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SCHUR GEOMETRIC CONVEXITY OF GNAN MEAN FOR TWO VARIABLES

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ABSTRACT. In this paper, the Schur geometric convexity of the Gnan mean and its dual form in two variables are discussed.

1. Introduction

For positive numbers a, b, let

$$I = I(a,b) = \begin{cases} \exp\left[\dfrac{b\ln b - a\ln a}{b-a} - 1\right], & a < b; \\ a, & a = b; \end{cases} \tag{1.1}$$

$$L = L(a,b) = \begin{cases} \dfrac{a-b}{\ln a - \ln b}, & a \neq b; \\ a, & a = b; \end{cases} \tag{1.2}$$

$$H = H(a,b) = \frac{a + \sqrt{ab} + b}{3}. \tag{1.3}$$

These are respectively called the Identric, Logarithmic and Heronian means.

In [**5**,**19**,**20**], V. Lokesha et al. studied extensively and obtained some remarkable results on the weighted Heron mean, the weighted Heron dual mean and the weighted product type means and their monotonicity.

In [**17**,**18**], Zhang et al. gave the generalizations of Heron mean, similar product type means and their dual forms.

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For two variables, the above means as follows:

(1.4)
$$I(a,b;k)=\prod_{i=1}^{k}\left(\frac{(k+1-i)a+ib}{k+1}\right)^{\frac{1}{k}},\quad I^{*}(a,b;k)=\prod_{i=0}^{k}\left(\frac{(k-i)a+ib}{k}\right)^{\frac{1}{k+1}},$$

and

$$H(a,b;k)=\frac{1}{k+1}\sum_{i=0}^{k}a^{\frac{k-i}{k}}b^{\frac{i}{k}},\quad h(a,b;k)=\frac{1}{k}\sum_{i=1}^{k}a^{\frac{k+1-i}{k+1}}b^{\frac{i}{k+1}}, \tag{1.5}$$

where k is a natural number. Authors proved that $H(a,b;k)$ and $I^{*}(a,b;k)$ are monotone decreasing functions, $h(a,b;k)$ and $I(a,b;k)$ are monotone increasing functions for any natural k and also established the following limitations:

$$\lim_{k\to+\infty}I(a,b;k)=\lim_{k\to+\infty}I^{*}(a,b;k)=I(a,b),$$

and

$$\lim_{k\to+\infty}H(a,b;k)=\lim_{k\to+\infty}h(a,b;k)=L(a,b).$$

In [**2**, **4**], the authors introduced the Gnan mean and its dual form for two variables. Also introduced the Gnan mean and its dual form for n variables. They obtained some interesting properties, monotonic results and its limitations. The definitions of Gnan mean and its dual form are given in next section. The Schur convex function was introduced by I. Schur, in 1923. It has many important applications in analytic inequalities. In 2003 X.M. Zhang propose the concept of "Schur-harmonically convex function" which is an extension of "Schur-convexity function". The detailed discussion on convexity and Schur-convexity can be found in [**1**]- [**12**].

2. Definitions and lemmas

In this section, we recall the definitions and lemmas which are essential to develop this paper.

Definition 2.1 ([**2**]). Let $a\geqslant 0$ and $b\geqslant 0$, and k be a non-negative integer, α,β two real numbers. The Gnan mean $G(a,b;k,\alpha,\beta)$ and its dual $g(a,b;k,\alpha,\beta)$ are forming as shown below:

$$\begin{aligned}
G(a,b;k,\alpha,\beta)&=\left[\frac{1}{k}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}},\\
G(a,b;k,0,\beta)&=\left[\frac{1}{k}\sum_{i=1}^{k}a^{\frac{(k+1-i)\beta}{k+1}}b^{\frac{i\beta}{k+1}}\right]^{\frac{1}{\beta}},\\
G(a,b;k,\alpha,0)&=\prod_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}},\\
G(a,b;k,0,0)&=\sqrt{ab};
\end{aligned}\tag{2.1}$$

and

$$
\begin{aligned}
g(a,b;k,\alpha,\beta) &= \left[\frac{1}{k+1}\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}},\\
g(a,b;k,0,\beta) &= \left[\frac{1}{k+1}\sum_{i=0}^{k}a^{\frac{(k-i)\beta}{k}}b^{\frac{i\beta}{k}}\right]^{\frac{1}{\beta}},\\
g(a,b;k,\alpha,0) &= \prod_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{1}{(k+1)\alpha}},\\
g(a,b;k,0,0) &= \sqrt{ab}.
\end{aligned}
\tag{2.2}
$$

Definition 2.2. ([**6**], [**14**]) Let $x=(x_1,x_2,...,x_n)$ and $y=(y_1,y_2,...,y_n)$ be arbitrary elements in $\mathbf{R}^n$. Then:

(1) x is majorized by y (in symbol $x \prec y$) if
$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]} \text{ and } \sum_{i=1}^{n} x_{[i]} \leqslant \sum_{i=1}^{n} y_{[i]},$$
where $x_{[1]} \geqslant,...,\geqslant x_{[n]}$ and $y_{[1]} \geqslant,...,\geqslant y_{[n]}$ are rearrangements of x and y in descending order.

(2) $x \geqslant y$ means $x_i \geqslant y_i$ for all $i=1,2,...,n$.
Let $\Omega \subseteq \mathbf{R}^n (n \geqslant 2)$. The function $\varphi:\Omega \to \mathbf{R}$ is said to be decreasing if and only if $-\varphi$ is increasing.

(3) $\Omega \subseteq \mathbf{R}^n$ is called a convex set if $(\alpha x_1+\beta y_1,....,\alpha x_n+\beta y_n)$ for every x, $y \in \Omega$ where $\alpha,\beta \in [0,1]$ with $\alpha+\beta=1$.

(4) Let $\Omega \subseteq \mathbf{R}^n$ the function $\varphi:\Omega \to \mathbf{R}$ be said to be a Schur-convex function on Ω if $x \leqslant y$ on Ω implies $\varphi(x) \leqslant \varphi(y)$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex.

Definition 2.3. ([**16**]) Let $x=(x_1,x_2,...,x_n)$ and $y=(y_1,y_2,...,y_n)$ be arbitrary elements in $\mathbf{R}_+^n$. Then for Ω $(\subseteq \mathbf{R}^n)$ is called harmonically convex set if $(x_1^{\alpha}y_1^{\beta},....,x_1^{\alpha}y_1^{\beta}) \in \Omega$ for all x, $y \in \Omega$ where $\alpha,\beta \in [0,1]$ with $\alpha+\beta=1$. For $\Omega \subseteq \mathbf{R}_+^n$, the function $\varphi:\Omega \to \mathbf{R}_+$ is said to be Schur-harmonically convex function on Ω if $(lnx_1,...,ln_n) \prec (lny_1,...,lny_n)$ on Ω implies $\varphi(x) \leqslant \varphi(y)$. φ is said to be a Schur-harmonically concave function on Ω if and only if $-\varphi$ is Schur-harmonically convex.

Definition 2.4. ([**6**], [**14**]) For $\Omega \subseteq \mathbf{R}^n$ is called symmetric set if $x \in \Omega$ implies $Px \in \Omega$ for every $n \times n$ matrix permutation P. A function $\varphi:\Omega \to \mathbf{R}$ is called symmetric if for every matrix permutation P, holds $\varphi(Px)=\varphi(x)$ for all $x \in \Omega$.

Definition 2.5. ([**6**], [**14**]) For $\Omega \subseteq \mathbf{R}^n$ if a function $\varphi:\Omega \to R$ is symmetric and convex function, then φ is called Schur-convex function on Ω.

Lemma 2.1 ([**16**]). *Let $\Omega \subseteq \boldsymbol{R}^n$ be a symmetric set with non empty interior geometrically convex set and let $\varphi:\Omega \to \boldsymbol{R}_+$ be continuous on Ω and differentiable* *in Ω^0. If φ is symmetric on Ω and*

$$(lnx_1 - lnx_2)\left(x_1\frac{\partial\varphi}{\partial x_1} - x_2\frac{\partial\varphi}{\partial x_2}\right) \geqslant 0\,(\leqslant 0). \tag{2.3}$$

holds for any $x = (x_1, x_2, ..., x_n) \in \Omega^0$, then φ is a Schur-geometrically convex (Schur-geometrically concave) function.

LEMMA 2.2 ([**13**]). *Let $a \leqslant b$ and $u(t) = ta + (1-t)b, \quad v(t) = tb + (1-t)a$, $\frac{1}{2} \leqslant t_2 \leqslant t_1 \leqslant 1 \quad or \quad 0 \leqslant t_1 \leqslant t_2 \leqslant \frac{1}{2} \quad$ then*

$$\frac{(a+b)}{2} \prec (u(t_2), v(t_2)) \prec (u(t_1), v(t_1)). \tag{2.4}$$

3. Main Results

In this section the Schur-geometric convexity of Gnan mean for two variables are established by the method of grouping the terms.

THEOREM 3.1. *Let $a \geqslant b \geqslant 0$ be arbitrary elements, k be a non-negative integer and α, β be two real numbers. Then:*

(1) *The Gnan mean $G(a, b, k; \alpha, \beta)$ is Schur-geometric convex by a and b if $\alpha \geqslant 0$ and $\beta \geqslant 0$.*
(2) *The Gnan mean $G(a, b, k; \alpha, \beta)$ is Schur-geometric concave by a and b if $\alpha \leqslant 0$ and $\beta \leqslant 0$.*

PROOF. **Proof of (1)**.

Case (i). Put $\alpha \neq \beta \neq 0$, $\alpha > 0$, $t > 0$ and $a > b$. We have the Gnan mean

$$G = G(a, b; k, \alpha, \beta) = \left[\frac{1}{k}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha + ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}} \tag{3.1}$$

Let us take log on both sides and differentiate partially with respect to a and multiply by a. Then we have

$$a\frac{\partial G}{\partial a} = G\frac{1}{\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha+ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}}}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha + ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}-1}\frac{k+1-i}{k+1}a^\alpha. \tag{3.2}$$

Similarly we have

$$b\frac{\partial G}{\partial b} = G\frac{1}{\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha+ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}}}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha + ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}-1}\frac{i}{k+1}b^\alpha. \tag{3.3}$$

Then

$$
\begin{aligned}
&(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = \\
&\frac{(lna - lnb)}{k}G^{1-\beta}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}}\frac{(k+1-i)a^{\alpha}-ib^{\alpha}}{(k+1-i)a^{\alpha}+ib^{\alpha}}.
\end{aligned}
\tag{3.4}
$$

i.e.

$$(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = [\Delta]\,[\Theta] \tag{3.5}$$

where

$$\Delta = \frac{(lna - lnb)}{k}G^{1-\beta}$$

and

$$\Theta = \sum_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha}+ib^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}}\frac{(k+1-i)a^{\alpha}-ib^{\alpha}}{(k+1-i)a^{\alpha}+ib^{\alpha}}.$$

For $k = 1$ we have

$$\Theta = \left(\frac{a^{\alpha}+b^{\alpha}}{2}\right)^{\frac{\beta}{\alpha}}\frac{a^{\alpha}-b^{\alpha}}{a^{\alpha}+b^{\alpha}} = (\cosh \alpha t)^{\frac{\beta}{\alpha}}\tanh \alpha t$$

where $a = e^t$ and $b = e^{-t}$. Then for all $\alpha, t > 0$ and $a > b$, we have

$$(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = [\Delta]\,[\Theta] > 0. \tag{3.6}$$

For $k = 2$ we have

$$\Theta = \left(\frac{2a^{\alpha}+b^{\alpha}}{3}\right)^{\frac{\beta}{\alpha}}\frac{2a^{\alpha}-b^{\alpha}}{2a^{\alpha}+b^{\alpha}} + \left(\frac{a^{\alpha}+2b^{\alpha}}{3}\right)^{\frac{\beta}{\alpha}}\frac{a^{\alpha}-2b^{\alpha}}{a^{\alpha}+2b^{\alpha}}$$

i.e.

$$\Theta = \left(\frac{1}{3}\right)^{\frac{\beta}{\alpha}}\left[(2a^{\alpha}+b^{\alpha})^{\frac{\beta}{\alpha}-1}(2a^{\alpha}-b^{\alpha}) + (a^{\alpha}+2b^{\alpha})^{\frac{\beta}{\alpha}-1}(a^{\alpha}-2b^{\alpha})\right] > 0$$

if it is

$$\left(\frac{2a^{\alpha}+b^{\alpha}}{a^{\alpha}+2b^{\alpha}}\right)^{\frac{\beta}{\alpha}-1}\frac{2a^{\alpha}-b^{\alpha}}{2b^{\alpha}-a^{\alpha}} > 1.$$

It is easy to prove that

$$\frac{2a^{\alpha}+b^{\alpha}}{a^{\alpha}+2b^{\alpha}} > 1 \quad \text{and} \quad \frac{2a^{\alpha}-b^{\alpha}}{2b^{\alpha}-a^{\alpha}} > 1.$$

Then for all $\alpha, t > 0$ and $a > b$ we have

$$(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = [\Delta]\,[\Theta] > 0. \tag{3.7}$$

For $k = 3$ we have

$$\Theta = \left(\frac{3a^\alpha + b^\alpha}{4}\right)^{\frac{\beta}{\alpha}} \frac{3a^\alpha - b^\alpha}{3a^\alpha + b^\alpha} + 4^{\frac{\beta}{\alpha}} (\cosh \alpha t)^{\frac{\beta}{\alpha}} \tanh \alpha t + \left(\frac{a^\alpha + 3b^\alpha}{4}\right)^{\frac{\beta}{\alpha}} \frac{a^\alpha - 3b^\alpha}{a^\alpha + 3b^\alpha}$$

where $a = e^t$ and $b = e^{-t}$. Thus

$$\Theta = \left(\frac{1}{4}\right)^{\frac{\beta}{\alpha}} \left[(3a^\alpha + b^\alpha)^{\frac{\beta}{\alpha}-1} (3a^\alpha - b^\alpha) + (a^\alpha + 3b^\alpha)^{\frac{\beta}{\alpha}-1} (a^\alpha - 3b^\alpha)\right] + 4^{\frac{\beta}{\alpha}} (\cosh \alpha t)^{\frac{\beta}{\alpha}} \tanh \alpha t > 0,$$

if it is

$$\left(\frac{3a^\alpha + b^\alpha}{a^\alpha + 3b^\alpha}\right)^{\frac{\beta}{\alpha}-1} \frac{3a^\alpha - b^\alpha}{3b^\alpha - a^\alpha} > 1.$$

It is easy to prove that

$$\frac{3a^\alpha + b^\alpha}{a^\alpha + 3b^\alpha} > 1 \quad \text{and} \quad \frac{3a^\alpha - b^\alpha}{3b^\alpha - a^\alpha} > 1.$$

Then for all $\alpha, t > 0$ and $a > b$ we have

$$(lna - lnb) \left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = [\Delta] [\Theta] > 0. \tag{3.8}$$

From above arguments we have two generalized cases as follows:

(a) When k is an even, on expanding the summation leads to even number of terms. Further, grouping the first and k^{th} term second and $(k-1)^{th}$ and so on, then we reached the required conclusion by proving as explained for $k = 2$.

(b) When k is an odd, on expanding the summation leads to odd number of terms. Further, grouping the first and k^{th} term, second and $(k-1)^{th}$ and middle term is $(\cosh \alpha t)^{\frac{\beta}{\alpha}} \tanh \alpha t$, then we reached the required conclusion by proving as explained for $k = 1$ and $k = 3$.

Case (ii). For all $\alpha = 0$, $\beta > 0$, $t > 0$ and $a > b$ we have the Gnan mean:

$$G = G(a, b; k, 0, \beta) = \left[\frac{1}{k} \sum_{i=1}^{k} a^{\frac{k+1-i}{k+1}\beta} b^{\frac{i}{k+1}\beta}\right]^{\frac{1}{\beta}}. \tag{3.9}$$

Then

$$(lna - lnb) \left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = \frac{(lna - lnb)}{k(k+1)} G^{1-\beta} \left[\sum_{i=1}^{k} a^{\frac{k+1-i}{k+1}\beta} b^{\frac{i}{k+1}\beta} (k+1-2i)\right] \tag{3.10}$$

i.e.

$$(lna - lnb) \left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = [\Delta] [\Theta], \tag{3.11}$$

where

$$\Delta = \frac{(lna - lnb)}{k(k+1)} G^{1-\beta} \quad \text{and} \quad \Theta = \left[\sum_{i=1}^{k} a^{\frac{k+1-i}{k+1}\beta} b^{\frac{i}{k+1}\beta} (k+1-2i) \right].$$

We want to prove

$$(lna - lnb) \left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b} \right) = [\Delta] [\Theta] > 0. \tag{3.12}$$

For $k = 1$

$$\Theta = 0;$$

for $k = 2$

$$\Theta = \left(a^{\frac{\beta}{3}} b^{\frac{\beta}{3}} \right) \left(a^{\frac{\beta}{3}} - b^{\frac{\beta}{3}} \right) > 0;$$

for $k = 3$

$$\Theta = \left(a^{\frac{3\beta}{4}} b^{\frac{\beta}{4}} \right) \left(a^{\frac{\beta}{2}} - b^{\frac{\beta}{2}} \right) > 0;$$

for $k = 4$

$$\Theta = \left(3a^{\frac{\beta}{5}} b^{\frac{\beta}{5}} \right) \left(a^{\frac{3\beta}{5}} - b^{\frac{3\beta}{5}} \right) + \left(a^{\frac{2\beta}{5}} b^{\frac{2\beta}{5}} \right) \left(a^{\frac{\beta}{5}} - b^{\frac{\beta}{5}} \right) > 0;$$

for $k = 5$

$$\Theta = \left(4a^{\frac{\beta}{6}} b^{\frac{\beta}{6}} \right) \left(a^{\frac{2\beta}{3}} - b^{\frac{2\beta}{3}} \right) + 2 \left(a^{\frac{\beta}{3}} b^{\frac{\beta}{3}} \right) \left(a^{\frac{\beta}{3}} - b^{\frac{\beta}{3}} \right) > 0$$

holds for all $\beta > 0$.

From above arguments we have the following generalization.

When $k = 1$, $\Theta = 0$. For $k = 2, 3$ after grouping Θ contains only one term. For $k = 4, 5$ after grouping Θ contains two terms. Grouping in general for all integral values k, $\beta > 0$ leads to the value of $\Theta > 0$. Hence, the following conclusion is proved

$$(lna - lnb) \left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b} \right) = [\Delta] [\Theta] > 0. \tag{3.13}$$

Case (iii). For all $\alpha > 0$, $\beta = 0$, $t > 0$ and $a > b$ we have the Gnan mean

$$G = G(a, b; k, \alpha, 0) = \prod_{i=1}^{k} \left(\frac{(k+1-i)a^{\alpha} + ib^{\alpha}}{k+1} \right)^{\frac{1}{k\alpha}} \tag{3.14}$$

equivalent to

$$\begin{aligned} G = &\left(\frac{ka^{\alpha} + b^{\alpha}}{k+1} \right)^{\frac{1}{k\alpha}} \left(\frac{(k-1)a^{\alpha} + 2b^{\alpha}}{k+1} \right)^{\frac{1}{k\alpha}} \cdots \\ &\left(\frac{2a^{\alpha} + (k-1)b^{\alpha}}{k+1} \right)^{\frac{1}{k\alpha}} \left(\frac{a^{\alpha} + kb^{\alpha}}{k+1} \right)^{\frac{1}{k\alpha}}. \end{aligned} \tag{3.15}$$

Then by grouping first and last term, second and second from the last and so on, it leads to:

$$
\begin{aligned}
&(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = \\
&(lna - lnb)\left[\frac{k}{k} + \frac{2(k-1)}{k} + \frac{3(k-2)}{k} + \frac{4(k-3)}{k} + ...\right]\left(\frac{a^{2\alpha} - b^{2\alpha}}{\times_i}\right)
\end{aligned} \tag{3.16}
$$

where

$$\times_i = ((k-i)a^\alpha + (i+1)b^\alpha)\,((i+1)a^\alpha + (k-i)b^\alpha) \geqslant 0\,.$$

For all integral values of $i = 0, 1, 2, ..., (k-1)$ holds

$$(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) > 0 \tag{3.17}$$

for all $\alpha > 0$.

Case (iv). For all $\alpha = 0$, $\beta = 0$, $t > 0$ and $a > b$, we have the Gnan mean

$$G = G(a, b; k, 0, 0) = \sqrt{ab}. \tag{3.18}$$

Then

$$(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = 0 \tag{3.19}$$

This completes the proof of (1).

Proof of (2).

Case (i). For all $\alpha \neq \beta \neq 0$, $\alpha < 0$, $t > 0$ and $a > b$, we have the Gnan mean

$$G = G(a, b; k, \alpha, \beta) = \left[\frac{1}{k}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha + ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}}. \tag{3.20}$$

Taking log on both sides and differentiate partially with respect to a and multiply by a gives

$$a\frac{\partial G}{\partial a} = G\frac{1}{\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha + ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}}}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha + ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}-1}\frac{k+1-i}{k+1}a^\alpha. \tag{3.21}$$

Similarly

$$b\frac{\partial G}{\partial b} = G\frac{1}{\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha + ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}}}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^\alpha + ib^\alpha}{k+1}\right)^{\frac{\beta}{\alpha}-1}\frac{i}{k+1}b^\alpha. \tag{3.22}$$

Then

$$(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = \tag{3.23}$$
$$\frac{(lna - lnb)}{k}G^{1-\beta}\sum_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha} + ib^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}}\frac{(k+1-i)a^{\alpha} - ib^{\alpha}}{(k+1-i)a^{\alpha} + ib^{\alpha}}$$

$$(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = [\Delta]\,[\Theta] \tag{3.24}$$

where

$$\Delta = \frac{(lna - lnb)}{k}G^{1-\beta}$$

and

$$\Theta = \sum_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha} + ib^{\alpha}}{k+1}\right)^{\frac{\beta}{\alpha}}\frac{(k+1-i)a^{\alpha} - ib^{\alpha}}{(k+1-i)a^{\alpha} + ib^{\alpha}}\,.$$

For $k = 1$

$$\Theta = \left(\frac{a^{\alpha} + b^{\alpha}}{2}\right)^{\frac{\beta}{\alpha}}\frac{a^{\alpha} - b^{\alpha}}{a^{\alpha} + b^{\alpha}} = (\cosh\alpha t)^{\frac{\beta}{\alpha}}\tanh\alpha t$$

where $a = e^{t}$ and $b = e^{-t}$. Then for all $\alpha < 0, t > 0$ and $a > b$

$$(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = [\Delta]\,[\Theta] < 0. \tag{3.25}$$

For $k = 2$

$$\Theta = \left(\frac{2a^{\alpha} + b^{\alpha}}{3}\right)^{\frac{\beta}{\alpha}}\frac{2a^{\alpha} - b^{\alpha}}{2a^{\alpha} + b^{\alpha}} + \left(\frac{a^{\alpha} + 2b^{\alpha}}{3}\right)^{\frac{\beta}{\alpha}}\frac{a^{\alpha} - 2b^{\alpha}}{a^{\alpha} + 2b^{\alpha}}.$$

Then

$$\Theta = \left(\frac{1}{3}\right)^{\frac{\beta}{\alpha}}\left[(2a^{\alpha} + b^{\alpha})^{\frac{\beta}{\alpha}-1}(2a^{\alpha} - b^{\alpha}) + (a^{\alpha} + 2b^{\alpha})^{\frac{\beta}{\alpha}-1}(a^{\alpha} - 2b^{\alpha})\right] < 0,$$

if it is

$$\left(\frac{2a^{\alpha} + b^{\alpha}}{a^{\alpha} + 2b^{\alpha}}\right)^{\frac{\beta}{\alpha}-1}\frac{2a^{\alpha} - b^{\alpha}}{2b^{\alpha} - a^{\alpha}} < 1.$$

It is easy to prove that

$$\frac{2a^{\alpha} + b^{\alpha}}{a^{\alpha} + 2b^{\alpha}} > 1 \quad \text{and} \quad \frac{2a^{\alpha} - b^{\alpha}}{2b^{\alpha} - a^{\alpha}} < 1.$$

Then for all $\alpha < 0$, $t > 0$ and $a > b$ we have

$$(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = [\Delta]\,[\Theta] < 0\,. \tag{3.26}$$

For $k = 3$

$$\Theta = \left(\frac{3a^{\alpha} + b^{\alpha}}{4}\right)^{\frac{\beta}{\alpha}}\frac{3a^{\alpha} - b^{\alpha}}{3a^{\alpha} + b^{\alpha}} + 4^{\frac{\beta}{\alpha}}(\cosh\alpha t)^{\frac{\beta}{\alpha}}\tanh\alpha t + \left(\frac{a^{\alpha} + 3b^{\alpha}}{4}\right)^{\frac{\beta}{\alpha}}\frac{a^{\alpha} - 3b^{\alpha}}{a^{\alpha} + 3b^{\alpha}}$$

where $a = e^t$ and $b = e^{-t}$. Then

$$\Theta = \left(\frac{1}{4}\right)^{\frac{\beta}{\alpha}} \left[(3a^\alpha + b^\alpha)^{\frac{\beta}{\alpha}-1}(3a^\alpha - b^\alpha) + (a^\alpha + 3b^\alpha)^{\frac{\beta}{\alpha}-1}(a^\alpha - 3b^\alpha)\right] +$$
$$4^{\frac{\beta}{\alpha}}(\cosh \alpha t)^{\frac{\beta}{\alpha}} \tanh \alpha t < 0,$$

if it is

$$\left(\frac{3a^\alpha + b^\alpha}{a^\alpha + 3b^\alpha}\right)^{\frac{\beta}{\alpha}-1} \frac{3a^\alpha - b^\alpha}{3b^\alpha - a^\alpha} < 1.$$

It is easy to prove that

$$\frac{3a^\alpha + b^\alpha}{a^\alpha + 3b^\alpha} > 1 \quad \text{and} \quad \frac{3a^\alpha - b^\alpha}{3b^\alpha - a^\alpha} < 1.$$

Then for all $\alpha < 0, t > 0$ and $a > b$ we have

$$(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = [\Delta]\,[\Theta] < 0\,. \tag{3.27}$$

From above arguments we have two generalized cases as follows:

(a)When k is an even, on expanding the summation leads to even number of terms. Further, grouping the first and k^{th} term, second and $(k-1)^{th}$ and so on, then we reached the required conclusion by proving as explained for $k = 2$.

(b)When k is an odd, on expanding the summation leads to odd number of terms. Further, grouping the first and k^{th} term, second and $(k-1)^{th}$ and middle term is $(\cosh \alpha t)^{\frac{\beta}{\alpha}} \tanh \alpha t$, then we reached the required conclusion by proving as explained for $k = 1$ and $k = 3$.

Case (ii). For all $\alpha = 0$, $\beta < 0$, $t > 0$ and $a > b$, we have the Gnan mean

$$G = G(a, b; k, 0, \beta) = \left[\frac{1}{k}\sum_{i=1}^{k} a^{\frac{k+1-i}{k+1}\beta} b^{\frac{i}{k+1}\beta}\right]^{\frac{1}{\beta}}. \tag{3.28}$$

Then

(3.29)

$$(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = \frac{(lna - lnb)}{k(k+1)}G^{1-\beta}\left[\sum_{i=1}^{k} a^{\frac{k+1-i}{k+1}\beta} b^{\frac{i}{k+1}\beta}(k+1-2i)\right]$$

i.e.

$$(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = [\Delta]\,[\Theta] \tag{3.30}$$

where

$$\Delta = \frac{(lna - lnb)}{k(k+1)}G^{1-\beta} \quad \text{and} \quad \Theta = \left[\sum_{i=1}^{k} a^{\frac{k+1-i}{k+1}\beta} b^{\frac{i}{k+1}\beta}(k+1-2i)\right].$$

We need to prove

$$(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = [\Delta]\,[\Theta] < 0\,. \tag{3.31}$$

For $k = 1$

$$\Theta = 0;$$

for $k = 2$

$$\Theta = \left(a^{\frac{\beta}{3}} b^{\frac{\beta}{3}}\right)\left(a^{\frac{\beta}{3}} - b^{\frac{\beta}{3}}\right) < 0;$$

for $k = 3$

$$\Theta = \left(a^{\frac{3\beta}{4}} b^{\frac{\beta}{4}}\right)\left(a^{\frac{\beta}{2}} - b^{\frac{\beta}{2}}\right) < 0;$$

For $k = 4$

$$\Theta = \left(3a^{\frac{\beta}{5}} b^{\frac{\beta}{5}}\right)\left(a^{\frac{3\beta}{5}} - b^{\frac{3\beta}{5}}\right) + \left(a^{\frac{2\beta}{5}} b^{\frac{2\beta}{5}}\right)\left(a^{\frac{\beta}{5}} - b^{\frac{\beta}{5}}\right) < 0;$$

for $k = 5$

$$\Theta = \left(4a^{\frac{\beta}{6}} b^{\frac{\beta}{6}}\right)\left(a^{\frac{2\beta}{3}} - b^{\frac{2\beta}{3}}\right) + 2\left(a^{\frac{\beta}{3}} b^{\frac{\beta}{3}}\right)\left(a^{\frac{\beta}{3}} - b^{\frac{\beta}{3}}\right) < 0$$

holds for all $\beta < 0$.

From above arguments we have the following generalization.

When $k = 1, \Theta = 0$. For $k = 2, 3$ after grouping Θ contains only one term. For $k = 4, 5$ after grouping Θ contains two terms. Grouping in general for all integral values $k > 0, \beta < 0$ leads to the value of $\Theta < 0$. Hence, the following conclusion is proved:

$$(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = [\Delta]\,[\Theta] < 0. \tag{3.32}$$

Case (iii). For all $\alpha < 0$, $\beta = 0$, $t > 0$ and $a > b$ we have the Gnan mean

$$G = G(a, b; k, \alpha, 0) = \prod_{i=1}^{k}\left(\frac{(k+1-i)a^{\alpha} + ib^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}} \tag{3.33}$$

equivalent to

$$\begin{aligned} G = &\left(\frac{ka^{\alpha} + b^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}}\left(\frac{(k-1)a^{\alpha} + 2b^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}} \cdots \\ &\left(\frac{2a^{\alpha} + (k-1)b^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}}\left(\frac{a^{\alpha} + kb^{\alpha}}{k+1}\right)^{\frac{1}{k\alpha}}. \end{aligned} \tag{3.34}$$

Then by grouping first and last term, second and second from the last and so on leads to

$$\begin{aligned}&(lna-lnb)\left(a\frac{\partial G}{\partial a}-b\frac{\partial G}{\partial b}\right)=\\&(lna-lnb)\left[\frac{k}{k}+\frac{2(k-1)}{k}+\frac{3(k-2)}{k}+\frac{4(k-3)}{k}+...\right]\left(\frac{a^{2\alpha}-b^{2\alpha}}{\times_i}\right)\end{aligned}\tag{3.35}$$

where

$$\times_i=((k-i)a^{\alpha}+(i+1)b^{\alpha})\,((i+1)a^{\alpha}+(k-i)b^{\alpha})\leqslant 0\,.$$

For all integral values of $i=0,1,2,....,(k-1)$

$$(lna-lnb)\left(a\frac{\partial G}{\partial a}-b\frac{\partial G}{\partial b}\right)<0\tag{3.36}$$

holds for all $\alpha<0$.

Case (iv). For all $\alpha=0$, $\beta=0$, $t>0$ and $a>b$ we have the Gnan mean

$$G=G(a,b;k,0,0)=\sqrt{ab}.\tag{3.37}$$

Then

$$(lna-lnb)\left(a\frac{\partial G}{\partial a}-b\frac{\partial G}{\partial b}\right)=0\,.\tag{3.38}$$

This completes the proof of (2). □

THEOREM 3.2. *Let $a\geqslant b\geqslant 0$ be arbitrary elements, k be a non-negative integer and α,β two real numbers. Then:*

(1) *The dual Gnan mean $g(a,b,k;\alpha,\beta)$ is Schur-geometric convex by a and b if $\alpha,\beta\geqslant 0$.*
(2) *The dual Gnan mean $g(a,b,k;\alpha,\beta)$ is Schur-geometric concave by a and b if $\alpha,\beta\leqslant 0$.*

PROOF. **Proof of (1)**.

Case (i). For all α, $\beta<0$, $t>0$ and $a>b$ we have the Gnan mean

$$g=g(a,b;k,\alpha,\beta)=\left[\frac{1}{k+1}\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}}\tag{3.39}$$

Taking log on both sides and differentiate partially with respect to a and multiply by a gives

$$a\frac{\partial g}{\partial a}=g\frac{1}{\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}}}\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}-1}\frac{k-i}{k}a^{\alpha}\,.\tag{3.40}$$

Similarly

$$b\frac{\partial g}{\partial b}=g\frac{1}{\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}}}\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}-1}\frac{i}{k}b^{\alpha}. \tag{3.41}$$

Then

$$\begin{aligned}&(lna-lnb)\left(a\frac{\partial g}{\partial a}-b\frac{\partial g}{\partial b}\right)=\\&\frac{(lna-lnb)}{k}g^{1-\beta}\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}}\frac{(k-i)a^{\alpha}-ib^{\alpha}}{(k-i)a^{\alpha}+ib^{\alpha}}\end{aligned} \tag{3.42}$$

i.e.

$$(lna-lnb)\left(a\frac{\partial g}{\partial a}-b\frac{\partial g}{\partial b}\right)=[\Delta]\,[\Theta] \tag{3.43}$$

where

$$\Delta=\frac{(lna-lnb)}{k}g^{1-\beta}\quad\text{and}\quad\Theta=\sum_{i=0}^{k}\left(\frac{(k-i)a^{\alpha}+ib^{\alpha}}{k}\right)^{\frac{\beta}{\alpha}}\frac{(k-i)a^{\alpha}-ib^{\alpha}}{(k-i)a^{\alpha}+ib^{\alpha}}$$

$a=e^{t}$ and $b=e^{-t}$. Then for all $\alpha, t>0$ and $a>b$ we have:
For $k=1$

$$\Theta=(a^{\alpha})^{\frac{\beta}{\alpha}-1}a^{\alpha}+(b^{\alpha})^{\frac{\beta}{\alpha}-1}(-b^{\alpha})=2\sinh\beta t.$$

Thus

$$(lna-lnb)\left(a\frac{\partial g}{\partial a}-b\frac{\partial g}{\partial b}\right)=[\Delta]\,[\Theta]>0. \tag{3.44}$$

For $k=2$

$$\begin{aligned}\Theta=(a^{\alpha})^{\frac{\beta}{\alpha}-1}a^{\alpha}+\left(\frac{a^{\alpha}+b^{\alpha}}{2}\right)^{\frac{\beta}{\alpha}-1}\left(\frac{a^{\alpha}-b^{\alpha}}{2}\right)+(b^{\alpha})^{\frac{\beta}{\alpha}-1}(-b^{\alpha})=\\2\sinh\beta t+M_{\alpha}^{\beta-\alpha}\sinh\alpha t.\end{aligned}$$

Thus

$$(lna-lnb)\left(a\frac{\partial g}{\partial a}-b\frac{\partial g}{\partial b}\right)=[\Delta]\,[\Theta]>0. \tag{3.45}$$

For $k=3$ we have

$$\begin{aligned}\Theta=(a^{\alpha})^{\frac{\beta}{\alpha}-1}a^{\alpha}+\left(\frac{2a^{\alpha}+b^{\alpha}}{3}\right)^{\frac{\beta}{\alpha}-1}\left(\frac{2a^{\alpha}-b^{\alpha}}{3}\right)\\+\left(\frac{a^{\alpha}+2b^{\alpha}}{3}\right)^{\frac{\beta}{\alpha}-1}\left(\frac{a^{\alpha}-2b^{\alpha}}{3}\right)+(b^{\alpha})^{\frac{\beta}{\alpha}-1}(-b^{\alpha}).\end{aligned}$$

Simplification leads to

$$\Theta = 2\sinh\beta t + \left(\frac{1}{3}\right)^{\frac{\beta}{\alpha}} \left(\frac{2a^\alpha + b^\alpha}{a^\alpha + 2b^\alpha}\right)^{\frac{\beta}{\alpha}-1} \left(\frac{2a^\alpha - b^\alpha}{2b^\alpha - a^\alpha}\right).$$

Thus

$$(lna - lnb)\left(a\frac{\partial g}{\partial a} - b\frac{\partial g}{\partial b}\right) = [\Delta]\,[\Theta] > 0. \tag{3.46}$$

Similarly for $k = 4$ we have

$$\Theta = 2\sinh\beta t + M_\alpha^{\beta-\alpha}\sinh\alpha t + \left(\frac{1}{4}\right)^{\frac{\beta}{\alpha}} \left(\frac{3a^\alpha + b^\alpha}{a^\alpha + 3b^\alpha}\right)^{\frac{\beta}{\alpha}-1} \left(\frac{3a^\alpha - b^\alpha}{3b^\alpha - a^\alpha}\right).$$

From above arguments we have two generalized cases as follows:

(a)When k is an even, on expanding the summation leads to odd number of terms. Further, grouping the first and k^{th} term gives $2\sinh\beta t$, middle term is $M_\alpha^{\beta-\alpha}\sinh\alpha t$ and rest of the terms are grouped as second and $(k-1)^{th}$ and so on. Then we reached the required conclusion by proving as explained for $k = 2, k = 4$.

(b)When k is an odd, on expanding the summation leads to even number of terms. Further, grouping the first and k^{th} term gives $2\sinh\beta t$, second and $(k-1)^{th}$ and so on. Then we reached the required conclusion by proving as explained for $k = 1$ and $k = 3$.

Case (ii). For all $\alpha = 0$, $\beta > 0$, $t > 0$ and $a > b$ we have the Gnan mean

$$g = g(a, b; k, 0, \beta) = \left[\frac{1}{k+1}\sum_{i=0}^{k} a^{\frac{k-i}{k}\beta} b^{\frac{i}{k}\beta}\right]^{\frac{1}{\beta}}. \tag{3.47}$$

Then

$$(lna - lnb)\left(a\frac{\partial g}{\partial a} - b\frac{\partial g}{\partial b}\right) = \frac{(lna - lnb)}{k(k+1)} G^{1-\beta}\left[\sum_{i=0}^{k} a^{\frac{k-i}{k}\beta} b^{\frac{i}{k}\beta}(k-2i)\right] \tag{3.48}$$

i.e.

$$(lna - lnb)\left(a\frac{\partial g}{\partial a} - b\frac{\partial g}{\partial b}\right) = [\Delta]\,[\Theta] \tag{3.49}$$

where

$$\Delta = \frac{(lna - lnb)}{k(k+1)} g^{1-\beta} \quad \text{and} \quad \Theta = \left[\sum_{i=0}^{k} a^{\frac{k-i}{k}\beta} b^{\frac{i}{k}\beta}(k-2i)\right].$$

We need to prove

$$(lna - lnb)\left(a\frac{\partial g}{\partial a} - b\frac{\partial g}{\partial b}\right) = [\Delta]\,[\Theta] > 0. \tag{3.50}$$

For $k=1$

$$\Theta = a^{\beta} - b^{\beta} > 0;$$

for $k=2$

$$\Theta = 2\left(a^{\beta} - b^{\beta}\right) > 0;$$

for $k=3$

$$\Theta = 3\left(a^{\beta} - b^{\beta}\right) + \left(a^{\frac{\beta}{3}} b^{\frac{\beta}{3}}\right)\left(a^{\frac{\beta}{3}} - b^{\frac{\beta}{3}}\right) > 0;$$

for $k=4$

$$\Theta = 4\left(a^{\beta} - b^{\beta}\right) + 2\left(a^{\frac{\beta}{4}} b^{\frac{\beta}{4}}\right)\left(a^{\frac{\beta}{2}} - b^{\frac{\beta}{2}}\right) > 0;$$

for $k=5$

$$\Theta = 5\left(a^{\beta} - b^{\beta}\right) + 3\left(a^{\frac{\beta}{5}} b^{\frac{\beta}{5}}\right)\left(a^{\frac{3\beta}{5}} - b^{\frac{3\beta}{5}}\right) + \left(a^{\frac{2\beta}{5}} b^{\frac{2\beta}{5}}\right)\left(a^{\frac{\beta}{5}} - b^{\frac{\beta}{5}}\right) > 0;$$

for $k=6$

$$\Theta = 6\left(a^{\beta} - b^{\beta}\right) + 4\left(a^{\frac{\beta}{6}} b^{\frac{\beta}{6}}\right)\left(a^{\frac{2\beta}{3}} - b^{\frac{2\beta}{3}}\right) + 2\left(a^{\frac{\beta}{3}} b^{\frac{\beta}{3}}\right)\left(a^{\frac{\beta}{3}} - b^{\frac{\beta}{3}}\right) > 0$$

holds for all $\beta > 0$.

From above arguments, we have the following generalization.

When $k = 1, 2$ after grouping Θ contains only one term. For $k = 3, 4$ after grouping Θ contains two terms. For $k = 5, 6$ after grouping Θ contains three terms. Grouping in general for all integral values k, $\beta > 0$ leads to the value of $\Theta > 0$. Hence, the following conclusion is proved:

$$(lna - lnb)\left(a\frac{\partial g}{\partial a} - b\frac{\partial g}{\partial b}\right) = [\Delta]\,[\Theta] > 0. \tag{3.51}$$

Case (iii). For all $\alpha > 0$, $\beta = 0$, $t > 0$ and $a > b$ we have the Gnan mean

$$g = g(a, b; k, \alpha, 0) = \prod_{i=0}^{k} \left(\frac{(k-i)a^{\alpha} + ib^{\alpha}}{k}\right)^{\frac{1}{(k+1)\alpha}} \tag{3.52}$$

equivalent to

(3.53)

$$g = (ka^{\alpha})^{\frac{1}{(k+1)\alpha}} \left(\frac{(k-1)a^{\alpha} + b^{\alpha}}{k}\right)^{\frac{1}{(k+1)\alpha}} \cdots \left(\frac{a^{\alpha} + (k-1)b^{\alpha}}{k}\right)^{\frac{1}{(k+1)\alpha}} (kb^{\alpha})^{\frac{1}{(k+1)\alpha}}.$$

Then by grouping first and last term, second and second from the last and so on leads to:

$$\begin{aligned} &(lna - lnb)\left(a\frac{\partial g}{\partial a} - b\frac{\partial g}{\partial b}\right) = \\ &(lna - lnb)\left[\frac{k-1}{k+1} + \frac{2(k-2)}{k+1} + \frac{3(k-3)}{k+1} + \frac{4(k-4)}{k+1} + \ldots\right]\left(\frac{a^{2\alpha} - b^{2\alpha}}{\times_i}\right) \end{aligned} \tag{3.54}$$

where

$$\times_i = ((k-i)a^\alpha + (i+1)b^\alpha)\left((i+1)a^\alpha + (k-i)b^\alpha\right) \geqslant 0$$

for all integral values of $i = 1, 2, ..., (k-1)$

$$(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) > 0 \tag{3.55}$$

holds for all $\alpha > 0$.

Case (iv). For all $\alpha = 0$, $\beta = 0$, $t > 0$ and $a > b$ we have the Gnan mean

$$G = G(a, b; k, 0, 0) = \sqrt{ab}. \tag{3.56}$$

Then

$$(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = 0 \tag{3.57}$$

This completes the proof of (1).

Proof of (2).

Case (i). For all α, $\beta < 0$, $t > 0$ and $a > b$ we have the Gnan mean

$$g = g(a, b; k, \alpha, \beta) = \left[\frac{1}{k+1}\sum_{i=0}^{k}\left(\frac{(k-i)a^\alpha + ib^\alpha}{k}\right)^{\frac{\beta}{\alpha}}\right]^{\frac{1}{\beta}}. \tag{3.58}$$

Taking log on both sides and differentiate partially with respect to a and multiply by a gives

$$a\frac{\partial g}{\partial a} = g\frac{1}{\sum_{i=0}^{k}\left(\frac{(k-i)a^\alpha + ib^\alpha}{k}\right)^{\frac{\beta}{\alpha}}}\sum_{i=0}^{k}\left(\frac{(k-i)a^\alpha + ib^\alpha}{k}\right)^{\frac{\beta}{\alpha}-1}\frac{k-i}{k}a^\alpha. \tag{3.59}$$

Similarly

$$b\frac{\partial g}{\partial b} = g\frac{1}{\sum_{i=0}^{k}\left(\frac{(k-i)a^\alpha + ib^\alpha}{k}\right)^{\frac{\beta}{\alpha}}}\sum_{i=0}^{k}\left(\frac{(k-i)a^\alpha + ib^\alpha}{k}\right)^{\frac{\beta}{\alpha}-1}\frac{i}{k}b^\alpha. \tag{3.60}$$

Then

$$\begin{gathered}(lna - lnb)\left(a\frac{\partial g}{\partial a} - b\frac{\partial g}{\partial b}\right) = \\ \frac{(lna - lnb)}{k}g^{1-\beta}\sum_{i=0}^{k}\left(\frac{(k-i)a^\alpha + ib^\alpha}{k}\right)^{\frac{\beta}{\alpha}}\frac{(k-i)a^\alpha - ib^\alpha}{(k-i)a^\alpha + ib^\alpha}\end{gathered} \tag{3.61}$$

i.e.

$$(lna - lnb)\left(a\frac{\partial g}{\partial a} - b\frac{\partial g}{\partial b}\right) = [\Delta]\,[\Theta] \tag{3.62}$$

where

$$\Delta = \frac{(lna - lnb)}{k} g^{1-\beta} \quad \text{and} \quad \Theta = \sum_{i=0}^{k} \left(\frac{(k-i)a^\alpha + ib^\alpha}{k} \right)^{\frac{\beta}{\alpha}} \frac{(k-i)a^\alpha - ib^\alpha}{(k-i)a^\alpha + ib^\alpha}$$

$a = e^t$ and $b = e^{-t}$. Then for all $\alpha < 0$, $t > 0$ and $a > b$, consider the following particular values of k we have:

For $k = 1$

$$\Theta = (a^\alpha)^{\frac{\beta}{\alpha}-1} a^\alpha + (b^\alpha)^{\frac{\beta}{\alpha}-1} (-b^\alpha) = 2\sinh \beta t.$$

Thus

$$(lna - lnb)\left(a\frac{\partial g}{\partial a} - b\frac{\partial g}{\partial b} \right) = [\Delta]\,[\Theta] < 0. \tag{3.63}$$

For $k = 2$

$$\Theta = (a^\alpha)^{\frac{\beta}{\alpha}-1} a^\alpha + \left(\frac{a^\alpha + b^\alpha}{2} \right)^{\frac{\beta}{\alpha}-1} \left(\frac{a^\alpha - b^\alpha}{2} \right) + (b^\alpha)^{\frac{\beta}{\alpha}-1} (-b^\alpha) =$$

$$2\sinh \beta t + M_\alpha^{\beta-\alpha} \sinh \alpha t.$$

Thus

$$(lna - lnb)\left(a\frac{\partial g}{\partial a} - b\frac{\partial g}{\partial b} \right) = [\Delta]\,[\Theta] < 0. \tag{3.64}$$

For $k = 3$

$$\Theta = (a^\alpha)^{\frac{\beta}{\alpha}-1} a^\alpha + \left(\frac{2a^\alpha + b^\alpha}{3} \right)^{\frac{\beta}{\alpha}-1} \left(\frac{2a^\alpha - b^\alpha}{3} \right)$$

$$+ \left(\frac{a^\alpha + 2b^\alpha}{3} \right)^{\frac{\beta}{\alpha}-1} \left(\frac{a^\alpha - 2b^\alpha}{3} \right) + (b^\alpha)^{\frac{\beta}{\alpha}-1} (-b^\alpha).$$

Simplification leads to

$$\Theta = 2\sinh \beta t + \left(\frac{1}{3} \right)^{\frac{\beta}{\alpha}} \left(\frac{2a^\alpha + b^\alpha}{a^\alpha + 2b^\alpha} \right)^{\frac{\beta}{\alpha}-1} \left(\frac{2a^\alpha - b^\alpha}{2b^\alpha - a^\alpha} \right).$$

Thus

$$(lna - lnb)\left(a\frac{\partial g}{\partial a} - b\frac{\partial g}{\partial b} \right) = [\Delta]\,[\Theta] < 0. \tag{3.65}$$

Similarly for $k = 4$ we have

$$\Theta = 2\sinh \beta t + M_\alpha^{\beta-\alpha} \sinh \alpha t + \left(\frac{1}{4} \right)^{\frac{\beta}{\alpha}} \left(\frac{3a^\alpha + b^\alpha}{a^\alpha + 3b^\alpha} \right)^{\frac{\beta}{\alpha}-1} \left(\frac{3a^\alpha - b^\alpha}{3b^\alpha - a^\alpha} \right).$$

From above arguments we have two generalized cases as follows:

(a) When k is an even, on expanding the summation leads to odd number of terms. Further, grouping the first and k^{th} term gives $2\sinh \beta t$, middle term is

$M_\alpha^{\beta-\alpha}\sinh\alpha t$ and rest of the terms are grouped as second and $(k-1)^{th}$ and so on. Then we reached the required conclusion by proving as explained for $k=2, k=4$.

(b) When k is an odd, on expanding the summation leads to even number of terms. Further, grouping the first and k^{th} term gives $2\sinh\beta t$, and rest of the terms are grouped as second and $(k-1)^{th}$ and so on. Then we reached the required conclusion by proving as explained for $k=1$ and $k=3$.

Case (ii). For all $\alpha=0$, $\beta<0$, $t>0$ and $a>b$ we have the Gnan mean

$$g=g(a,b;k,0,\beta)=\left[\frac{1}{k+1}\sum_{i=0}^{k}a^{\frac{k-i}{k}\beta}b^{\frac{i}{k}\beta}\right]^{\frac{1}{\beta}}. \tag{3.66}$$

Then

$$(lna-lnb)\left(a\frac{\partial g}{\partial a}-b\frac{\partial g}{\partial b}\right)=\frac{(lna-lnb)}{k(k+1)}G^{1-\beta}\left[\sum_{i=0}^{k}a^{\frac{k-i}{k}\beta}b^{\frac{i}{k}\beta}(k-2i)\right] \tag{3.67}$$

i.e.

$$(lna-lnb)\left(a\frac{\partial g}{\partial a}-b\frac{\partial g}{\partial b}\right)=[\Delta]\,[\Theta] \tag{3.68}$$

where

$$\Delta=\frac{(lna-lnb)}{k(k+1)}g^{1-\beta}\quad\text{and}\quad\Theta=\left[\sum_{i=0}^{k}a^{\frac{k-i}{k}\beta}b^{\frac{i}{k}\beta}(k-2i)\right].$$

We need to prove

$$(lna-lnb)\left(a\frac{\partial g}{\partial a}-b\frac{\partial g}{\partial b}\right)=[\Delta]\,[\Theta]<0. \tag{3.69}$$

For $k=1$

$$\Theta=a^\beta-b^\beta<0.$$

For $k=2$

$$\Theta=2\left(a^\beta-b^\beta\right)<0.$$

For $k=3$

$$\Theta=3\left(a^\beta-b^\beta\right)+\left(a^{\frac{\beta}{3}}b^{\frac{\beta}{3}}\right)\left(a^{\frac{\beta}{3}}-b^{\frac{\beta}{3}}\right)<0.$$

For $k=4$

$$\Theta=4\left(a^\beta-b^\beta\right)+2\left(a^{\frac{\beta}{4}}b^{\frac{\beta}{4}}\right)\left(a^{\frac{\beta}{2}}-b^{\frac{\beta}{2}}\right)<0.$$

For $k=5$

$$\Theta=5\left(a^\beta-b^\beta\right)+3\left(a^{\frac{\beta}{5}}b^{\frac{\beta}{5}}\right)\left(a^{\frac{3\beta}{5}}-b^{\frac{3\beta}{5}}\right)+\left(a^{\frac{2\beta}{5}}b^{\frac{2\beta}{5}}\right)\left(a^{\frac{\beta}{5}}-b^{\frac{\beta}{5}}\right)<0.$$

For $k=6$

$$\Theta=6\left(a^\beta-b^\beta\right)+4\left(a^{\frac{\beta}{6}}b^{\frac{\beta}{6}}\right)\left(a^{\frac{2\beta}{3}}-b^{\frac{2\beta}{3}}\right)+2\left(a^{\frac{\beta}{3}}b^{\frac{\beta}{3}}\right)\left(a^{\frac{\beta}{3}}-b^{\frac{\beta}{3}}\right)<0$$

holds for all $\beta<0$.

From above arguments we have the following generalization.

When $k = 1, 2$ after grouping Θ contains only one term. For $k = 3, 4$ after grouping Θ contains two terms. For $k = 5, 6$ after grouping Θ contains three terms. Grouping in general for all integral values $k > 0, \beta < 0$ leads to the value of $\Theta < 0$. Hence, the following conclusion is proved:

$$(lna - lnb)\left(a\frac{\partial g}{\partial a} - b\frac{\partial g}{\partial b}\right) = [\Delta]\,[\Theta] < 0. \tag{3.70}$$

Case (iii). For all $\alpha < 0$, $\beta = 0$, $t > 0$ and $a > b$ we have the Gnan mean

$$g = g(a, b; k, \alpha, 0) = \prod_{i=0}^{k}\left(\frac{(k-i)a^{\alpha} + ib^{\alpha}}{k}\right)^{\frac{1}{(k+1)\alpha}} \tag{3.71}$$

equivalent to

$$g = (ka^{\alpha})^{\frac{1}{(k+1)\alpha}}\left(\frac{(k-1)a^{\alpha} + b^{\alpha}}{k}\right)^{\frac{1}{(k+1)\alpha}} \cdots \left(\frac{a^{\alpha} + (k-1)b^{\alpha}}{k}\right)^{\frac{1}{(k+1)\alpha}}(kb^{\alpha})^{\frac{1}{(k+1)\alpha}}. \tag{3.72}$$

Then by grouping first and last term, second and second from the last and so on leads to

$$\begin{aligned}&(lna - lnb)\left(a\frac{\partial g}{\partial a} - b\frac{\partial g}{\partial b}\right) = \\ &(lna - lnb)\left[\frac{k-1}{k+1} + \frac{2(k-2)}{k+1} + \frac{3(k-3)}{k+1} + \frac{4(k-4)}{k+1} + \ldots\right]\left(\frac{a^{2\alpha} - b^{2\alpha}}{\times_i}\right)\end{aligned} \tag{3.73}$$

where

$$\times_i = ((k-i)a^{\alpha} + (i+1)b^{\alpha})\,((i+1)a^{\alpha} + (k-i)b^{\alpha}) \leqslant 0.$$

For all integral values of $i = 1, 2, \ldots., (k-1)$

$$(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) < 0 \tag{3.74}$$

holds for all $\alpha < 0$.

Case (iv). For all $\alpha = 0$, $\beta = 0$, $t > 0$ and $a > b$ we have the Gnan mean

$$G = G(a, b; k, 0, 0) = \sqrt{ab}. \tag{3.75}$$

Then

$$(lna - lnb)\left(a\frac{\partial G}{\partial a} - b\frac{\partial G}{\partial b}\right) = 0 \tag{3.76}$$

This completes the proof of (2). □

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