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CARLESON MEASURE CHARACTERIZATION ON ANALYTIC $Q_K(p,q)$ SPACES

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ABSTRACT. In this paper, we study composition operators on Bloch space and $Q_K(p,q)$ spaces. We give a Carleson measure characterization on $Q_K(p,q)$ spaces, then we use this Carleson measure characterization of the compact compositions on $Q_K(p,q)$ spaces to show that every compact composition operator on $Q_K(p,q)$ spaces is compact on Bloch space. Moreover, necessary and sufficient condition for C_ϕ from the Bloch space $\mathcal{B}$ to a general class of analytic functions $Q_K(p,q)$ to be compact is given in terms of the map ϕ.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and let $\partial\mathbb{D}$ be its boundary. Let $H(\mathbb{D})$ denote the class of all holomorphic functions on $\mathbb{D}$. The Bloch space $\mathcal{B}$ is the space of holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

It becomes a Banach space with norm

$$|f(0)| + \|f\|_{\mathcal{B}}.$$

For $a \in \mathbb{D}$ the Möbius transformation $\varphi_a(z)$ is defined by

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad \text{for } z \in \mathbb{D}.$$

For a point $a \in \mathbb{D}$ and $0 < r < 1$, the pseudo-hyperbolic disk $D(a,r)$ with pseudo-hyperbolic center a and pseudo-hyperbolic radius r is defined by $D(a,r) = \varphi_a(rD)$. The pseudo-hyperbolic disk $D(a,r)$ is also an Euclidean disk: its Euclidean center and Euclidean radius are $\frac{(1-r^2)a}{1-r^2|a|^2}$ and $\frac{(1-|a|^2)r}{1-r^2|a|^2}$, respectively (see [**37**]). Let A

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denote the normalized Lebesgue area measure on $\mathbb{D}$, and for a Lebesgue measurable set $K_1 \subset \mathbb{D}$, denote by $|K_1|$ the measure of K_1 with respect to A. It follows immediately that:

$$|D(a,r)| = \frac{(1-|a|^2)^2}{(1-r^2|a|^2)^2} r^2.$$

The following identity is easily verified:

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2} = (1-|z|^2)|\varphi_a'(z)|.$$

For $a \in \mathbb{D}$, the substitution $z = \varphi_a(w)$ results in the Jacobian change in measure given by $dA(w) = |\varphi_a'(z)|^2 dA(z)$. For a Lebesgue integrable or a non-negative Lebesgue measurable function h on $\mathbb{D}$, we thus have the following change-of-variable formula:

$$\int\limits_{D(0,r)} h(\varphi_a(w))dA(w) = \int\limits_{D(a,r)} h(z)\left(\frac{1-|\varphi_a(z)|^2}{1-|z|^2}\right)^2 dA(z)\ . \tag{1.1}$$

Note that $\varphi_a(\varphi_a(z)) = z$, thus $\varphi_a^{-1}(z) = \varphi_a(z)$. For $a, z \in \mathbb{D}$ and $0 < r < 1$, the pseudo-hyperbolic disc $D(a,r)$ is defined by $D(a,r) = \{z \in \mathbb{D} : |\varphi_a(z)| < r\}$. Denote by

$$g(z,a) = \log\left|\frac{1-\bar{a}z}{z-a}\right| = \log\frac{1}{|\varphi_a(z)|}$$

the Green's function of $\mathbb{D}$ with logarithmic singularity at $a \in \mathbb{D}$.
Two quantities A_f and B_f, both depending on an analytic function f on $\mathbb{D}$, are said to be equivalent, written as $A_f \approx B_f$, if there exists a finite positive constant C not depending on f such that for every analytic function f on $\mathbb{D}$ we have:

$$\frac{1}{C} B_f \leqslant A_f \leqslant C B_f.$$

If the quantities A_f and B_f, are equivalent, then in particular we have $A_f < \infty$ if and only if $B_f < \infty$.
Note: we say $K_1 \lesssim K_2$ (for two functions K_1 and K_2) if there is a constant $C > 0$ such that $K_1 \leqslant C K_2$.

Definition 1.1. [**31**] If E is any set, we define the characteristic function χ_E of the set E to be the function given by

$$\chi_E(z) = \begin{cases} 1 & if \quad z \in E \\ 0 & if \ \ z \notin E. \end{cases}$$

The function $\chi_E(z)$ is measurable if and only if E is measurable.

Definition 1.2. [**34**] Let f be an analytic function on $\mathbb{D}$ and let $0 < p < \infty$. If

$$\|f\|_p^p = \sup_{0<r<1} \frac{1}{2\pi} \int\limits_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty,$$

then f belongs to the Hardy space H^p. If $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)| < \infty$, then f belongs to the Hardy space H^∞. Moreover, $f \in H^2$ if and only if

$$\int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)dA(z) < \infty.$$

Definition 1.3. [**44**] Let f be an analytic function in $\mathbb{D}$ and let $0 < \alpha < \infty$. If

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f'(z)| < \infty,$$

then f belongs to the α-Bloch space $\mathcal{B}^\alpha$. The space $\mathcal{B}^1$ is called the Bloch space $\mathcal{B}$.

Definition 1.4. [**37**,**38**] Let f be an analytic function in $\mathbb{D}$ and let $1 < p < \infty$. If

$$\|f\|_{B_p}^p = \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2}dA(z) < \infty,$$

then f belongs to the Besov space B_p .

Definition 1.5. (see [**12**] and [**13**]) For $0 \leqslant p < \infty$, the spaces Q_p are defined by

$$Q_p = \{f \in H(\mathbb{D}) : \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g^p(z,a)dA(z) < \infty\},$$

where the weight function $g(z,a) = \log\left|\frac{1-\bar{a}z}{a-z}\right|$ is defined as the composition of the Möbius transformation φ_a and the fundamental solution of the two-dimensional real Laplacian. These spaces have been studied extensively (see [**11**,**12**,**18**,**42**] and others).

Definition 1.6. [**40**] Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function and let f be an analytic function in $\mathbb{D}$ then $f \in Q_K$ if

$$\|f\|_{Q_K}^2 \quad = \quad \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'(z)\right|^2 K(g(z,a))dA(z) < \infty.$$

Definition 1.7. [**41**] Let $K : [0,\infty) \to [0,\infty)$ be a right continuous and nondecreasing function. For $0 < p < \infty$ and $-2 < q < \infty$, we say that a function f analytic in $\mathbb{D}$ belongs to the space $Q_K(p,q)$ if

$$\|f\|_{Q_K(p,q)}^p \quad = \quad \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|^2\right)^q K(g(z,a))dA(z) < \infty$$

where $dA(z)$ is the Euclidean area element on $\mathbb{D}$.

Remark 1.1. It should be noted that $Q_K(p,q)$ spaces are more general many classes of analytic functions. If $p=2, q=0$, we have that $Q_K(p,q)=Q_K$ (see [**19**, **40**]). If $K(t)=t^s$, then $Q_K(p,q)=F(p,q,s)$ (see [**43**]) that $F(p,q,s)$ is contained in $\frac{q+2}{p}$-Bloch space.

In this paper, we will study $Q_K(p,q)$ spaces with a right continuous and nondecreasing function $K:[0,\infty)\to[0,\infty)$. This choice for the weight function is due to some technical reasons. Also, we assume throughout the paper that

$$\int_0^1 (1-r^2)^{-2}K(\log\frac{1}{r})rdr<\infty.$$

We can define an auxiliary function as follows:

$$\varphi_K(s)=\sup_{0<t\leqslant 1}\frac{K(st)}{K(t)},\qquad 0<s<\infty,$$

we assume that

$$\int_0^1 \varphi_K(s)\frac{ds}{s}<\infty \qquad \text{(see [20])}, \tag{1.2}$$

and

$$\int_1^\infty \varphi_K(s)\frac{ds}{s^2}<\infty \qquad \text{(see [20])}. \tag{1.3}$$

From now we take the above weight function K satisfies the following properties :
(a) K is nondecreasing on $[0,\infty)$,
(b) K is second differentiable on $(0,1)$,
(c) $\int_0^{\frac{1}{e}} K(\log(\frac{1}{r}))rdr<\infty$,
(d) $K(t)=K(1)>0,\ t\geqslant 1$ and
(e) $K(2t)\approx K(t),\ \ t\geqslant 0$.
For a subarc $I\subset\partial\mathbb{D}$, let

$$S(I)=\{r\xi\in\mathbb{D}:1-|I|<r<1,\ \xi\in I\}.$$

If $|I|\geqslant 1$ then we set $S(I)=\mathbb{D}$. For $0<p<\infty$, we say that a positive measure $d\mu$ is a p-Carleson measure on $\mathbb{D}$ if

$$\sup_{I\subset\partial\mathbb{D}}\frac{\mu(S(I))}{|I|^p}<\infty.$$

Here and henceforth $\sup_{I\subset\partial\mathbb{D}}$ indicates the supremum taken over all subarcs I of $\partial\mathbb{D}$. Note that $p=1$, gives the classical Carleson measure (cf. [**16**]). For several studies about Carleson measure and p-Carleson measure on some different classes of holomorphic Banach function spaces, we refer to [**5**, **10**, **11**, **14**, **21**, **22**, **42**].

From [**20**], we know that a positive Borel measure μ on $\mathbb{D}$ is called a K-Carleson measure if

$$\|\mu\|_K = \sup_{I\subset\partial\mathbb{D}} \mu(S(I)) < \infty,$$

where the supremum is taken over all subarcs I of $\partial\mathbb{D}$, and

$$\mu(S(I)) = \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right) d\mu(z).$$

Also, μ is said to be a compact K-Carleson measure if

$$\|\mu\|_K < \infty \text{ and } \lim_{|I|\to 0} \mu(S(I)) = 0,$$

where the supremum is taken over all subarcs $I \subset \partial\mathbb{D}$. Here, for the subarc I of $\partial\mathbb{D}$, $|I|$ is the length of I and

$$S(I) = \{r\xi : \xi \in I, 1-|I| < r < 1\}$$

is the corresponding Carleson box based on I.
A linear subspace X of $\mathcal{B}$ with a semi norm $\|.\|_X$ is Möbius invariant if for all Möbius transformation ϕ and all $f \in X$, $f\circ\phi \in X$ and $\|f\circ\phi\|_X = \|f\|_X$, there exists a positive constant λ such that

$$\|f\|_{\mathcal{B}} \leqslant \lambda\|f\|_X.$$

It is easy to see that $\mathcal{B}$ is a Möbius invariant space.
A Möbius invariant Banach space X is a Möbius invariant subspace of the Bloch space with a seminorm $\|.\|_X$, whose norm is

$$f \to \|f\|_X \quad \text{or} \quad f \to |f(0)| + \|f\|_X.$$

Rubel and Timoney showed in [**30**] that $\mathcal{B}$ is the largest Möbius invariant Banach space that possesses a decent linear functional. It is clear that $Q_K(p,q)$ is a Banach space with the norm $\|f\| = |f(0)| + \|f\|_{K,p,q}$ where $p \geqslant 1$. If $q+2=p$, $Q_K(p,q)$ is Möbius invariant, i.e.,

$$\|f\circ\varphi_a\| = \|f\|_{K,p,q} \quad \text{for all } a \in \mathbb{D}.$$

Let ϕ be an analytic self-map of unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ and let $dA(z)$ be the Euclidean area element on $\mathbb{D}$. Associated with ϕ, the composition operator C_ϕ is defined by

$$C_\phi f = f\circ\phi.$$

The problem of boundedness and compactness of C_ϕ has been studied in many Banach spaces of analytic functions and the study of such operators has recently attracted the most attention.
Shapiro in [**32**], using Nevanlinna counting function, characterized the compact composition operator on H^2 as follows:
C_ϕ is a compact operator on H^2 if and only if

$$\lim_{|w|\to 1} \frac{N_\phi(w)}{-\log|w|} = 0.$$

MacCluer in [**24**], Madigan in [**26**], Roan in [**29**], and Shapiro in [**33**] have characterized the boundedness and compactness of C_ϕ in "small" spaces. In "large" spaces, MacCluer and Shapiro proved in [**25**] that C_ϕ is compact on Bergman spaces if and only if ϕ does not have an angular derivative at any point of $\partial\mathbb{D}$. Madigan and Matheson proved in [**27**] that C_ϕ is compact on the Bloch space if and only if

$$\lim_{|\phi(z)|\to 1} \frac{|\phi'(z)|(1-|z|^2)}{1-|\phi(z)|} = 0.$$

They also proved that if C_ϕ is compact on $\mathcal{B}$ then it can not have an angular derivative at any point of $\partial\mathbb{D}$. Tjani (see [**39**]) studied compact composition operators on the Besov spaces. Bourdon, Cima and Matheson in [**15**] and Smith in [**36**] investigated the same problem on $BMOA$. Li and Wulan in [**23**] gave some characterizations of compact composition operators on Q_K and $F(p,q,s)$ spaces. Very recently in [**4–6**] the authors studied boundedness and compactness of composition operators on some analytic weighted Besov spaces. In this paper we study compact composition operator on the spaces $Q_K(p,q)$. Also we will discuss some important properties of these spaces, then we give a Carleson measure characterization of the compact composition operator C_ϕ on $Q_K(p,q)$ spaces.
Now, we give the following theorem:

Theorem 1.1. *Let $0<r<1$, $0<p<\infty$, $-2<q<\infty$ and $K:[0,\infty)\to[0,\infty)$. Also, let f be an analytic function on $\mathbb{D}$. Then the following quantities are equivalent:*

(A) $\|f\|^p_{\mathcal{B}}<\infty$.

(B) For $0<p<\infty$, we have

$$\sup_{a\in\mathbb{D}} \frac{1}{\big|D(a,r)\big|^{1-\frac{p}{2}}} \int\limits_{D(a,r)} |f'(z)|^p dA(z)<\infty.$$

(C) For $0<p<\infty$, we have

$$\sup_{a\in\mathbb{D}} \int\limits_{D(a,r)} |f'(z)|^p \Big(1-|z|\Big)^{p-2} dA(z)<\infty.$$

(D) For $0<p<\infty$ and $-2<q<\infty$, we have

$$\sup_{a\in\mathbb{D}} \int\limits_{D(a,r)} |f'(z)|^p \big(1-|z|\big)^{p-2} K(1-|\varphi_a(z)|)\, dA(z)<\infty.$$

(E) For $0<p<\infty$, we have

$$\sup_{a\in\mathbb{D}} \int\limits_{\mathbb{D}} |f'(z)|^p \left(\log\frac{1}{|z|}\right)^p |\varphi'_a(z)|^2\, dA(z)<\infty.$$

(F)

$$\sup_{a\in\mathbb{D}}\int\limits_{\mathbb{D}} |f'(z)|^p \,\big(1-|z|\big)^{p-2} K\big(g(z,a)\big)\, dA(z) < \infty, \quad \textit{if and only if}$$

$$\int\limits_0^1 (1-r^2)^{-2} K(\log\frac{1}{r}) r dr < \infty. \tag{1.4}$$

Proof. Let $0<r<1$, $0<p<\infty$ and $K:[0,\infty)\to[0,\infty)$. Because for every analytic function g on $\mathbb{D}$, $|g|^p$ is a subharmonic function, we have

$$\big|g(0)\big|^p \leqslant \frac{1}{\pi r^2}\int\limits_{D(0,r)} \big|g(w)\big|^p\, dA(w).$$

Set $g = f'\circ\varphi_a$, we obtain that

$$\begin{aligned}\big|f'(a)\big|^p &\leqslant \frac{1}{\pi r^2}\int\limits_{D(0,r)} \big|f'\circ\varphi_a(w)\big|^p dA(w)\\ &= \frac{1}{\pi r^2}\int\limits_{D(a,r)} \big|f'(z)\big|^p \frac{(1-|\varphi_a(z)|^2)^2}{(1-|z|^2)^2} dA(z).\end{aligned}$$

Since,

$$\frac{1-|\varphi_a(z)|^2}{1-|z|^2} = |\varphi_a'(z)|\,, \quad \text{where } \frac{1-|\varphi_a(z)|^2}{1-|z|^2} \leqslant \frac{4}{1-|a|^2} \quad a,z\in\mathbb{D} \;\; (\text{ see [37]}).$$

Then, we obtain that

$$\big|f'(a)\big|^p \leqslant \frac{16}{\pi r^2(1-|a|^2)^2}\int\limits_{D(a,r)} \big|f'(z)\big|^p dA(z).$$

Therefore, by $(1-|a|^2)^2 \approx (1-|z|^2)^2 \approx |D(a,r)|$, for $z\in D(a,r)$, we deduce that

$$|f'(a)|^p(1-|a|)^p \leqslant \frac{16(1-|a|)^p}{\pi r^2(1-|a|^2)^2}\int\limits_{D(a,r)} |f'(z)|^p dA(z).$$

Since $(1-|a|)^2 \approx (1-|a|^2)^2$, then

$$\begin{aligned}|f'(a)|^p(1-|a|)^p &\leqslant \frac{16}{\pi r^2(1-|a|)^{2-p}}\int\limits_{D(a,r)} |f'(z)|^p dA(z)\\ &\leqslant \frac{16\lambda}{\pi r^2\big|D(a,r)\big|^{1-\frac{p}{2}}}\int\limits_{\mathbb{D}} |f'(z)|^p dA(z)\\ &= \frac{M(r)}{\big|D(a,r)\big|^{1-\frac{p}{2}}}\int\limits_{\mathbb{D}} |f'(z)|^p dA(z),\end{aligned}$$

where λ is a positive constant and $M(r)=\frac{16\lambda}{\pi r^2}$ is a constant depending on r. Thus the quantity (A) is less than or equal to a constant times the quantity (B).

From $|D(a,r)| \approx (1-|z|^2)^2$ for all $z \in D(a,r)$, it is obvious that $(B) \approx (C)$.
By $1-|\varphi_a(z)|^2 > 1-r^2$ and $1-|\varphi_a(z)| > 1-r$ for $z \in D(a,r)$, we thus obtain

$$\begin{aligned}
&\int\limits_{D(a,r)} |f'(z)|^p(1-|z|)^{p-2}dA(z)\\
&= \int\limits_{D(a,r)} |f'(z)|^p(1-|z|)^{p-2}\ \frac{K(1-|\varphi_a(z)|^2)}{K(1-|\varphi_a(z)|^2)}dA(z)\\
&\leqslant \frac{1}{K(1-r^2)}\int\limits_{D(a,r)} |f'(z)|^p(1-|z|)^{p-2}\ K(1-|\varphi_a(z)|^2)dA(z).
\end{aligned}$$

Hence, the quality (C) is less than or equal to a constant times (D). By $1-|\varphi_a(z)|^2 \leqslant 2g(z,a)$ for all $z, a \in \mathbb{D}$, we obtain that the quantity (D) is less than or equal to a constant times (F).
The equivalent between the quantity (F) and quantity (A) follows from Wulan and Zhou (see [**41**]).
Now, from the inequality $1-|z|^2 \leqslant 2\log\frac{1}{|z|}$ for every $z \in \mathbb{D}$, putting $K(1-|\varphi_a(z)|) = (1-|\varphi_a(z)|)^2$ in (D), we see the quantity (D) is less than or equal to (E). Finally, let

$$\begin{aligned}
I(a) &= \int\limits_{D(a,r)} \left|f'(z)\right|^p\left(\log\frac{1}{|z|}\right)^p|\varphi'_a(z)|^2\,dA(z)\\
&= \Bigg(\int\limits_{\mathbb{D}_{\frac{1}{4}}} + \int\limits_{\mathbb{D}\setminus\mathbb{D}_{\frac{1}{4}}}\Bigg)\left|f'(z)\right|^p\left(\log\frac{1}{|z|}\right)^p|\varphi'_a(z)|^2\,dA(z)\\
&= I_1(a) + I_2(a),
\end{aligned}$$

where for $z \in \mathbb{D}_{\frac{1}{4}} = \{z : |z| < \frac{1}{4}\}$, $|\varphi'_a(z)|^2 = \frac{(1-|a|^2)}{|1-\bar{a}z|^4} \leqslant \frac{1}{(1-|z|)^4} \leqslant (\frac{4}{3})^4$, then we obtain

$$\begin{aligned}
I_1(a) &= \int\limits_{\mathbb{D}_{\frac{1}{4}}} \left|f'(z)\right|^p\left(\log\frac{1}{|z|}\right)^p|\varphi'_a(z)|^2\,dA(z)\\
&\leqslant \|f\|^p_{\mathcal{B}}\int\limits_{\mathbb{D}_{\frac{1}{4}}}\left(\frac{\log\frac{1}{|z|}}{(1-|z|)}\right)^p|\varphi'_a(z)|^2\,dA(z)\\
&\leqslant \|f\|^p_{\mathcal{B}}(\frac{4}{3})^{p+4}\int\limits_{\mathbb{D}_{\frac{1}{4}}}\left(\log\frac{1}{|z|}\right)^p dA(z)\\
&= (\frac{4}{3})^{p+4}C(p)\|f\|^p_{\mathcal{B}},
\end{aligned}$$

where

$$C(p)=\int\limits_{\mathbb{D}_{\frac14}}\left(\log\frac{1}{|z|}\right)^p dA(z)<\infty.$$

Now, for $z\in\mathbb{D}\backslash\mathbb{D}_{\frac14}$, we know that $\log\frac{1}{|z|}\leqslant 4(1-|z|^2)\leqslant 8(1-|z|)$, then

$$\begin{aligned} I_2(a)&\leqslant 8\int\limits_{\mathbb{D}\backslash\mathbb{D}_{\frac14}}\left|f'(z)\right|^p\left(\log\frac{1}{|z|}\right)^p|\varphi_a'(z)|^2\,dA(z)\\ &\leqslant 8^p\|f\|_{\mathcal{B}}^p\int\limits_{\mathbb{D}\backslash\mathbb{D}_{\frac14}}|\varphi_a'(z)|^2dA(z)\leqslant\lambda\|f\|_{\mathcal{B}}^p\end{aligned}$$

where λ is a positive constant. Hence, the quantity (E) is less than or equal to a constant times (A). The proof is complete. □

For $\mathcal{B}_0^\alpha$, we have the corresponding result with Theorem 1.1.
The following lemma was proved by Tjani in [**39**]:

Lemma 1.1. *[**39**] Let X, Y be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose*

(i) the point evaluation functionals on X are continuous.

(ii) the closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.

(iii) $T: X\to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.
Then T is a compact operator if and only if given a bounded sequence (f_n) in X such that $f_n\to 0$ uniformly on compact sets, then the sequence (Tf_n) converges to zero in the norm of Y.

Recall that a linear operator $T: X\to Y$ is said to be compact if it takes bounded sets in X to sets in Y which have compact closure. For Banach spaces X and Y of the space of all analytic functions $H(\mathbb{D})$, we call that T is compact from X to Y if and only if for each bounded sequence (x_n) in X, the sequence $(Tx_n)\in Y$ contains a subsequence converging to some limit in Y.

2. Modified Nevanlinna counting function and composition operators

Using Riesz Factorization theorem and Vitali's convergence theorem, Shapiro and Taylor showed in [**35**] that, C_ϕ is compact on H^p, for some $0<p<\infty$ if and only if C_ϕ is compact on H^2. Moreover, Shapiro solved the compactness problem for composition operators on H^p using the Nevanlinna counting function

$$N_\phi(w)=\sum_{\phi(z)=w}-\log|w|\quad\text{(see [26])}.$$

The counting function for the Besov space B_p is

$$N_p(w,\phi)=\sum_{\phi(z)=w}\Big(|\phi'(z)|(1-|z|^2)\Big)^{p-2}\quad \text{for } w\in\mathbb{D},\ p>1 \quad (\text{see [39]}).$$

In [**23**], Li and Wulan gave a modification of the Nevanlinna type counting function on $F(p,q,s)$ spaces as follows:

$$N_{p,q,s,\phi}(w)=\sum_{\phi(z)=w}|\phi'(z)|^{p-2}(1-|z|^2)^q g^s(z,a) \quad (\text{see [15]})$$

for $w\in\phi(\mathbb{D})$, $2\leqslant p<\infty$, $-2<q<\infty$ and $0<s<\infty$. Now, we introduce the following definition:

Definition 2.1. The counting function for the $Q_K(p,q)$ spaces is

$$N_{K,p,q,\phi}(w)=\sum_{\phi(z)=w}|\phi'(z)|^{p-2}(1-|z|^2)^q K(g(z,a)),$$

for $w\in\phi(\mathbb{D})$, $2\leqslant p<\infty$, $-2<q<\infty$ and $K:[0,\infty)\to[0,\infty)$.

The above counting functions come up in the change of variables formula in the respective spaces as follows:

For $f\in Q_K(p,q)$, $2\leqslant p<\infty$, $-2<q<\infty$ and $K:[0,\infty)\to[0,\infty)$, we have

$$\begin{aligned}\|C_\phi f\|^p_{Q_K(p,q)} &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|(f\circ\phi)'(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)\\ &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(\phi(z))|^p|\phi'(z)|^2|\phi'(z)|^{p-2}(1-|z|^2)^q K(g(z,a))dA(z)\end{aligned}$$

By making a non-univalent change of variables, we obtain that

$$\|C_\phi f\|^p_{Q_K(p,q)}=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(w)|^p N_{K,p,q,\phi}(w)dA(w). \tag{2.1}$$

Now we consider the restriction of C_ϕ to $Q_K(p,q)$. Then C_ϕ is bounded operator if and only if there is a positive constant λ such that

$$\|C_\phi f\|^p_{Q_K(p,q)}\leqslant\lambda\|f\|^p_{Q_K(p,q)} \tag{2.2}$$

for all $f\in Q_K(p,q)$ or, equivalently by (2.1),

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(w)|^p N_{K,p,q,\phi}(w)dA(w)\leqslant\lambda\|f\|^p_{Q_K(p,q)}$$

Here, we shall show that the measures which obey a "generalized" Carleson condition play a role in understanding which analytic function ϕ mapping $\mathbb{D}$ into $\mathbb{D}$ produce bounded composition operators on certain Möbius invariant spaces $X=(Q_K(p,q)$ or $\mathcal{B}^\alpha)$. This leads, as in [**10**], to the following definition of generalized Carleson type measure. Since we are interested in characterizing the compact composition operators, we will also talk about vanishing Carleson measure.

Definition 2.2. Let μ be a positive measure on $\mathbb{D}$ and let $X=\mathcal{B}^\alpha$ or $Q_K(p,q)$ for $0<p<\infty$, $-2<q<\infty$ and $K:[0,\infty)\to[0,\infty)$. Then μ is an (X,K)-Carleson measure if there is a constant $A>0$ such that

$$\int_{\mathbb{D}} |f'(w)|^p d\mu(w) \leqslant A\|f\|_X^p,$$

for all $f\in X$, holds.

In view of (2.2), we see that C_ϕ is a bounded operator on $Q_K(p,q)$ if and only if the measure $N_{K,p,q,\phi}(w)dA(w)$ is a $(Q_K(p,q),K)$-Carleson measure. Now, we give characterization of compact composition operator on $Q_K(p,q)$ spaces in terms of K-Carleson measure.

Theorem 2.1. *Let* $0<p<\infty$ *and* $K:[0,\infty)\to[0,\infty)$. *The following are equivalent:*

(i) *μ is a $(Q_K(p,p-2),K)$-Carleson measure,*

(ii) *there is a constant A such that $\mu(S(I))\leqslant A|I|^p$ for a subarc $I\subset\partial\mathbb{D}$,*

(iii) *there is a constant C such that*

$$\int_{\mathbb{D}} |\varphi_a'(z)|^p d\mu(z) \ \leqslant\ C \qquad for\ \ all\ \ a\in\mathbb{D}.$$

Proof. Suppose (i) holds. Then using Theorem 1.1 and Definition 2.2, we obtain

$$\int_{\mathbb{D}} |f'(z)|^p d\mu(z) \leqslant C\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2}K(g(z,a))dA(z),$$

for all $f\in Q_K(p,p-2)$. In particular this holds for $f(z)=\varphi_a(z)=\frac{a-z}{1-\bar{a}z}$. Hence

$$\begin{aligned}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |\varphi_a'(z)|^p d\mu(z) &\leqslant C\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |\varphi_a'(z)|^p(1-|z|^2)^{p-2}K(g(z,a))dA(z)\\ &\leqslant C\ \|\varphi_a\|^p_{Q_K(p,p-2)} \ \leqslant\ C\ \lambda,\end{aligned}$$

for all $a\in\mathbb{D}$. This gives (iii).
Suppose that (iii) holds, we shall show that (ii) is true, hence

$$\begin{aligned}C &\geqslant \int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p d\mu(z) \geqslant \int_{S(I)}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p d\mu(z),\\ &\gtrsim \frac{\mu(S(I))}{|I|^p} \geqslant \frac{\lambda}{|I|^p}\mu(S(I))\end{aligned}$$

we have

$$\mu(S(I))<A|I|^p$$

This gives (ii).
Suppose now that (ii) holds, we shall show that (i) is true, thus completing the

implications. For $z = re^{i\theta}$, let

$$E_1(z) = \left\{ w : |w - z| < \frac{1 - |z|}{2} \right\},$$

$$E_2(z) = \left\{ w : |w - z| < 1 - |z| \right\}.$$

Then

$$E_1(z) \subseteq E_2(z) \subseteq S(2(1 - |z|), \theta).$$

Further, if $w \in E_1(z)$, then

$$\frac{1}{2} \leqslant \frac{1 - |w|}{1 - |z|} \leqslant \frac{3}{2}.$$

Let $f \in Q_K(p, q)$; because f is analytic we have

$$f'(z) = \frac{4}{\pi(1 - |z|)^2} \int\limits_{E_1(z)} f'(w) dA(w).$$

Therefore by Jensen's inequality (see [**31**]),

$$|f'(z)|^p \leqslant \frac{4}{\pi(1 - |z|)^2} \int\limits_{E_1(z)} |f'(w)|^p dA(w).$$

Thus,

$$\begin{aligned}
\int\limits_{\mathbb{D}} |f'(z)|^p d\mu(z) &\leqslant \int\limits_{\mathbb{D}} \frac{4}{\pi(1 - |z|)^2} \Bigg(\int\limits_{E_1(z)} |f'(w)|^p dA(w) \Bigg) d\mu(z) \\
&\leqslant \frac{4}{\pi} \int\limits_{\mathbb{D}} \Bigg(\int\limits_{E_1(z)} |f'(w)|^p \left(\frac{3}{2(1 - |w|)} \right)^2 dA(w) \Bigg) d\mu(z) \\
&\leqslant \frac{9}{\pi} \int\limits_{\mathbb{D}} \int\limits_{\mathbb{D}} |f'(w)|^p \chi_{E_1(z)}(w) (1 - |w|)^{-2} dA(w) d\mu(z) \\
&\leqslant \frac{9}{\pi} \int\limits_{\mathbb{D}} |f'(w)|^p (1 - |w|)^{-2} \int\limits_{\mathbb{D}} \chi_{E_1(z)}(w) d\mu(z) dA(w).
\end{aligned}$$

However, $\chi_{E_1(z)}(w) \leqslant \chi_{S(2(1-|z|),\theta)}(w)$, $z = |z|e^{i\theta}$, since $w \in E_1(z)$ implies that

$$|w - e^{i\theta}| < 2(1 - |w|).$$

Now applying (ii) and using condition (e), we have

$$\int\limits_{\mathbb{D}} \chi_{E_1(z)} d\mu(z) \leqslant \mu(S(2(1 - |w|), \theta)) \leqslant A2^q (1 - |w|)^p.$$

Therefore,

$$\begin{aligned}\int_{\mathbb{D}} |f'(z)|^p d\mu(z) &\leqslant \frac{9}{\pi} A 2^q \int_{\mathbb{D}} |f'(w)|^p (1-|w|)^{p-2} K(g(z,a)) dA(w)\\ &\leqslant C \int_{\mathbb{D}} |f'(w)|^p (1-|w|)^{p-2} K(g(z,a)) dA(w),\end{aligned}$$

where C is a constant. By Theorem 1.1 the quantities (C) and (E) are equivalent so, we have

$$\begin{aligned}\int_{\mathbb{D}} |f'(z)|^p d\mu(z) &\leqslant C \int_{\mathbb{D}} |f'(w)|^p (1-|w|)^{p-2} K(g(z,a)) dA(w)\\ &\leqslant C \ \|f\|^p_{Q_K(p,p-2)},\end{aligned}$$

then,

$$\int_{\mathbb{D}} |f'(z)|^p d\mu(z) \leqslant C \ \|f\|^p_{Q_K(p,p-2)}$$

which is (i). This finishes the proof. □

Hence Theorem 2.1 yields:

THEOREM 2.2. *Let ϕ be an analytic function on $\mathbb{D}$, $0 < p < \infty$ and $K : [0,\infty) \to [0,\infty)$. Then C_ϕ is a bounded operator on $Q_K(p,p-2)$ if and only if*

$$\sup_{a\in\mathbb{D}} \|C_\phi \ \varphi_a\|_{Q_K(p,p-2)} < \infty.$$

PROPOSITION 2.1. *Let $K : [0,\infty) \to [0,\infty)$ be nondecreasing function, for $I \subset \partial\mathbb{D}$. Then for $0 < p < \infty$, the following are equivalent:*
(i) μ is a vanishing K-Carleson measure.
(ii) $\int_{\mathbb{D}} |\varphi_a'(z)|^p d\mu(z) \to 0$, as $|a| \to 1^-$.

PROOF. First, suppose that (ii) holds. Then, given an $\varepsilon > 0$ there is a $\delta > 0$ such that for $1-\delta < |a| < 1$,

$$\int_{\mathbb{D}} |\varphi_a'(z)|^p d\mu(z) < \varepsilon.$$

Fix $\varepsilon > 0$ and let $\delta > 0$ be as above. Consider any $0 < |I| < \delta$, $\theta \in [0,2\pi)$, and let $a = e^{i\theta}(1 - \frac{|I|}{2\pi})$ and $z \in S(|I|,\theta)$. Then,

$$\frac{1-|a|^2}{|1-\bar{a}z|^2} \gtrsim \frac{1}{|I|}, \quad z \in S(I).$$

Hence, $z \in S(|I|,\theta)$ implies that $|\varphi_a'(z)| \gtrsim \frac{1}{|I|}$. Then by our hypothesis,

$$\varepsilon \geqslant \int_{\mathbb{D}} |\varphi_a'(z)|^p d\mu(z) \geqslant \int_{S(I)} \left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p d\mu(z) \gtrsim \frac{\mu(S(I))}{|I|^p}.$$

This proves (i).
Conversely, suppose that (i) holds. Then, given an $\varepsilon > 0$ there is a $\delta > 0$ such that for any $0 < |I| < \delta$ and any $\theta \in [0, 2\pi)$,

$$\mu(S(|I|, \theta)) < \varepsilon |I|^p. \tag{2.3}$$

Fix $\varepsilon > 0$, and let δ be as above. Fix $|I_0| < \delta$ such that (2.3) holds. Also, fix $a = |a|e^{i\theta} \in \mathbb{D}$ with $|a| > 1 - |I_0|$. We will show that for a large,

$$\int\limits_{\mathbb{D}} |\varphi_a'(z)|^p d\mu(z) < \varepsilon.$$

Let $E = \Big\{ z \in \mathbb{D} : \big| e^{i\theta} - |a|z \big| \geqslant |I_0| \Big\}$. Then for each $a \in \mathbb{D}$,

$$\int\limits_{\mathbb{D}} |\varphi_a'(z)|^p d\mu(z) = \int\limits_{E} |\varphi_a'(z)|^p d\mu(z) + \int\limits_{E^c} |\varphi_a'(z)|^p d\mu(z). \tag{2.4}$$

We will estimate each of the integrals above.
First if $z \in E$,

$$|\varphi_a'(z)|^p = \left(\frac{1 - |a|^2}{\big| e^{i\theta} - |a|z \big|} \right)^p \leqslant \left(\frac{1 - |a|^2}{|I_0|^2} \right)^p < \varepsilon \tag{2.5}$$

for a large. Therefore (2.5) yields that for a large,

$$\int\limits_{E} |\varphi_a'(z)|^p d\mu(z) < \varepsilon \mu(E) < \varepsilon \mu(\mathbb{D}) < \varepsilon\, \lambda. \tag{2.6}$$

Let $n_0 = n_0(a)$ be the smallest positive integer such that

$$2^{n_0}(1 - |a|) < |I_0| \leqslant 2^{n_0+1}(1 - |a|). \tag{2.7}$$

We will show that

$$E^c \subset S(2^{n_0}(1 - |a|), \theta) \subset S(|I_0|, \theta). \tag{2.8}$$

Let $z \in E^c$. Then,

$$\begin{array}{rcl} \big| z - e^{i\theta} \big| & = & \big| z - e^{i\theta} + |a|z - |a|z \big| \\ & \leqslant & \big| z - |a|z \big| + \big| e^{i\theta} - |a|z \big| \\ & < & 1 - |a| + |I_0| \\ & < & 1 - |a| + 2^{n_0}(1 - |a|) \\ & \leqslant & 2^{n_0}(1 - |a|). \end{array}$$

This proves that $E^c \subset S(2^{n_0}(1 - |a|), \theta)$. Next let $z \in S(2^{n_0}(1 - |a|), \theta)$. Then, by (2.7),

$$|z - e^{i\theta}| \leqslant 2^{n_0}(1 - |a|) < |I_0|.$$

Hence $S(2^{n_0}(1 - |a|), \theta) \subset S(|I_0|, \theta)$. Thus, (2.8) is proved.
Let $E_k = S(2^k(1 - |a|), \theta),\ k = 0, 1, 2, ..., n_0$. It is clear that

$$E_0 \subset E_1 \subset ... \subset E_{n_0} \subset S(|I_0|, \theta).$$

Then,

$$\int\limits_{E^c} |\varphi_a'(z)|^p d\mu(z) \leqslant \int\limits_{S(2^{n_0}(1-|a|),\theta)} |\varphi_a'(z)|^p d\mu(z)$$

$$(2.9)\qquad = \int\limits_{E_0} + \int\limits_{E_1 \setminus E_0} + ... + \int\limits_{E_{n_0}\setminus E_{n_0-1}} |\varphi_a'(z)|^p d\mu(z).$$

We will estimate each of the integral above.
First, if $z \in E_0$, then $|z - e^{i\theta}| < 1 - |a|$ and

$$|\varphi_a'(z)| \leqslant \frac{1-|a|^2}{(1-|a|)^2} \leqslant \frac{2}{1-|a|}.$$

Since $1 - |a| < |I_0| < \delta$, (2.3) yields

$$\int\limits_{E_0} |\varphi_a'(z)|^p d\mu(z) \leqslant \frac{2^p}{(1-|a|)^p}\mu(E_0) \leqslant \varepsilon\, \lambda.$$

Next if $z \in E_k \setminus E_{k-1}$ for some $k = 2, 3, ..., n_0$, then

$$|\varphi_a'(z)| = \frac{1-|a|^2}{|e^{i\theta} - |a|z|^2} \leqslant \frac{1-|a|^2}{|e^{i\theta} - |a|z|^2} \leqslant \frac{1-|a|^2}{(|z-e^{i\theta}| - |z|(1-|a|))^2}$$

$$(2.10)\qquad \leqslant \frac{\lambda}{4^k(1-|a|)}.$$

Hence, (2.3), (2.7), and (2.10) yield

$$\int\limits_{E_k\setminus E_{k-1}} |\varphi_a'(z)|^p d\mu(z) \leqslant \frac{\lambda}{4^{kp}(1-|a|)^p}\mu(E_k)$$

$$\leqslant \frac{\varepsilon\lambda 2^{kp}(1-|a|)^p}{4^{kp}(1-|a|)^p}$$

$$(2.11)\qquad = \frac{\varepsilon\lambda}{2^{kp}}.$$

Therefore (2.6), (2.9), (2.10) and (2.11) imply that

$$\int\limits_{\mathbb{D}} |\varphi_a'(z)|^p d\mu(z) < \varepsilon\, \lambda + \varepsilon \left(\sum_{k=0}^{n_0} \frac{1}{2^{kp}}\right)\lambda < \varepsilon\, \lambda$$

for a large. This proves (ii). □

Now, we give a K-Carleson measure characterization of the compact composition operator on $Q_K(p,q)$ spaces.

THEOREM 2.3. *Let μ be a positive measure on $\mathbb{D}$, and assume that K satisfies* (2.1).
For $0 < p < \infty, -2 < q < \infty$. Then

(i) $\mu(p,q)$ *is a* K*-Carleson measure if and only if*

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} K(1-|\varphi_a(z)|^2)d\mu(z)<\infty. \tag{2.12}$$

(ii) $\mu(p,q)$ *is a compact* K*-Carleson measure if and only if* (2.12) *holds and*

$$\lim_{|a|\to 1^-}\int_{\mathbb{D}} K(1-|\varphi_a(z)|^2)d\mu(z)=0.$$

Proof. The prove is very similar as in [**28**]. Arazy, Fisher and Peetre in [**10**], Cima and Wogen in [**16**], Tjani in [**39**] gave the characterization of p-Carlson measure on Besov spaces. Also Ahmed and Bakhit [**5**] gave the characterization of p-Carlson measure on $F(p,q,s)$ spaces. □

We will prove the following lemmas on $Q_K(p,q)$ spaces:

Lemma 2.1. *Let* $K:[0,\infty)\to[0,\infty)$ *and* $X=Q_K(p,q)$ *where* $0<p<\infty$ *and* $-2<q<\infty$, *we have*

(i) *Every bounded sequence* $(f_n)\in X$ *is uniformly bounded on compact sets.*

(ii) *For any sequence* (f_n) *on* X *such that* $\|f_n\|_X\to 0$, $f_n-f_n(0)\to 0$ *uniformly on compact sets.*

Proof. From [**41**], we have

$$\pi r^2 K(\log\frac{1}{r})(1-|a|^2)^{q+2}|f'(a)|^p\leqslant\int_{\mathbb{D}}|f'(z)|^p\big(1-|z|^2\big)^q K(g(z,a))dA(z).$$

We have

$$\|f\|_{\mathcal{B}^{\frac{q+2}{p}}}\leqslant\frac{1}{\pi r^2K(\log\frac{1}{r})}\|f\|_{Q_K(p,q)},$$

where $\frac{1}{\pi r^2K(\log\frac{1}{r})}$ is a constant depending on K.

If $z\in D(0,r)$, $0<r<1$, then we have

$$\begin{aligned}|f_n-f_n(0)|=\left|\int_0^1 f'(zt)zdt\right| &\leqslant \|f\|_{\mathcal{B}^{\frac{q+2}{p}}}\int_0^1\frac{|z|dt}{(1-|z|^2t^2)^{\frac{q+2}{p}}}\\ &\leqslant \frac{1}{(1-|r|^2)^{\frac{q+2}{p}}}\|f\|_{\mathcal{B}^{\frac{q+2}{p}}}\\ &\leqslant \frac{1}{\pi r^2K(\log\frac{1}{r})}\frac{1}{(1-|r|^2)^{\frac{q+2}{p}}}\|f\|_{Q_K(p,q)}.\end{aligned}$$

Hence the result follows. □

LEMMA 2.2. *Let $K:[0,\infty)\to[0,\infty)$ be nondecreasing function and let $X,Y=Q_K(p,q)$ or $\mathcal{B}^\alpha$. Also suppose that $0<p<\infty,-2<q<\infty$ and $0<\alpha<\infty$. Then $C_\phi:X\to Y$ is a compact operator if and only if for any bounded sequence $(f_n)\in X$ with $f_n\to 0$ uniformly on compact sets as $n\to\infty$, $\|C_\phi f_n\|_Y\to 0$ as $n\to\infty$.*

PROOF. We will show that (i), (ii), and (iii) of Lemma 1.1 hold for our spaces. By Lemma 2.1 it is easy to see that (i) and (iii) holds. To show that (ii) holds, let (f_n) be a sequence in the closed unit ball of X. Then by Lemma 2.1, (f_n) is a uniformly bounded on compact sets. Therefore, by Montel's theorem (see [**17**]), there is a subsequence $(f_{n_k}),(n_1<n_2<\ldots)$ such that $f_{n_k}\to h$ uniformly bounded on compact sets, for some $h\in H(\mathbb{D})$. Thus we only need to show that $h\in X$.

(a) If $X=Q_K(p,q)$, then

$$\begin{aligned}\int_{\mathbb{D}}|h'(z)|^p(1-|z|^2)^qK(g(z,a))dA(z) &= \int_{\mathbb{D}}\lim_{k\to\infty}|f'_{n_k}(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)\\ &\leqslant \lim_{k\to\infty}\inf\int_{\mathbb{D}}|f'_{n_k}(z)|^p(1-|z|^2)^qK(g(z,a))dA(z),\\ &= \lim_{k\to\infty}\inf\|f_{n_k}\|^p_{Q_K(p,q)} < \infty,\end{aligned}$$

where we used Fatou's theorem [**31**] and our hypothesis.

(b) If $X=\mathcal{B}^\alpha$, we have that

$$\begin{aligned}|h'(z)|(1-|z|^2)^\alpha &= \lim_{k\to\infty}|f'_{n_k}(z)|(1-|z|^2)^\alpha\\ &\leqslant \lim_{k\to\infty}\|f_{n_k}\|_{\mathcal{B}^\alpha}<\infty,\end{aligned}$$

this by our hypothesis. Therefore, Lemma 2.1 yields that $C_\phi:X\to Y$ is a compact operator if and only if for any bounded sequence $(f_n)\in X$ with $f_n\to 0$ uniformly on compact sets as $n\to\infty$, $|f_n(\phi(0))|+\|C_\phi f_n\|_Y\to 0$, as $n\to\infty$, which is clearly equivalent to the statement of this lemma. This completes the proof of the lemma.

□

We prove the following theorem for compact composition operators on $Q_{K^*}(p,q)$ spaces.

THEOREM 2.4. *Let $0\leqslant p<p^*<\infty$, $-2<q<\infty$ and $K:[0,\infty)\to[0,\infty)$. Then the following are equivalent:*

(i) *$C_\phi:Q_K(p,q)\to Q_K(p^*,q)$ is a compact operator.*

(ii) *$N_{K,p^*,q,\phi}(w)dA(w)$ is a vanishing K-Carleson measure.*

(iii) *$\|C_\phi\,\varphi_a\|_{Q_K(p^*,q)}\to 0$ as $|a|\to 1^-$.*

PROOF. By (2.1)

$$\|C_\phi\,\varphi_a\|^{p^*}_{Q_K(p^*,q)}=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|\varphi'_a(w)|^{p^*}N_{K,p^*,q,\phi}(w)dA(w)$$

Thus Theorem 2.3 yields (ii)$\Longleftrightarrow$(iii). Next we show that (i)$\Longrightarrow$(iii).
We assume that $C_\phi : Q_K(p,q) \to Q_K(p^*,q)$ is a compact operator. Note that $\{\varphi_a : a \in \mathbb{D}\}$ is a bounded set in $Q_K(p,q)$. Since

$$\|\varphi_a\|_{Q_K(p,q)} = \|z \circ \varphi_a\|_{Q_K(p,q)},$$

the norm of φ_a in $Q_K(p,q)$ is

$$|\varphi_a(0)| + \|\varphi_a\|_{Q_K(p,q)} < 1 + \|\varphi_a\|_{Q_K(p,q)} < \infty.$$

Also $(\varphi_a - a) \longrightarrow 0$ as $|a| \to 1$, uniformly on compact sets, since

$$|\varphi_a - a| = |z|\left(\frac{1-|a|^2}{|1-\bar{a}z|}\right), \text{ where } |z| = r < 1.$$

Hence by Lemma 2.2, we obtain that

$$\|C_\phi(\varphi_a - a)\|_{Q_K(p^*,q)} \to 0, \text{ as } |a| \to 1^-.$$

Finally, let us show that (ii)$\Rightarrow$(i). Let (f_n) be a bounded sequence in $Q_K(p,q)$ that converges to 0 uniformly on compact sets. Then the mean value property for the analytic function f_n' yields that

$$f_n'(w) = \frac{4}{\pi(1-|w|)^2} \int\limits_{|w-z|<\frac{1-|w|}{2}} |f_n'(z)| dA(z). \tag{2.13}$$

Therefore by Jensen's inequality (see [**31**], Theorem 3.3) and (2.13), we obtain

$$|f_n'(w)|^{p^*} \leqslant \frac{4}{\pi(1-|w|)^2} \int\limits_{E_1(w)} |f_n'(z)|^{p^*} dA(z), \tag{2.14}$$

where

$$E_1(w) = \left\{ z : |w - z| < \frac{1-|w|}{2} \right\}.$$

Then by (2.15) and Fubini's Theorem (see [**31**], Theorem 8.8),

$$\begin{aligned} \|C_\phi f_n\|^{p^*}_{Q_K(p^*,q)} &= \sup_{a\in\mathbb{D}} \int\limits_{\mathbb{D}} |f_n'(w)|^{p^*} N_{K,p^*,q,\phi}(w) dA(w) \\ &\leqslant \sup_{a\in\mathbb{D}} \int\limits_{\mathbb{D}} \frac{4}{\pi(1-|w|)^2} \Bigg(\int\limits_{E_1(w)} |f_n'(z)|^{p^*} dA(z) \Bigg) N_{K,p^*,q,\phi}(w) dA(w). \end{aligned}$$

Then,

$$\|C_\phi f_n\|^{p^*}_{Q_K(p^*,q)} \leqslant \frac{4}{\pi} \sup_{a\in\mathbb{D}} \int\limits_{\mathbb{D}} |f_n'(z)|^{p^*} \Bigg(\int\limits_{\mathbb{D}} \frac{\chi_{E_1(w)}(z) N_{K,p^*,q,\phi}(w) dA(w)}{(1-|w|)^2} \Bigg) dA(z), \tag{2.15}$$

Note that if $|w-z| < \frac{1-|w|}{2}$, then $w \in S(2(1-|z|),\theta)$, where $z = |z|e^{i\theta}$, since

$$|w - e^{i\theta}| \leqslant |z - w| + |e^{i\theta} - z| < \frac{1-|w|}{2} + |\frac{z}{|z|} - z| < 2(1-|z|).$$

Moreover, if $|w-z| < \frac{1-|w|}{2}$, then $\frac{1}{(1-|w|)^2} < \frac{\lambda}{(1-|z|)^2}$. Therefore (2.14) yields

$$\begin{aligned}
\|C_\phi f_n\|^{p^*}_{Q_K(p^*,q)} &\leqslant \lambda \sup_{a\in\mathbb{D}} \int\limits_{\mathbb{D}} \frac{|f_n'(z)|^{p^*}}{(1-|z|)^2}\Bigg(\int\limits_{S(2(1-|z|),\theta)} N_{K,p^*,q,\phi}(w)dA(w)\Bigg)dA(z)\\
&= \lambda \sup_{a\in\mathbb{D}}\Bigg(\int\limits_{|z|>1-\frac{\delta}{2}} + \int\limits_{|z|\leqslant 1-\frac{\delta}{2}} \frac{|f_n'(z)|^{p^*}}{(1-|z|)^2}\Bigg(\int\limits_{S(2(1-|z|),\theta)} N_{K,p^*,q,\phi}(w)dA(w)\Bigg)dA(z)\Bigg)\\
&= \lambda(J_1+J_2),
\end{aligned}$$

for any $0<\delta<1$, Fix $\varepsilon>0$ and let $\delta>0$ be such that for any $\theta\in[0,2\pi]$ and any $|I|<\delta$,

$$\sup_{a\in\mathbb{D}} \int\limits_{S(I)} N_{K^*,p^*,q,\phi}(w)dA(w) \leqslant \varepsilon\, |I|^{p^*}. \tag{2.16}$$

By (2.16) we have

$$\begin{aligned}
J_1 &\leqslant \varepsilon\, 2^{p^*} \int\limits_{|z|>1-\frac{\delta}{2}} \frac{|f_n'(z)|^{p^*}}{(1-|z|)^2}(1-|z|)^{p^*}dA(z)\\
&\leqslant \varepsilon\, 2^{p^*} \int\limits_{|z|>1-\frac{\delta}{2}} |f_n'(z)|^{p^*}(1-|z|)^{p^*-2}dA(z)\\
&\leqslant \varepsilon\ \lambda\ \|f_n\|^{p^*}_{B_{p^*}} < \varepsilon\ \lambda,
\end{aligned}$$

and

$$\begin{aligned}
J_2 &\leqslant \lambda \int\limits_{|z|\leqslant 1-\frac{\delta}{2}} \frac{|f_n'(z)|^{p^*}}{(1-|z|)^2}\Bigg(\int\limits_{S(2(1-|z|),\theta)} N_{K,p^*,q,\phi}(w)dA(w)\Bigg)dA(z)\\
&= \lambda \left(\int\limits_{\mathbb{D}} N_{K,p^*,q,\phi}(w)dA(w)\right) \int\limits_{|z|\leqslant 1-\frac{\delta}{2}} |f_n'(z)|^{p^*}dA(z) < \lambda,
\end{aligned}$$

for n large enough, since $f_n'\to 0$ uniformly on compact sets. We obtain that

$$\|C_\phi f_n\|^{p^*}_{Q_K(p^*,q)} < \lambda(J_1+J_2) < \varepsilon\,\lambda,$$

for n large enough. Therefore, $\|C_\phi f_n\|^{p^*}_{Q_K(p^*,q)} \to 0$, as $n\to\infty$ and Lemma 2.2 yields that
$C_\phi : Q_K(p,q)\to Q_K(p^*,q)$ is a compact operator. This finishes the proof of the theorem. □

Remark 2.1. *It is still an open problem to obtain similar results that obtained in this paper for hyperbolic and quaternion classes. For more studies of composition operators on hyperbolic classes, we refer to [**3**, **7**] and others. For details of quaternion function spaces, we refer to [**1**, **2**, **8**, **9**] and others.*

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