:

$$\sigma_R(F)=\begin{cases} (0,\infty), \text{ if } \ a(s)>0, \forall s\in\mathbb{N} \\ [0,\infty), \text{ if } (a(s)\geqslant 0)\wedge(\exists a(s)=0) \\ (-\infty,0), \text{ if } a(s)<0, \forall s\in\mathbb{N} \\ (-\infty,0], \text{if } (a(s)\leqslant 0)\wedge(\exists a(s)=0) \\ \mathbb{R}\setminus\{0\}, \text{ if } (\exists i,j\in\mathbb{N})(a(i)>0\wedge a(j)<0), a(s)\neq 0, \ \forall s\in\mathbb{N} \\ \mathbb{R}, \text{ if } (\exists i,j,k\in\mathbb{N})\,(a(i)>0, a(j)<0, a(k)=0). \end{cases}$$

The point spectrum is the same as the Rhodius spectrum. The Rhodius and the Neuberger spectrum are the same in the following cases:

a) $(a(s)\geqslant 0)\wedge(\exists a(s)=0)$, $\sigma_R(F)=\sigma_N(F)=[0,\infty)$

b) $(a(s)\leqslant 0)\wedge(\exists a(s)=0)$, $\sigma_R(F)=\sigma_N(F)=(-\infty,0]$

c) $(\exists i,j,k\in\mathbb{N})\,(a(i)>0, a(j)<0, a(k)=0)$, $\sigma_R(F)=\sigma_N(F)=\mathbb{R}$.

In other three cases we get $\sigma_p(F)=\sigma_R(F)\subset\sigma_N(F)=\sigma_R(F)\cup\{0\}$.

The asymptotic spectrum has been defined by Furi, Martelli and Vignoli [12] in 1978. We call a continuous operator $F : X \to Y$ stably solvable if given any compact operator $G : X \to Y$ with

$$[G]_Q = \limsup_{\|x\|\to\infty} \frac{\|G(x)\|}{\|x\|} = 0,$$

the equation $F(x) = G(x)$ has a solution $x \in X$. A stably solvable operator $F \in \mathfrak{C}(X,Y)$ is said to be FMV-regular if both $[F]_q > 0$ and $[F]_a > 0$ ($[F]_q = \liminf\limits_{\|x\|\to\infty} \frac{\|G(x)\|}{\|x\|}$; $[F]_a = \inf\limits_{\alpha(M)>0} \frac{\alpha(F(M))}{\alpha(M)}$, $M \subseteq X$ bounded, $\alpha(M)$ is the measure of noncompactness), see [2]. Given $F \in \mathfrak{C}(X)$, the set

$$\rho_{FMV}(F) = \{\lambda \in \mathbb{K} : \lambda I - F \text{ is FMV-regular}\}$$

is called the Furi-Martelli-Vignoli resolvent set and its complement $\sigma_{FMV}(F) = \mathbb{K}\backslash\rho_{FMV}(F)$ the Furi-Martelli-Vignoli spectrum of F, or FMV-spectrum, for short. Intuitively speaking, if a point $\lambda \in \mathbb{K}$ belongs to $\sigma_{FMV}(F)$, then the operator $\lambda I - F$ is characterized by some lack of surjectivity, properness or boundedness. This spectrum is based on the notion of stable solvability of operators, a nonlinear analogue of surjectivity and it takes into account the asymptotic properties of an operator. For a bounded linear operator all these spectra (Rhodius, Neuberger and FMV) gives precisely the familiar spectrum. In contrast to other two nonlinear spectra, the FMV-spectrum, in general,does not contain the point spectrum. On the other hand, the FMV-spectrum has a nice property which the Rhodius and Neuberger do not have in general: the FMV-spectrum is always closed.

Let the superposition operator $F : l_p \to l_p$ be generated by the function $f(s,u) = a(s)\cdot u^n$, $((a(s))_s \in l_\infty)$ or by the function $f(s,u) = a(s)+u^n$ $((a(s))_s \in l_\infty, 1 \leqslant t \leqslant p \leqslant \infty)$. If n is even and $\mathbb{K} = \mathbb{R}$, then $\lambda I - F$ is not surjective for any real λ, so $\lambda I - F$ is not stably solvable for any real λ. Hence the FMV-spectrum of F is $\sigma_{FMV}(F) = \mathbb{R}$. If the superposition operator $F : l_p \to l_p$ is generated by the function $f(s,u) = u^n$, where n is odd then $\sigma_{FMV}(F) = \emptyset$.

We can conclude that for our superposition operators $F : l_p \to l_p$, generated by the functions $f(s,u) = a(s)\cdot u^n$ or $f(s,u) = a(s)\cdot u^n$, $(1 \leqslant p \leqslant \infty, \mathbb{K} = \mathbb{R})$, all these nonlinear spectra of F (the Rhodius spectrum, the Neuberger spectrum and FMV-spectrum) coincide if n is even. The FMV-spectrum has various applications to integral equations, boundary value problems and bifurcation theory.Eigenvalues plays an important role in classical linear spectral theory. In contrast to the other two spectra, the FMV-spectrum in general does not contain the point spectrum. The role of the point spectrum now may be substituted by the asymtotic approximate point spectrum (see [2]). Many concepts in nonlinear analysis are in fact of local nature (such as the derivative of a map at some point) and so recently a new notion called spectrum of a nonlinear operator at some point, has been defined by Calamai, Furi and Vignoli [14], (CFV-spectrum $\sigma_{CFV}(F)$, or $\sigma(f,p)$).This spectrum is close in spirit to the FMV-spectrum. Nevertheless, while the asymptotic spectrum is related to the asymptotic behaviour of a map, $\sigma(f,p)$ depends only on the germ of f at p. In [13] authors also introduced and study a spectrum called small Calamai-Furi-Vignoli spectrum and denoted by $\sigma_{cfv}(F)$. In view of the Theorem 3.7. from[13], we may easily find out the small Calamai-Furi-Vignoli spectrum $\sigma_{cfv}(F)$ for our superposition operators F $(F : l_p \to l_p, (1 \leqslant p \leqslant \infty)$ generated by the functions $f(s,u) = a(s) + u^n$ or $f(s,u) = a(s) \cdot u^n$; $n \in \mathbb{N}, n \geqslant 2)$. These operators are Fréchet differentiable at 0 and $F'(0)(x_1, x_2, x_3, ...) = (0, 0, 0, ...)$, so $\sigma_{cfv}(F) = \sigma(F'(0)) = \{0\}$.

These results of the Fréchet differentiability and the Neuberger spectrum (and other notions of spectra) for nonlinear superposition operators may be used in solving some nonlinear operator equations and eigenvalue problems. We are interested in a solvability of nonlinear systems of equations (of Hammerstein type), i.e. operator equations with a superposition operator F in a space of sequences l_p and $l_{p,\sigma}$(see [16]). These systems often occur in a chaos theory and theory of stochastic processes.

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