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SCHUR HARMONIC CONVEXITY OF GNAN MEAN FOR TWO VARIABLES

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Abstract. In this paper, the Schur harmonic convexity of the Gnan mean and its dual form in two variables are discussed.

1. Introduction

For $a, b > 0$ let

$$M_r = M_r(a,b) = \begin{cases} \left(\frac{a^r+b^r}{2}\right)^{\frac{1}{r}}, & a \neq b, r \neq 0; \\ \sqrt{ab}, & r = 0; \end{cases} \tag{1.1}$$

$$I = I(a,b) = \begin{cases} \exp\left[\dfrac{b\ln b - a\ln a}{b-a} - 1\right], & a \neq b; \\ a, & a = b; \end{cases} \tag{1.2}$$

$$L = L(a,b) = \begin{cases} \dfrac{a-b}{\ln a - \ln b}, & a \neq b; \\ a, & a = b; \end{cases} \tag{1.3}$$

$$H = H(a,b) = \frac{a + \sqrt{ab} + b}{3}. \tag{1.4}$$

These are respectively called: Power mean, Identric mean, Logarithmic and Heron means. In [**4**, **19**, **20**] Lokesha et al. studied extensively and obtained some remarkable results on the weighted Heron mean and its dual form. Also, they discussed some results on the weighted product type means and their monotonicity.

In [**17**, **18**] Zhang et al. gave the generalizations of Heron mean, similar product type means and their dual forms.

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For two variables, the above means can be written as follows:

(1.5)
$$I(a,b;k)=\prod_{i=1}^{k}\left(\frac{(k+1-i)a+ib}{k+1}\right)^{\frac{1}{k}},\quad I^{*}(a,b;k)=\prod_{i=0}^{k}\left(\frac{(k-i)a+ib}{k}\right)^{\frac{1}{k+1}}$$

and

$$H(a,b;k)=\frac{1}{k+1}\sum_{i=0}^{k}a^{\frac{k-i}{k}}b^{\frac{i}{k}},\quad h(a,b;k)=\frac{1}{k}\sum_{i=1}^{k}a^{\frac{k+1-i}{k+1}}b^{\frac{i}{k+1}}, \tag{1.6}$$

where k is a natural number. Authors proved that $H(a,b;k)$ and $I^{*}(a,b;k)$ are monotone decreasing functions (but $h(a,b;k)$ and $I(a,b;k)$ monotone increasing functions) with k and established the following limitations:

$$\lim_{k\to+\infty}I(a,b;k)=\lim_{k\to+\infty}I^{*}(a,b;k)=I(a,b),$$

and

$$\lim_{k\to+\infty}H(a,b;k)=\lim_{k\to+\infty}h(a,b;k)=L(a,b).$$

In [**2**] Lokesha et al. defined the Gnan mean and its dual form for two variables. Also, they obtained some interesting properties, monotonic results and its limitations. The definitions of Gnan mean and its dual form are given in the following section. The notion of Schur convex function was introduced by I. Schur in 1923 and has had interesting applications in analytic inequalities, elementary quantum mechanics and quantum information theory (See [**5**]). In [**16**] Zhang proposed the concept of *Schur-harmonically convex function* which is an extension of Schur-convexity function. The detailed discussion on convexity and Schur convexity one can be found in [**1**] – [**12**].

2. Definition and Lemmas

For readers convenience we recall some definitions as follows:

Definition 2.1. ([**2**]) Let $a,b\geqslant 0$, and let k be a non-negative integer, and let α,β be two real numbers. The Gnan mean and its dual forms are as follows:

$$\begin{aligned}
G(a,b;k,\alpha,\beta)&=\left[\frac{1}{k}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}}\\
G(a,b;k,0,\beta)&=\left[\frac{1}{k}\sum_{i=1}^{k}a^{\frac{(k+1-i)\beta}{k+1}}b^{\frac{i\beta}{k+1}}\right]^{\frac{1}{\beta}}\\
G(a,b;k,\alpha,0)&=\prod_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}}\\
G(a,b;k,0,0)&=\sqrt{ab};
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned} g(a,b;k,\alpha,\beta) &= \left[\frac{1}{k+1}\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}} \\ g(a,b;k,0,\beta) &= \left[\frac{1}{k+1}\sum_{i=0}^{k}a^{\frac{(k-i)\beta}{k}}b^{\frac{i\beta}{k}}\right]^{\frac{1}{\beta}} \\ g(a,b;k,\alpha,0) &= \prod_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{1}{(k+1)\alpha}} \\ g(a,b;k,0,0) &= \sqrt{ab}. \end{aligned} \tag{2.2}$$

Definition 2.2. ([**5**], [**14**]) Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ be arbitrary elements in $\mathbf{R}^n$ $(n \geqslant 2)$.

(1) x is said to by majorized by y, (in symbol $x \prec y$), if

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]} \text{ for } 1 \leqslant k \leqslant n-1,$$

and

$$\sum_{i=1}^{n} x_{[i]} \leqslant \sum_{i=1}^{n} y_{[i]},$$

where $x_{[1]} \geqslant \cdots \geqslant x_{[n]}$ and $y_{[1]} \geqslant \cdots \geqslant y_{[n]}$ are rearrangements of x and y in a descending order.

(2) $x \geqslant y$ means $x_i \geqslant y_i$ for all $i = 1, 2, \cdots, n$.
Let $\Omega \subseteq \mathbf{R}^n$ $(n \geqslant 2)$. The function $\varphi : \Omega \to \mathbf{R}$ is said to be decreasing if and only if $-\varphi$ is increasing.

(3) $\Omega \subseteq \mathbf{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \cdots, \alpha x_n + \beta y_n) \in \Omega$ for all x and y, where α, $\beta \in [0, 1]$ with $\alpha + \beta = 1$.

(4) Let $\Omega \subseteq \mathbf{R}^n$ $(n \geqslant 2)$ be a set with nonempty interior. Then the function $\varphi : \Omega \to \mathbf{R}$ be said to be a Schur-convex if $x \prec y$ on Ω implies $\varphi(x) \leqslant \varphi(y)$. φ is said to be a Schur-concave if $-\varphi$ is a Schur-convex.

Definition 2.3. ([**16**]) Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ be arbitrary elements in $\mathbf{R}^n_+$. Then $\Omega\,(\subseteq \mathbf{R}^n)$ is called a harmonically convex set if $(x_1^{\alpha} y_1^{\beta}, ..., x_1^{\alpha} y_1^{\beta}) \in \Omega$, for all $x, y \in \Omega$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$. For $\Omega\,(\subseteq \mathbf{R}^n_+)$, the function $\varphi : \Omega \to \mathbf{R}_+$ is said to be a Schur harmonically convex function on Ω if $(lnx_1, ..., ln_n) \prec (lny_1, ..., lny_n)$ on Ω implies $\varphi(x) \leqslant \varphi(y)$. φ is said to be Schur harmonically concave if $-\varphi$ is Schur Harmonically convex.

Definition 2.4. ([**5**], [**14**]) Let $\Omega\,(\subseteq \mathbf{R}^n)$ be called symmetric set if $x \in \Omega$ implies $Px \in \Omega$ for every $n \times n$ permutation matrix P. The function $\varphi : \Omega \to \mathbf{R}$ is called symmetric if $\varphi(Px) = \varphi(x)$ holds for every permutation matrix P and for all $x \in \Omega$.

DEFINITION 2.5. ([**5**], [**14**]) For $\Omega\,(\subseteq \mathbf{R}^n)$, the function $\varphi : \Omega \to \mathbf{R}$ is called symmetric and convex function if φ is Schur-convex on Ω.

LEMMA 2.1 ([**16**]). *Let $\Omega\,(\subseteq \boldsymbol{R}^n)$ be a symmetric set with a nonempty interior harmonic convex set Ω^0. Let $\varphi : \Omega \to \boldsymbol{R}_+$ be continuous on Ω and differentiable in Ω^0. Then φ Schur-harmonic convex (Schur-harmonic concave) on Ω and*

$$(x_1 - x_2)(x_1^2\frac{\partial\varphi}{\partial x_1} - x_2^2\frac{\partial\varphi}{\partial x_2}) \geqslant 0\ (\leqslant 0) \tag{2.3}$$

holds for any $x = (x_1, x_2, \cdots, x_n) \in \Omega^0$.

3. Main Results

In this section, the Schur-harmonic convexity of Gnan mean for two variables are given.

THEOREM 3.1. *Let $a \geqslant b \geqslant 0$ be arbitrary elements, k be a nonnegative integer and α, β two real numbers. Then:*

(1) *The Gnan mean $G(a, b, k; \alpha, \beta)$ is Schur-harmonic convex by a and b if $\alpha + 1$, $\beta + 1 \geqslant 0$.*
(2) *The Gnan mean $G(a, b, k; \alpha, \beta)$ is Schur-harmonic concave by a and b if $\alpha + 1$, $\beta + 1 \leqslant 0$.*

PROOF. **Proof of (1):**
Case (1). Let α and β be any two non-zero distinct real numbers with $\alpha + 1 > 0$, $t > 0$ and $a > b$. The Gnan mean can be defined as:

$$G = G(a, b; k, \alpha, \beta) = \left[\frac{1}{k}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha + ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}} \tag{3.1}$$

Take log on both sides and differentiate partially with respect to a and multiply by a^2. Then we have

(3.2)
$$a^2\frac{\partial G}{\partial a} = G\frac{1}{\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha+ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}}}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha + ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}-1}\frac{k+1-i}{k+1}a^{\alpha+1}.$$

Similarly we have

(3.3)
$$b^2\frac{\partial G}{\partial b} = G\frac{1}{\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha+ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}}}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha + ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}-1}\frac{i}{k+1}b^{\alpha+1}.$$

Then

$$(a - b)\left(a^2\frac{\partial G}{\partial a} - b^2\frac{\partial G}{\partial b}\right) = \tag{3.4}$$

$$\frac{(a-b)}{k}G^{1-\beta}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}}\frac{(k+1-i)a^{\alpha+1}-ib^{\alpha+1}}{(k+1-i)a^{\alpha}+ib^{\alpha}}$$

i.e.

$$(a-b)\left(a^2\frac{\partial G}{\partial a}-b^2\frac{\partial G}{\partial b}\right)=[\Delta]\,[\Theta] \tag{3.5}$$

where,

$$\Delta=\frac{(a-b)}{k}G^{1-\beta}$$

and

$$\Theta=\sum_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}}\frac{(k+1-i)a^{\alpha+1}-ib^{\alpha+1}}{(k+1-i)a^{\alpha}+ib^{\alpha}}.$$

For $k=1$ we have

$$\Theta=\left(\frac{a^{\alpha}+b^{\alpha}}{2}\right)^{\frac{\beta}{\alpha}}\frac{a^{\alpha+1}-b^{\alpha+1}}{a^{\alpha}+b^{\alpha}}=2(Cosh\alpha t)^{\frac{\beta}{\alpha}-1}Sinh(\alpha+1)t.$$

where $a=e^t$ and $b=e^-t$. Then for all $\alpha+1$, $t>0$ and $a>b$, we have

$$(a-b)\left(a^2\frac{\partial G}{\partial a}-b^2\frac{\partial G}{\partial b}\right)=[\Delta]\,[\Theta]>0. \tag{3.6}$$

For $k=2$ we have

$$\Theta=\left(\frac{2a^{\alpha}+b^{\alpha}}{3}\right)^{\frac{\beta}{\alpha}}\frac{2a^{\alpha+1}-b^{\alpha+1}}{2a^{\alpha}+b^{\alpha}}+\left(\frac{a^{\alpha}+2b^{\alpha}}{3}\right)^{\frac{\beta}{\alpha}}\frac{a^{\alpha+1}-2b^{\alpha+1}}{a^{\alpha}+2b^{\alpha}}$$

i.e.

$$\Theta=\left(\frac{1}{3}\right)^{\frac{\beta}{\alpha}}\left[(2a^{\alpha}+b^{\alpha})^{\frac{\beta}{\alpha}-1}(2a^{\alpha+1}-b^{\alpha+1})+(a^{\alpha}+2b^{\alpha})^{\frac{\beta}{\alpha}-1}(a^{\alpha+1}-2b^{\alpha+1})\right]>0$$

if it is

$$\left(\frac{2a^{\alpha}+b^{\alpha}}{a^{\alpha}+2b^{\alpha}}\right)^{\frac{\beta}{\alpha}-1}\frac{2a^{\alpha+1}-b^{\alpha+1}}{2b^{\alpha+1}-a^{\alpha+1}}>1.$$

It is easy to prove that:

$$\frac{2a^{\alpha}+b^{\alpha}}{a^{\alpha}+2b^{\alpha}}>1 \quad \text{and} \quad \frac{2a^{\alpha+1}-b^{\alpha+1}}{2b^{\alpha+1}-a^{\alpha+1}}>1.$$

Then for all $\alpha+1$, $t>0$ and $a>b$ we have

$$(a-b)\left(a^2\frac{\partial G}{\partial a}-b^2\frac{\partial G}{\partial b}\right)=[\Delta]\,[\Theta]>0. \tag{3.7}$$

For $k=3$ we have

$$\Theta=$$

$$\left(\tfrac{3a^{\alpha}+b^{\alpha}}{4}\right)^{\frac{\beta}{\alpha}}\tfrac{3a^{\alpha+1}-b^{\alpha+1}}{3a^{\alpha}+b^{\alpha}}+2(4)^{\frac{\beta}{\alpha}}(Cosh\alpha t)^{\frac{\beta}{\alpha}-1}Sinh(\alpha+1)t+\left(\tfrac{a^{\alpha}+3b^{\alpha}}{4}\right)^{\frac{\beta}{\alpha}}\tfrac{a^{\alpha+1}-3b^{\alpha+1}}{a^{\alpha}+3b^{\alpha}}$$

where $a = e^t$ and $b = e^{-t}$. Thus

$$\Theta = \left(\frac{1}{4}\right)^{\frac{\beta}{\alpha}} \left[(3a^\alpha + b^\alpha)^{\frac{\beta}{\alpha}-1}(3a^{\alpha+1} - b^{\alpha+1}) + (a^\alpha + 3b^\alpha)^{\frac{\beta}{\alpha}-1}(a^{\alpha+1} - 3b^{\alpha+1})\right]$$

$$+2(4)^{\frac{\beta}{\alpha}}(Cosh\alpha t)^{\frac{\beta}{\alpha}-1} Sinh(\alpha+1)t > 0$$

if it is

$$\left(\frac{3a^\alpha + b^\alpha}{a^\alpha + 3b^\alpha}\right)^{\frac{\beta}{\alpha}-1} \frac{3a^{\alpha+1} - b^{\alpha+1}}{3b^{\alpha+1} - a^{\alpha+1}} > 1.$$

It is easy to prove that:

$$\frac{3a^\alpha + b^\alpha}{a^\alpha + 3b^\alpha} > 1 \quad \text{and} \quad \frac{3a^{\alpha+1} - b^{\alpha+1}}{3b^{\alpha+1} - a^{\alpha+1}} > 1.$$

Then for all $\alpha + 1$, $t > 0$ and $a > b$ we have

$$(a-b)\left(a^2 \frac{\partial G}{\partial a} - b^2 \frac{\partial G}{\partial b}\right) = [\Delta]\,[\Theta] > 0. \tag{3.8}$$

From above arguments we have two generalized cases as follows:
(i) When k is an even, on expanding the summation leads to even number of terms. Further, grouping the first and k^{th} term, second and $(k-1)^{\text{th}}$ term, and so on. Hence, we can arrive at the required conclusion by proving as explained for $k = 2$.
(ii) When k is an odd, on expanding the summation leads to odd number of terms. Further, grouping first and k^{th} term, second and $(k-1)^{\text{th}}$, and so on. In expansion, the middle term is $2(\cosh \alpha t)^{\frac{\beta}{\alpha}-1} \sinh(\alpha+1)t$. Hence, we can arrive at the required conclusion by proving as explained for $k = 1$ and $k = 3$.

Case (2). For $\alpha = 0$, $\beta + 1 > 0$, $t > 0$ and $a > b$ we have:

$$G = G(a, b; k, 0, \beta) = \left[\frac{1}{k}\sum_{i=1}^{k} a^{\frac{k+1-i}{k+1}\beta} b^{\frac{i}{k+1}\beta}\right]^{\frac{1}{\beta}}. \tag{3.9}$$

Then

(3.10)
$$(a-b)\left(a^2 \frac{\partial G}{\partial a} - b^2 \frac{\partial G}{\partial b}\right) = \frac{(a-b)}{k(k+1)} G^{1-\beta}\sqrt{ab}\left[\sum_{i=1}^{k} a^{\frac{k+1-i}{k+1}\beta} b^{\frac{i}{k+1}\beta}(k+1-2i)\right]$$

i.e.

$$(a-b)\left(a^2 \frac{\partial G}{\partial a} - b^2 \frac{\partial G}{\partial b}\right) = [\Delta]\,[\Theta] \tag{3.11}$$

where,

$$\Delta = \frac{(a-b)}{k(k+1)} G^{1-\beta}\sqrt{ab} \quad \text{and} \quad \Theta = \left[\sum_{i=1}^{k} a^{\frac{k+1-i}{k+1}\beta} b^{\frac{i}{k+1}\beta}(k+1-2i)\right].$$

We want to prove

$$(a-b)\left(a^2\frac{\partial G}{\partial a}-b^2\frac{\partial G}{\partial b}\right)=[\Delta]\,[\Theta]>0. \tag{3.12}$$

For $k=1$,

$$\Theta=0.$$

For $k=2$,

$$\Theta=\left(a^{\frac{\beta}{3}}b^{\frac{\beta}{3}}\right)\left(a^{\frac{\beta}{3}}-b^{\frac{\beta}{3}}\right)>0.$$

For $k=3$,

$$\Theta=\left(a^{\frac{3\beta}{4}}b^{\frac{\beta}{4}}\right)\left(a^{\frac{\beta}{2}}-b^{\frac{\beta}{2}}\right)>0.$$

For $k=4$,

$$\Theta=\left(3a^{\frac{\beta}{5}}b^{\frac{\beta}{5}}\right)\left(a^{\frac{3\beta}{5}}-b^{\frac{3\beta}{5}}\right)+\left(a^{\frac{2\beta}{5}}b^{\frac{2\beta}{5}}\right)\left(a^{\frac{\beta}{5}}-b^{\frac{\beta}{5}}\right)>0.$$

For $k=5$,

$$\Theta=\left(4a^{\frac{\beta}{6}}b^{\frac{\beta}{6}}\right)\left(a^{\frac{2\beta}{3}}-b^{\frac{2\beta}{3}}\right)+2\left(a^{\frac{\beta}{3}}b^{\frac{\beta}{3}}\right)\left(a^{\frac{\beta}{3}}-b^{\frac{\beta}{3}}\right)>0,\text{holds for all }\beta>0.$$

From the above cases, in general, we have the following:

$$(a-b)\left(a^2\frac{\partial G}{\partial a}-b^2\frac{\partial G}{\partial b}\right)=[\Delta]\,[\Theta]>0. \tag{3.13}$$

Case (3). For all $\alpha+1>0$, $\beta=0$, $t>0$ and $a>b$ we have the Gnan mean

$$G=G(a,b;k,\alpha,0)=\prod_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}} \tag{3.14}$$

equivalent to

$$G= \tag{3.15}$$
$$\left(\frac{ka^{\alpha}+b^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}}\left(\frac{(k-1)a^{\alpha}+2b^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}}\cdots\left(\frac{2a^{\alpha}+(k-1)b^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}}\left(\frac{a^{\alpha}+kb^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}}.$$

Then by grouping first and last term, second and second from the last and so, it leads to:

$$(a-b)\left(a^2\frac{\partial G}{\partial a}-b^2\frac{\partial G}{\partial b}\right)=$$

$$(a-b)\left[\frac{k}{k}+\frac{2(k-1)}{k}+\frac{3(k-2)}{k}+\frac{4(k-3)}{k}+\ldots\right](ab)^{\alpha}\left(\frac{a^{\alpha}-b^{\alpha}}{\times_i}\right) \tag{3.16}$$
$$+\left[\frac{1+2+3+4+5+\ldots.}{k}\right]\left(\frac{a^{2\alpha+1}-b^{2\alpha+1}}{\times_i}\right)$$

where

$$\times_i=\left((k-i)a^{\alpha}+(i+1)b^{\alpha}\right)\left((i+1)a^{\alpha}+(k-i)b^{\alpha}\right)\geqslant 0.$$

For all integral values of $i = 0, 1, 2, ..., (k-1)$ holds

$$(a-b)\left(a^2\frac{\partial G}{\partial a} - b^2\frac{\partial G}{\partial b}\right) > 0. \tag{3.17}$$

Case (4). For $\alpha = 0$, $\beta = 0$, $t > 0$ and $a > b$ we have the Gnan mean

$$G = G(a, b; k, 0, 0) = \sqrt{ab}. \tag{3.18}$$

Then

$$(a-b)\left(a^2\frac{\partial g}{\partial a} - b^2\frac{\partial g}{\partial b}\right) = \frac{1}{2}\sqrt{ab}(a-b) > 0. \tag{3.19}$$

Proof of (2):

Case (1). Let α, β be any two non-zero distinct real numbers with $\alpha+1 < 0$, $t > 0$ and $a > b$. The Gnan mean can be defined as:

$$G = G(a, b; k, \alpha, \beta) = \left[\frac{1}{k}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha + ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}} \tag{3.20}$$

Taking log on both sides and differentiate partially with respect to a and multiply by a^2 gives

$$a^2\frac{\partial G}{\partial a} =$$

$$G\frac{1}{\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha + ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}}}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha + ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}-1}\frac{k+1-i}{k+1}a^{\alpha+1}. \tag{3.21}$$

Similarly

$$b^2\frac{\partial G}{\partial b} = G\frac{1}{\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha + ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}}}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha + ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}-1}\frac{i}{k+1}b^{\alpha+1}. \tag{3.22}$$

Then

$$(a-b)\left(a^2\frac{\partial G}{\partial a} - b^2\frac{\partial G}{\partial b}\right) =$$

$$\frac{(a-b)}{k}G^{1-\beta}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha + ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}}\frac{(k+1-i)a^{\alpha+1} - ib^{\alpha+1}}{(k+1-i)a^\alpha + ib^\alpha} \tag{3.23}$$

$$(a-b)\left(a^2\frac{\partial G}{\partial a} - b^2\frac{\partial G}{\partial b}\right) = [\Delta]\,[\Theta] \tag{3.24}$$

where

$$\Delta = \frac{(a-b)}{k}G^{1-\beta}$$

and

$$\Theta = \sum_{i=1}^{k} \left(\frac{(k+1-i)a^{\alpha} + ib^{\alpha}}{k+1} \right)^{\frac{\beta}{\alpha}} \frac{(k+1-i)a^{\alpha+1} - ib^{\alpha+1}}{(k+1-i)a^{\alpha} + ib^{\alpha}}.$$

For $k = 1$

$$\Theta = \left(\frac{a^{\alpha} + b^{\alpha}}{2} \right)^{\frac{\beta}{\alpha}} \frac{a^{\alpha+1} - b^{\alpha+1}}{a^{\alpha} + b^{\alpha}} = 2(\cosh \alpha t)^{\frac{\beta}{\alpha}-1} \sinh(\alpha+1)t.$$

where $a = e^t$ and $b = e^{-t}$. Then for all $\alpha + 1 < 0$, $t > 0$ and $a > b$

$$(a-b)\left(a^2 \frac{\partial G}{\partial a} - b^2 \frac{\partial G}{\partial b} \right) = [\Delta]\,[\Theta] < 0. \tag{3.25}$$

For $k = 2$

$$\Theta = \left(\frac{2a^{\alpha} + b^{\alpha}}{3} \right)^{\frac{\beta}{\alpha}} \frac{2a^{\alpha+1} - b^{\alpha+1}}{2a^{\alpha} + b^{\alpha}} + \left(\frac{a^{\alpha} + 2b^{\alpha}}{3} \right)^{\frac{\beta}{\alpha}} \frac{a^{\alpha+1} - 2b^{\alpha+1}}{a^{\alpha} + 2b^{\alpha}}.$$

Then

$$\Theta = \left(\frac{1}{3} \right)^{\frac{\beta}{\alpha}} \left[(2a^{\alpha} + b^{\alpha})^{\frac{\beta}{\alpha}-1} (2a^{\alpha+1} - b^{\alpha+1}) + (a^{\alpha} + 2b^{\alpha})^{\frac{\beta}{\alpha}-1} (a^{\alpha+1} - 2b^{\alpha+1}) \right] < 0$$

if it is

$$\left(\frac{2a^{\alpha} + b^{\alpha}}{a^{\alpha} + 2b^{\alpha}} \right)^{\frac{\beta}{\alpha}-1} \frac{2a^{\alpha+1} - b^{\alpha+1}}{2b^{\alpha+1} - a^{\alpha+1}} < 1.$$

It is easy to prove that:

$$\frac{2a^{\alpha} + b^{\alpha}}{a^{\alpha} + 2b^{\alpha}} > 1 \quad \text{and} \quad \frac{2a^{\alpha+1} - b^{\alpha+1}}{2b^{\alpha+1} - a^{\alpha+1}} < 1.$$

Then for all $\alpha + 1 < 0$, $t > 0$ and $a > b$ we have

$$(a-b)\left(a^2 \frac{\partial G}{\partial a} - b^2 \frac{\partial G}{\partial b} \right) = [\Delta]\,[\Theta] < 0. \tag{3.26}$$

For $k = 3$

$$\Theta = \left(\tfrac{3a^{\alpha}+b^{\alpha}}{4}\right)^{\frac{\beta}{\alpha}} \tfrac{3a^{\alpha+1}-b^{\alpha+1}}{3a^{\alpha}+b^{\alpha}} + 2(4)^{\frac{\beta}{\alpha}} (\cosh \alpha t)^{\frac{\beta}{\alpha}-1} \sinh(\alpha+1)t$$
$$+ \left(\tfrac{a^{\alpha}+3b^{\alpha}}{4}\right)^{\frac{\beta}{\alpha}} \tfrac{a^{\alpha+1}-3b^{\alpha+1}}{a^{\alpha}+3b^{\alpha}}$$

where $a = e^t$ and $b = e^{-t}$. Then

$$\Theta = \left(\frac{1}{4} \right)^{\frac{\beta}{\alpha}} \left[(3a^{\alpha} + b^{\alpha})^{\frac{\beta}{\alpha}-1} (3a^{\alpha+1} - b^{\alpha+1}) + (a^{\alpha} + 3b^{\alpha})^{\frac{\beta}{\alpha}-1} (a^{\alpha+1} - 3b^{\alpha+1}) \right]$$
$$+2(4)^{\frac{\beta}{\alpha}} (\cosh \alpha t)^{\frac{\beta}{\alpha}-1} \sinh(\alpha+1)t < 0,$$

if it is

$$\left(\frac{3a^{\alpha} + b^{\alpha}}{a^{\alpha} + 3b^{\alpha}} \right)^{\frac{\beta}{\alpha}-1} \frac{3a^{\alpha+1} - b^{\alpha+1}}{3b^{\alpha+1} - a^{\alpha+1}} < 1.$$

It is easy to prove that:

$$\frac{3a^{\alpha}+b^{\alpha}}{a^{\alpha}+3b^{\alpha}}>1 \quad \text{and} \quad \frac{3a^{\alpha+1}-b^{\alpha+1}}{3b^{\alpha+1}-a^{\alpha+1}}<1.$$

Then for all $\alpha+1<0$, $t>0$ and $a>b$ we have

$$(a-b)\left(a^2\frac{\partial G}{\partial a}-b^2\frac{\partial G}{\partial b}\right)=[\Delta]\,[\Theta]<0. \tag{3.27}$$

From above arguments we have two generalized cases as follows:
(i) When k is an even, on expanding the summation leads to even number of terms. Further, grouping the first and k^{th} term, second and $(k-1)^{\text{th}}$ term, and so on. Hence, we can arrive at the required conclusion by proving as explained for $k=2$.
(ii) When k is an odd, on expanding the summation leads to odd number of terms. Further, grouping first and k^{th} term, second and $(k-1)^{\text{th}}$, and so on. In expansion, the middle term is $2(\cosh\alpha t)^{\frac{\beta}{\alpha}-1}\sinh(\alpha+1)t$. Hence, we can arrive at the required conclusion by proving as explained for $k=1$ and $k=3$.

Case(2). For $\alpha=0$, $\beta+1<0$, $t>0$ and $a>b$ we have:

$$G=G(a,b;k,0,\beta)=\left[\frac{1}{k}\sum_{i=1}^{k}a^{\frac{k+1-i}{k+1}\beta}b^{\frac{i}{k+1}\beta}\right]^{\frac{1}{\beta}}. \tag{3.28}$$

Then

$$(a-b)\left(a^2\frac{\partial G}{\partial a}-b^2\frac{\partial G}{\partial b}\right)=\frac{(a-b)}{k(k+1)}G^{1-\beta}\sqrt{ab}\left[\sum_{i=1}^{k}a^{\frac{k+1-i}{k+1}\beta}b^{\frac{i}{k+1}\beta}(k+1-2i)\right] \tag{3.29}$$

i.e.

$$(a-b)\left(a^2\frac{\partial G}{\partial a}-b^2\frac{\partial G}{\partial b}\right)=[\Delta]\,[\Theta] \tag{3.30}$$

where

$$\Delta=\frac{(a-b)}{k(k+1)}G^{1-\beta}\sqrt{ab} \quad \text{and} \quad \Theta=\left[\sum_{i=1}^{k}a^{\frac{k+1-i}{k+1}\beta}b^{\frac{i}{k+1}\beta}(k+1-2i)\right]$$

We need to prove that

$$(a-b)\left(a^2\frac{\partial G}{\partial a}-b^2\frac{\partial G}{\partial b}\right)=[\Delta]\,[\Theta]<0. \tag{3.31}$$

for $k=1$

$$\Theta=0;$$

for $k=2$

$$\Theta=\left(a^{\frac{\beta}{3}}b^{\frac{\beta}{3}}\right)\left(a^{\frac{\beta}{3}}-b^{\frac{\beta}{3}}\right)<0;$$

for $k=3$

$$\Theta=\left(a^{\frac{3\beta}{4}}b^{\frac{\beta}{4}}\right)\left(a^{\frac{\beta}{2}}-b^{\frac{\beta}{2}}\right)<0;$$

for $k=4$

$$\Theta = \left(3a^{\frac{\beta}{5}}b^{\frac{\beta}{5}}\right)\left(a^{\frac{3\beta}{5}} - b^{\frac{3\beta}{5}}\right) + \left(a^{\frac{2\beta}{5}}b^{\frac{2\beta}{5}}\right)\left(a^{\frac{\beta}{5}} - b^{\frac{\beta}{5}}\right) < 0;$$

for $k=5$

$$\Theta = \left(4a^{\frac{\beta}{6}}b^{\frac{\beta}{6}}\right)\left(a^{\frac{2\beta}{3}} - b^{\frac{2\beta}{3}}\right) + 2\left(a^{\frac{\beta}{3}}b^{\frac{\beta}{3}}\right)\left(a^{\frac{\beta}{3}} - b^{\frac{\beta}{3}}\right) < 0$$

holds for any $\beta < 0$.

From the above cases, in general, we have the following:

$$(a-b)\left(a^2\frac{\partial G}{\partial a} - b^2\frac{\partial G}{\partial b}\right) = [\Delta]\,[\Theta] < 0. \tag{3.32}$$

Case (3). For all $\alpha+1<0$, $\beta=0$, $t>0$ and $a>b$ we have the Gnan mean

$$G = G(a,b;k,\alpha,0) = \prod_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha} + ib^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}} \tag{3.33}$$

equivalent to

$$G = \tag{3.34}$$
$$\left(\frac{ka^{\alpha}+b^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}}\left(\frac{(k-1)a^{\alpha}+2b^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}}\cdots\left(\frac{2a^{\alpha}+(k-1)b^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}}\left(\frac{a^{\alpha}+kb^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}}.$$

Then by grouping first and k^{th}, second and $(k-1)^{\text{th}}$ term, and so on. Hence, it leads to:

$$(a-b)\left(a^2\frac{\partial G}{\partial a} - b^2\frac{\partial G}{\partial b}\right) =$$

$$(a-b)\left[\frac{k}{k} + \frac{2(k-1)}{k} + \frac{3(k-2)}{k} + \frac{4(k-3)}{k} + \ldots\right](ab)^{\alpha}\left(\frac{a^{\alpha}-b^{\alpha}}{\times_i}\right) \tag{3.35}$$
$$+\left[\frac{1+2+3+4+5+\ldots.}{k}\right]\left(\frac{a^{2\alpha+1}-b^{2\alpha+1}}{\times_i}\right)$$

where

$$\times_i = ((k-i)a^{\alpha} + (i+1)b^{\alpha})\,((i+1)a^{\alpha} + (k-i)b^{\alpha}) \leqslant 0$$

For all integral values of $i = 0,1,2,\ldots.,(k-1)$

$$(a-b)\left(a^2\frac{\partial G}{\partial a} - b^2\frac{\partial G}{\partial b}\right) < 0 \tag{3.36}$$

holds for all $\alpha+1<0$.

Case (4). For $\alpha=0$, $\beta=0$, $t>0$ and $a>b$ we have the Gnan mean

$$G = G(a,b;k,0,0) = \sqrt{ab} \tag{3.37}$$

then,

$$(a-b)\left(a^2\frac{\partial g}{\partial a} - b^2\frac{\partial g}{\partial b}\right) = \frac{1}{2}\sqrt{ab}(a-b) < 0. \tag{3.38}$$

This completes the proof of (2) and 3.1. ∎

THEOREM 3.2. *Let $a \geqslant b \geqslant 0$ be arbitrary elements, let k be a non-negative integer and α, β two real numbers. Then:*

(1) *The dual Gnan mean $g(a,b,k;\alpha,\beta)$ is Schur-harmonic convex by a and b if $\alpha+1$, $\beta+1 \geqslant 0$.*
(2) *The dual Gnan mean $g(a,b,k;\alpha,\beta)$ is Schur-harmonic concave by a and b if $\alpha+1$, $\beta+1 \leqslant 0$.*

PROOF. **Proof of (1):**
Case (1). Let α and β be any two non-zero distinct real numbers with $\alpha+1$, $t<0$ and $a>b$. Then the Gnan mean can be defined as:

$$g = g(a,b;k,\alpha,\beta) = \left[\frac{1}{k+1}\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}} \tag{3.39}$$

Taking log on both sides and differentiate partially with respect to a and multiply by a^2 gives

$$a^2\frac{\partial g}{\partial a} = g\frac{1}{\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}}}\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}-1}\frac{k-i}{k}a^{\alpha+1}. \tag{3.40}$$

Similarly

$$b^2\frac{\partial g}{\partial b} = g\frac{1}{\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}}}\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}-1}\frac{i}{k}b^{\alpha+1}. \tag{3.41}$$

Then

$$(a-b)\left(a^2\frac{\partial g}{\partial a} - b^2\frac{\partial g}{\partial b}\right) =$$

$$\frac{(a-b)}{k}g^{1-\beta}\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}}\frac{(k-i)a^{\alpha+1}-ib^{\alpha+1}}{(k-i)a^{\alpha}+ib^{\alpha}} \tag{3.42}$$

i.e.

$$(a-b)\left(a^2\frac{\partial g}{\partial a} - b^2\frac{\partial g}{\partial b}\right) = [\Delta]\,[\Theta] \tag{3.43}$$

where

$$\Delta = \frac{(a-b)}{k}g^{1-\beta} \quad \text{and} \quad \Theta = \sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}}\frac{(k-i)a^{\alpha+1}-ib^{\alpha+1}}{(k-i)a^{\alpha}+ib^{\alpha}}$$

$a=e^t$ and $b=e^{-t}$. Then for all $\beta+1$, $t>0$ and $a>b$ we have:
For $k=1$

$$\Theta = (a^{\alpha})^{\frac{\beta}{\alpha}-1}a^{\alpha+1} + (b^{\alpha})^{\frac{\beta}{\alpha}-1}(-b^{\alpha+1}) = 2\sinh(\beta+1)t.$$

Thus

$$(a-b)\left(a^2\frac{\partial g}{\partial a}-b^2\frac{\partial g}{\partial b}\right)=[\Delta]\,[\Theta]>0. \tag{3.44}$$

For $k=2$

$$\Theta=(a^\alpha)^{\frac{\beta}{\alpha}-1}\,a^{\alpha+1}+\left(\frac{a^\alpha+b^\alpha}{2}\right)^{\frac{\beta}{\alpha}-1}\left(\frac{a^\alpha-b^\alpha}{2}\right).$$

Thus

$$+\,(b^\alpha)^{\frac{\beta}{\alpha}-1}\,(-b^{\alpha+1})=2\sinh(\beta+1)t+M_\alpha^{\beta-\alpha}\sinh\alpha t.$$

$$(a-b)\left(a^2\frac{\partial g}{\partial a}-b^2\frac{\partial g}{\partial b}\right)=[\Delta]\,[\Theta]>0. \tag{3.45}$$

For $k=3$ we have

$$\Theta=(a^\alpha)^{\frac{\beta}{\alpha}-1}\,a^{\alpha+1}+\left(\frac{2a^\alpha+b^\alpha}{3}\right)^{\frac{\beta}{\alpha}-1}\left(\frac{2a^{\alpha+1}-b^{\alpha+1}}{3}\right)$$

$$+\left(\frac{a^\alpha+2b^\alpha}{3}\right)^{\frac{\beta}{\alpha}-1}\left(\frac{a^{\alpha+1}-2b^{\alpha+1}}{3}\right)+(b^\alpha)^{\frac{\beta}{\alpha}-1}\,(-b^{\alpha+1}).$$

Simplification leads to

$$\Theta=2\sinh(\beta+1)t+\left(\frac{1}{3}\right)^{\frac{\beta}{\alpha}}\left(\frac{2a^\alpha+b^\alpha}{a^\alpha+2b^\alpha}\right)^{\frac{\beta}{\alpha}-1}\left(\frac{2a^{\alpha+1}-b^{\alpha+1}}{2b^{\alpha+1}-a^{\alpha+1}}\right).$$

Then

$$(a-b)\left(a^2\frac{\partial g}{\partial a}-b^2\frac{\partial g}{\partial b}\right)=[\Delta]\,[\Theta]>0. \tag{3.46}$$

Similarly for $k=4$ we have

$$\Theta=2\sinh(\beta+1)t+M_\alpha^{\beta-\alpha}\sinh\alpha t+\left(\frac{1}{4}\right)^{\frac{\beta}{\alpha}}\left(\frac{3a^\alpha+b^\alpha}{a^\alpha+3b^\alpha}\right)^{\frac{\beta}{\alpha}-1}\left(\frac{3a^{\alpha+1}-b^{\alpha+1}}{3b^{\alpha+1}-a^{\alpha+1}}\right).$$

From above arguments, we have the following generalized cases:

(i) When k is an even, on expanding the summation leads to odd number of terms then grouping first and last term together gives $2\sinh(\beta+1)t$. In the expansion, the middle term is $M_\alpha^{\beta-\alpha}\sinh\alpha t$ and rest of the terms are grouped as second term and $(k-1)^{\text{th}}$ and so on. Hence, we can arrive at the required conclusion by proving as explained for $k=2, k=4$.

(ii) When k is an odd, on expanding the summation leads to even number of terms then grouping first and k^{th} term gives $2\sinh(\beta+1)t$, second and $(k-1)^{\text{th}}$ and so on. Hence, we can arrive at the required conclusion by proving as explained for $k=1$ and $k=3$.

Case (2). For $\alpha=0$, $\beta+1>0$, $t>0$ and $a>b$ we have:

$$g=g(a,b;k,0,\beta)=\left[\frac{1}{k+1}\sum_{i=0}^{k}a^{\frac{k-i}{k}\beta}b^{\frac{i}{k}\beta}\right]^{\frac{1}{\beta}}. \tag{3.47}$$

Then

$$(a-b)\left(a^2\frac{\partial g}{\partial a}-b^2\frac{\partial g}{\partial b}\right)=\frac{(a-b)}{k(k+1)}G^{1-\beta}\sqrt{ab}\left[\sum_{i=0}^{k}a^{\frac{k-i}{k}\beta}b^{\frac{i}{k}\beta}(k-2i)\right] \tag{3.48}$$

i.e.

$$(a-b)\left(a^2\frac{\partial g}{\partial a}-b^2\frac{\partial g}{\partial b}\right)=[\Delta]\,[\Theta] \tag{3.49}$$

where

$$\Delta=\frac{(a-b)}{k(k+1)}g^{1-\beta}\sqrt{ab}\quad\text{and}\quad\Theta=\left[\sum_{i=0}^{k}a^{\frac{k-i}{k}\beta}b^{\frac{i}{k}\beta}(k-2i)\right].$$

We need to prove

$$(a-b)\left(a^2\frac{\partial g}{\partial a}-b^2\frac{\partial g}{\partial b}\right)=[\Delta]\,[\Theta]>0. \tag{3.50}$$

For $k=1$

$$\Theta=a^\beta-b^\beta;$$

for $k=2$

$$\Theta=2\left(a^\beta-b^\beta\right)>0;$$

for $k=3$

$$\Theta=3\left(a^\beta-b^\beta\right)+\left(a^{\frac{\beta}{3}}b^{\frac{\beta}{3}}\right)\left(a^{\frac{\beta}{3}}-b^{\frac{\beta}{3}}\right)>0;$$

for $k=4$

$$\Theta=4\left(a^\beta-b^\beta\right)+2\left(a^{\frac{\beta}{4}}b^{\frac{\beta}{4}}\right)\left(a^{\frac{\beta}{2}}-b^{\frac{\beta}{2}}\right)>0;$$

For $k=5$

$$\Theta=5\left(a^\beta-b^\beta\right)+3\left(a^{\frac{\beta}{5}}b^{\frac{\beta}{5}}\right)\left(a^{\frac{3\beta}{5}}-b^{\frac{3\beta}{5}}\right)+\left(a^{\frac{2\beta}{5}}b^{\frac{2\beta}{5}}\right)\left(a^{\frac{\beta}{5}}-b^{\frac{\beta}{5}}\right)>0;$$

for $k=6$

$$\Theta=6\left(a^\beta-b^\beta\right)+4\left(a^{\frac{\beta}{6}}b^{\frac{\beta}{6}}\right)\left(a^{\frac{2\beta}{3}}-b^{\frac{2\beta}{3}}\right)+2\left(a^{\frac{\beta}{3}}b^{\frac{\beta}{3}}\right)\left(a^{\frac{\beta}{3}}-b^{\frac{\beta}{3}}\right)>0$$

holds for all $\beta+1>0$.

From above arguments we have:

$$(a-b)\left(a^2\frac{\partial g}{\partial a}-b^2\frac{\partial g}{\partial b}\right)=[\Delta]\,[\Theta]>0. \tag{3.51}$$

Case (3). For all $\alpha>0$, $\beta=0$, $t>0$ and $a>b$ we have the Gnan mean

$$g=g(a,b;k,\alpha,0)=\prod_{i=0}^{k}\left(\frac{(k-i)a^\alpha+ib^\alpha}{k}\right)^{\frac{1}{(k+1)\alpha}} \tag{3.52}$$

equivalent to

(3.53)

$$g=(ka^\alpha)^{\frac{1}{(k+1)\alpha}}\left(\frac{(k-1)a^\alpha+b^\alpha}{k}\right)^{\frac{1}{(k+1)\alpha}}\cdots\left(\frac{a^\alpha+(k-1)b^\alpha}{k}\right)^{\frac{1}{(k+1)\alpha}}(kb^\alpha)^{\frac{1}{(k+1)\alpha}}.$$

Then by grouping first and k^{th} term, second and $(k-1)^{\text{th}}$, and so on leads to

$$(a-b)\left(a^2\frac{\partial g}{\partial a}-b^2\frac{\partial g}{\partial b}\right)=$$

$$(a-b)\left[\frac{k-1}{k+1}+\frac{2(k-2)}{k+1}+\frac{3(k-3)}{k+1}+\frac{4(k-4)}{k+1}+...\right](ab)^{\alpha}\left(\frac{a^{\alpha}-b^{\alpha}}{\times_i}\right) \tag{3.54}$$
$$+\left[\frac{1+2+3+4+5+....}{k}\right]\left(\frac{a^{2\alpha+1}-b^{2\alpha+1}}{\times_i}\right)$$

where

$$\times_i=((k-i)a^{\alpha}+(i+1)b^{\alpha})\,((i+1)a^{\alpha}+(k-i)b^{\alpha})\geqslant 0$$

for all integral values of $i=1,2,\cdots,(k-1)$

$$(a-b)\left(a^2\frac{\partial G}{\partial a}-b^2\frac{\partial G}{\partial b}\right)>0 \tag{3.55}$$

holds for all $\alpha+1>0$.

Case (4). For $\alpha=0$, $\beta=0$, $t>0$ and $a>b$ we have

$$G=G(a,b;k,0,0)=\sqrt{ab}. \tag{3.56}$$

Then

$$(a-b)\left(a^2\frac{\partial g}{\partial a}-b^2\frac{\partial g}{\partial b}\right)=\frac{1}{2}\sqrt{ab}(a-b)>0. \tag{3.57}$$

Proof of (2):

Case (1). Let α, β be any two non-zero distinct real numbers with $\alpha+1<0$, $t>0$ and $a>b$. The Gnan mean can be defined as

$$g=g(a,b;k,\alpha,\beta)=\left[\frac{1}{k+1}\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}}. \tag{3.58}$$

Taking log on both sides and differentiate partially with respect to a and multiply by a^2 we have

$$a^2\frac{\partial g}{\partial a}=g\frac{1}{\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}}}\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}-1}\frac{k-i}{k}a^{\alpha+1}. \tag{3.59}$$

Similarly

$$b^2\frac{\partial g}{\partial b}=g\frac{1}{\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}}}\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}-1}\frac{i}{k}b^{\alpha+1}. \tag{3.60}$$

Then

$$(a-b)\left(a^2\frac{\partial g}{\partial a}-b^2\frac{\partial g}{\partial b}\right)=$$

$$\frac{(a-b)}{k} g^{1-\beta} \sum_{i=0}^{k} \left(\frac{(k-i)a^{\alpha} + ib^{\alpha}}{k} \right)^{\frac{\beta}{\alpha}} \frac{(k-i)a^{\alpha+1} - ib^{\alpha+1}}{(k-i)a^{\alpha} + ib^{\alpha}} \tag{3.61}$$

i.e.

$$(a-b)\left(a^2 \frac{\partial g}{\partial a} - b^2 \frac{\partial g}{\partial b} \right) = [\Delta]\,[\Theta] \tag{3.62}$$

where

$$\Delta = \frac{(a-b)}{k} g^{1-\beta} \quad \text{and} \quad \Theta = \sum_{i=0}^{k} \left(\frac{(k-i)a^{\alpha} + ib^{\alpha}}{k} \right)^{\frac{\beta}{\alpha}} \frac{(k-i)a^{\alpha+1} - ib^{\alpha+1}}{(k-i)a^{\alpha} + ib^{\alpha}}$$

$a = e^t$ and $b = e^{-t}$. Then for all $\beta + 1 < 0$, $t > 0$ and $a > b$ considering the following particular values of k, we have:

For $k = 1$

$$\Theta = (a^{\alpha})^{\frac{\beta}{\alpha}-1} a^{\alpha+1} + (b^{\alpha})^{\frac{\beta}{\alpha}-1} (-b^{\alpha+1}) = 2\sinh(\beta+1)t.$$

Thus

$$(a-b)\left(a^2 \frac{\partial g}{\partial a} - b^2 \frac{\partial g}{\partial b} \right) = [\Delta]\,[\Theta] < 0. \tag{3.63}$$

For $k = 2$

$$\Theta = (a^{\alpha})^{\frac{\beta}{\alpha}-1} a^{\alpha+1} + \left(\frac{a^{\alpha} + b^{\alpha}}{2} \right)^{\frac{\beta}{\alpha}-1} \left(\frac{a^{\alpha} - b^{\alpha}}{2} \right).$$

$$+ (b^{\alpha})^{\frac{\beta}{\alpha}-1} (-b^{\alpha+1}) = 2\sinh(\beta+1)t + M_{\alpha}^{\beta-\alpha} \sinh \alpha t.$$

Thus

$$(a-b)\left(a^2 \frac{\partial g}{\partial a} - b^2 \frac{\partial g}{\partial b} \right) = [\Delta]\,[\Theta] < 0. \tag{3.64}$$

For $k = 3$

$$\Theta = (a^{\alpha})^{\frac{\beta}{\alpha}-1} a^{\alpha+1} + \left(\frac{2a^{\alpha} + b^{\alpha}}{3} \right)^{\frac{\beta}{\alpha}-1} \left(\frac{2a^{\alpha+1} - b^{\alpha+1}}{3} \right)$$

$$+ \left(\frac{a^{\alpha} + 2b^{\alpha}}{3} \right)^{\frac{\beta}{\alpha}-1} \left(\frac{a^{\alpha+1} - 2b^{\alpha+1}}{3} \right) + (b^{\alpha})^{\frac{\beta}{\alpha}-1} (-b^{\alpha+1}).$$

Simplification leads to

$$\Theta = 2\sinh(\beta+1)t + \left(\frac{1}{3} \right)^{\frac{\beta}{\alpha}} \left(\frac{2a^{\alpha} + b^{\alpha}}{a^{\alpha} + 2b^{\alpha}} \right)^{\frac{\beta}{\alpha}-1} \left(\frac{2a^{\alpha+1} - b^{\alpha+1}}{2b^{\alpha+1} - a^{\alpha+1}} \right).$$

Thus

$$(a-b)\left(a^2 \frac{\partial g}{\partial a} - b^2 \frac{\partial g}{\partial b} \right) = [\Delta]\,[\Theta] < 0. \tag{3.65}$$

Similarly for $k = 4$ we have

$$\Theta = 2\sinh(\beta+1)t + M_{\alpha}^{\beta-\alpha}\sinh\alpha t + \left(\frac{1}{4}\right)^{\frac{\beta}{\alpha}}\left(\frac{3a^{\alpha}+b^{\alpha}}{a^{\alpha}+3b^{\alpha}}\right)^{\frac{\beta}{\alpha}-1}\left(\frac{3a^{\alpha+1}-b^{\alpha+1}}{3b^{\alpha+1}-a^{\alpha+1}}\right).$$

From above arguments we have two generalized cases as follows:

(i) When k is an even, expanding the summation leads to odd number of terms then grouping first and k^{th} term leads to $2\sinh(\beta+1)t$ and the middle term is $M_{\alpha}^{\beta-\alpha}\sinh\alpha t$. The rest of the terms are grouped as second and $(k-1)^{\text{th}}$ term, and so on. Hence, we get the required conclusion by proving as explained for $k = 2$ and $k = 4$.

(ii) When k is an odd, expanding the summation leads to even number of terms then grouping first and k^{th} term leads to $2\sinh(\beta+1)t$, second $(k-1)^{\text{th}}$ term, and so on. Hence, we get the required conclusion by proving as explained for $k = 1$ and $k = 3$.

Case (2). For $\alpha = 0$, $\beta + 1 < 0$, $t > 0$ and $a > b$ we have

$$g = g(a,b;k,0,\beta) = \left[\frac{1}{k+1}\sum_{i=0}^{k} a^{\frac{k-i}{k}\beta} b^{\frac{i}{k}\beta}\right]^{\frac{1}{\beta}}. \tag{3.66}$$

Then

$$(a-b)\left(a^2\frac{\partial g}{\partial a} - b^2\frac{\partial g}{\partial b}\right) = \frac{(a-b)}{k(k+1)}G^{1-\beta}\sqrt{ab}\left[\sum_{i=0}^{k} a^{\frac{k-i}{k}\beta} b^{\frac{i}{k}\beta}(k-2i)\right] \tag{3.67}$$

i.e.

$$(a-b)\left(a^2\frac{\partial g}{\partial a} - b^2\frac{\partial g}{\partial b}\right) = [\Delta]\,[\Theta] \tag{3.68}$$

where

$$\Delta = \frac{(a-b)}{k(k+1)}g^{1-\beta}\sqrt{ab} \quad \text{and} \quad \Theta = \left[\sum_{i=0}^{k} a^{\frac{k-i}{k}\beta} b^{\frac{i}{k}\beta}(k-2i)\right].$$

We need to prove

$$(a-b)\left(a^2\frac{\partial g}{\partial a} - b^2\frac{\partial g}{\partial b}\right) = [\Delta]\,[\Theta] < 0. \tag{3.69}$$

For $k = 1$

$$\Theta = a^{\beta} - b^{\beta} < 0.$$

For $k = 2$

$$\Theta = 2\left(a^{\beta} - b^{\beta}\right) < 0.$$

For $k = 3$

$$\Theta = 3\left(a^{\beta} - b^{\beta}\right) + \left(a^{\frac{\beta}{3}} b^{\frac{\beta}{3}}\right)\left(a^{\frac{\beta}{3}} - b^{\frac{\beta}{3}}\right) < 0.$$

For $k = 4$

$$\Theta = 4\left(a^{\beta} - b^{\beta}\right) + 2\left(a^{\frac{\beta}{4}} b^{\frac{\beta}{4}}\right)\left(a^{\frac{\beta}{2}} - b^{\frac{\beta}{2}}\right) < 0.$$

For $k = 5$

$$\Theta = 5\left(a^{\beta} - b^{\beta}\right) + 3\left(a^{\frac{\beta}{5}} b^{\frac{\beta}{5}}\right)\left(a^{\frac{3\beta}{5}} - b^{\frac{3\beta}{5}}\right) + \left(a^{\frac{2\beta}{5}} b^{\frac{2\beta}{5}}\right)\left(a^{\frac{\beta}{5}} - b^{\frac{\beta}{5}}\right) < 0.$$

For $k = 6$

$$\Theta = 6\left(a^{\beta} - b^{\beta}\right) + 4\left(a^{\frac{\beta}{6}} b^{\frac{\beta}{6}}\right)\left(a^{\frac{2\beta}{3}} - b^{\frac{2\beta}{3}}\right) + 2\left(a^{\frac{\beta}{3}} b^{\frac{\beta}{3}}\right)\left(a^{\frac{\beta}{3}} - b^{\frac{\beta}{3}}\right) < 0$$

holds for all $\beta < 0$.

From above arguments, we have the conclusion:

$$(a-b)\left(a^2 \frac{\partial g}{\partial a} - b^2 \frac{\partial g}{\partial b}\right) = [\Delta]\,[\Theta] < 0. \tag{3.70}$$

Case (3). For $\alpha + 1 < 0$, $\beta = 0$, $t > 0$ and $a > b$ we have

$$g = g(a, b; k, \alpha, 0) = \prod_{i=0}^{k} \left(\frac{(k-i)a^{\alpha} + ib^{\alpha}}{k}\right)^{\frac{1}{(k+1)\alpha}} \tag{3.71}$$

equivalent to

(3.72)

$$g = (ka^{\alpha})^{\frac{1}{(k+1)\alpha}} \left(\frac{(k-1)a^{\alpha} + b^{\alpha}}{k}\right)^{\frac{1}{(k+1)\alpha}} \cdots \left(\frac{a^{\alpha} + (k-1)b^{\alpha}}{k}\right)^{\frac{1}{(k+1)\alpha}} (kb^{\alpha})^{\frac{1}{(k+1)\alpha}}.$$

Then by grouping first and last term, second and second from the last, and so on leads to:

$$(a-b)\left(a^2 \frac{\partial g}{\partial a} - b^2 \frac{\partial g}{\partial b}\right) =$$

$$(a-b)\left[\frac{k-1}{k+1} + \frac{2(k-2)}{k+1} + \frac{3(k-3)}{k+1} + \frac{4(k-4)}{k+1} + \ldots\right](ab)^{\alpha}\left(\frac{a^{\alpha} - b^{\alpha}}{\times_i}\right) + \left[\frac{1+2+3+4+5+\ldots.}{k}\right]\left(\frac{a^{2\alpha+1} - b^{2\alpha+1}}{\times_i}\right) \tag{3.73}$$

where

$$\times_i = ((k-i)a^{\alpha} + (i+1)b^{\alpha})\,((i+1)a^{\alpha} + (k-i)b^{\alpha}) \leqslant 0.$$

For all integral values of $i = 1, 2, \ldots., (k-1)$

$$(a-b)\left(a^2 \frac{\partial G}{\partial a} - b^2 \frac{\partial G}{\partial b}\right) < 0 \tag{3.74}$$

holds for all $\alpha < 0$.

Case (4). For $\alpha = 0$, $\beta = 0$, $t > 0$ and $a > b$ we have the Gnan mean as:

$$G = G(a, b; k, 0, 0) = \sqrt{ab}. \tag{3.75}$$

Then

$$(a-b)\left(a^2 \frac{\partial g}{\partial a} - b^2 \frac{\partial g}{\partial b}\right) = \frac{1}{2}\sqrt{ab}(a-b) < 0. \tag{3.76}$$

This completes the proof of (2) and the proof of Theorem 3.2. $\square$

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