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SEMI COMPLETE GRAPHS - III

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Abstract. A Further study about semi-complete graph is made. Path connector set, Edge path connector set, Path-critical edges and neighbourhood sets in this graph are introduced and interesting results are developed.

1. Introduction

In the earlier papers [2], [3] the utility of semi-complete graphs is mentioned. As there is wide application of these graphs in computers and defence problems further useful concepts, namely Path connector set, Edge path connector set, Path-critical edge, neighbourhood set with regard to these graphs are introduced and useful study about these is made.

2. Preliminaries

We, first give a few definitions, observations and results that are useful for development in the succeeding articles.

Definitions 2.1([2]). (i) A graph G is said to be semi-complete(SC) iff (if and only if) it is simple and for any two vertices u, v of G there is a vertex w of G such that $\{u, w, v\}$ is a path in G.
(ii) A graph G is said to be purely semi-complete iff G is semi-complete but not complete.

Theorem 2.1. *([2]) G is a semi-complete graph. Then there exists a unique path of length 2 between any two vertices of G iff the edge set of G can be partitioned into edge disjoint triangles.*

Theorem 2.2. *([2]) G is a union of triangles such that no two triangles have a common edge; then all the triangles have a common vertex.*

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Definition 2.2 ([3]). A semi-complete(SC) graph G is said to be strong semi-complete (S.S.C)iff there is atleast one edge of G whose removal from G does not affect the semi-complete property(i.e it results in a semi-complete graph).

A characterization result for a semi-complete graph to be strong semi-complete graph is the following:

THEOREM 2.3. *([3]) A semi-complete graph G is strong semi-complete iff there is an edge uv of G such that there are atleast two paths of length 2 from u to any point of $N(v) - \{u\}$ and v to any point of $N(u) - \{v\}$.*

E.Sampath Kumar [4] introduced the concept of neighbourhood sets as follows:

Definition ([4]). (i) A set S of vertices in a graph G is said to be a neighbourhood set of G iff $G = \bigcup_{v \in S} < N[v] >$, where $< N[v] >$ is the subgraph of G induced by "v" and all its neighbours(adjacent vertices) in G.

For convenience, a neighbourhood set of G is referred as n-set of G.

Since the vertex set of G is itself an n-set of G, there is no interest to discuss about maximum n-set in a graph.

(ii)The minimum among the cardinalities of all n-sets in a graph G is called the neighbourhood number of G and is denoted by $n(G)$.

A characterization result for a subset of the vertex set of a graph to be an $n - set$ is the following:

Result 2.1. ([4]) A subset S of the vertex set V is an n-set of G iff each edge in $< V - S >$ (the subgraph induced by V-S in G) is in $< N[v] >$ for some $v \in S$.

To avoid trivialities, we consider only nonempty graphs. Now we introduce path connector set in a graph.

3. PATH CONNECTOR SET

Definitions 3.1. (i) A Path connector set(pc-set) in a graph G is a subset V' of the vertex set V of G such that for any distinct pair of non-adjacent vertices in G there is a shortest path whose internal vertices are from V'.

(ii) A path connector set in G is said to be a minimum path connector set(mpc-set) in G iff(if and only if) it has the minimum cardinality among all the pc-sets in G.

EXAMPLE 3.1. For the graph given in Figure 1 $\{v_3, v_5, v_6\}, \{v_3, v_5, v_8\}$ are mpc-sets.

Observations 3.1. (i) As there are no non-adjacent vertices in the complete graph K_n, it follows that any subset of the vertex set of K_n is a pc-set. In particular, the empty set is also a pc-set(infact mpc-set). So there is no interest in complete graphs with regard to this aspect.

(ii) As there are atleast two non-adjacent vertices in a disconnected graph such that there is no path between them it follows that pc-sets do not exist for such graphs. Clearly

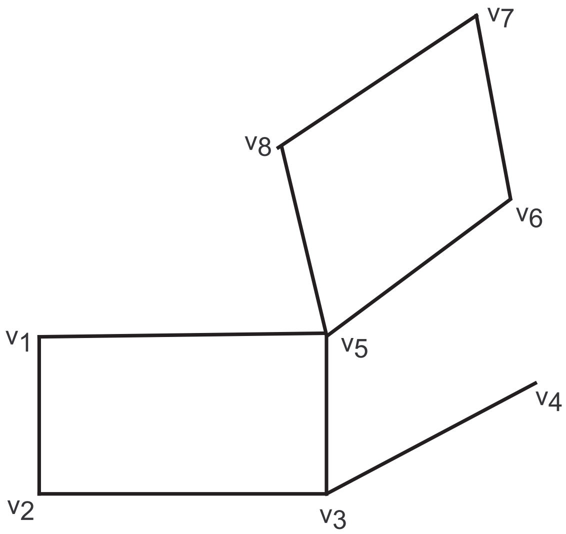

FIGURE 1

Result 3.1. A non empty graph is connected iff it admits pc-sets.
Proof: For, if G is such a graph its vertex set itself is a pc-set(so there is no interest to discuss about maximum pc-sets).

Conversely, if G admits pc-sets,then by definition, it follows that G is connected.

Note 3.1. Any nonempty connected graph admits mpc-set.

For, if V is the vertex set of G then $\wp$, the class of all pc-sets in G is nonempty, since $V \in \wp$.Hence $\wp$ admits an element S with minimum cardinality$\Rightarrow$ S is a mpc-set in G.

THEOREM 3.1. *(Characterization Result) G is a purely semi-complete graph with vertex set V. Then $S \subseteq V$ is a pc-set in G iff for every distinct pair of non-adjacent vertices u and v in G there is a $w \in S$ which is adjacent to both u and v in G.*

PROOF. Since G is semi-complete there is a path of length two between any two vertices in G. Then the shortest path between any two non-adjacent vertices is of length two in such a graph. If S is a pc-set in G, by definition, follows the necessary part.Conversely, if S has the property stated then clearly S is a pc-set in G. □

THEOREM 3.2. *G is a purely semi-complete graph with vertex set V. Then*
(a)Any pc-set in G is a dominating set in G.
(b)Further, if $\mid S \mid \geqslant 2$ then S is a total dominating set in G.

Proof. Since any semi-complete graph is connected, it follows that the graph G admits a nonempty pc-set, say S. If S is singleton say $\{v_0\}$,then since G is semi-complete follows that every vertex of G is adjacent with v_0. Thus S is a dominating set in G.Now, assume that $\mid S \mid \geqslant 2$. Let $u \in V$ and $v \in S - \{u\}$. If u is adjacent to v then we are through; otherwise since G is semi-complete, there is a $w \in S$ such that $\{u, w, v\}$ is a shortest path in G. Now u is adjacent to $w \in S$
$\Rightarrow S$ is a total dominating set in G.

This completes the proof of the theorem. □

Observations 3.2. (i) If S of the above theorem has exactly two elements(vertices) then they are adjacent in G.
(ii)S of the above theorem is an independent set iff $|S| = 1$.

Remark 3.1. The converse of Theorem.(3.2(a)) is true iff the cardinality of the dominating set is 1.

For, that single vertex set is clearly a pc-set (infact a mpc-set) in G.

If the cardinality of the dominating set is > 1 then it need not be a pc-set in view of the following:

Example 3.2. Consider the following graph G, in Figure 2

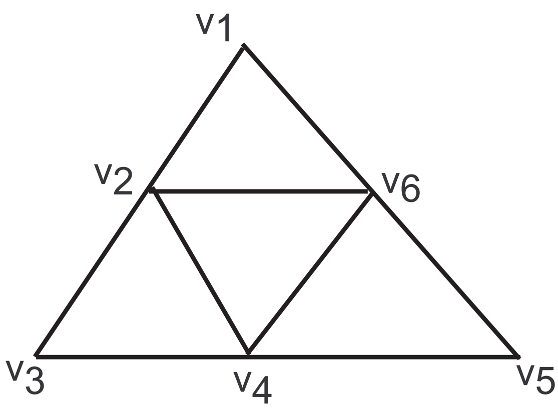

Figure 2

clearly $\{v_2, v_6\}$ is a (total) dominating set in G; but this is not a pc-set in G, since there is only one shortest path between v_3 and v_5,namely $\{v_3, v_4, v_5\}$ and v_4 is not in $\{v_2, v_6\}$.
Infact, $\{v_2, v_4, v_6\}$ is a pc-set (further mpc-set) in G.

Theorem 3.3. *G is purely semi-complete graph with n vertices. Then the domination number $\gamma(G) = 1 \Leftrightarrow |mpcs(G)| = 1$.*

PROOF. Since G is purely semi-complete, it follows that $n \geqslant 4$. Let $\gamma(G) = 1$. So there is a $v_0 \in V(G)$ such that $d_G(v_0) = n - 1$. Denote $S = \{v_0\}$. Let $v_1, v_2 \in V(G)$ such that v_1 and v_2 are not adjacent in G. Now follows that $\{v_1, v_0, v_2\}$ is a shortest $v_1 - v_2$ path in $G \Rightarrow$ S is a pc-set in G. Since $|S| = 1$, it follows that S is a mpc-set in $G \Rightarrow |mpcs(G)| = 1$

Conversely, assume that $|mpcs(G)| = 1$. So there is a pc-set S in G with $|S| = 1$. Now, by Theorem.(3.2(a)), it follows that S is a dominating set in G $\Rightarrow \gamma(G) = 1$.

This completes the proof of the theorem. □

Observations 3.3. (i) From Theorem.(3.3) and observation (3.2.(ii)), it follows that for any such graph G,$\gamma(G) = 1 \Leftrightarrow$ any mpc-set in G is an independent set in G.

(ii) From Theorem.(3.2), Remark (3.1) and Theorem.(3.3),we have

A purely semi-complete graph is a union of triangles, where all the triangles have a common vertex iff $|mpcs(G)| = 1 \Leftrightarrow$ any mpc-set in G is an independent set in G.

(iii) If G is a semi-complete graph such that there is a unique path of length two between every pair of non-adjacent vertices in G, then $|mpcs(G)| = 1 \Rightarrow$ there is a unique mpc-set and it is independent set in G.

For, by Theorem.(2.2), it follows that,the edge set of G is a union of edge disjoint triangles where all the triangles have a common vertex

$\Rightarrow \gamma(G) = 1$

$\Leftrightarrow |mpcs(G)| = 1$.

The converse of (iii) is false in view of the following:

EXAMPLE 3.3. Consider the following graph G in Figure 3

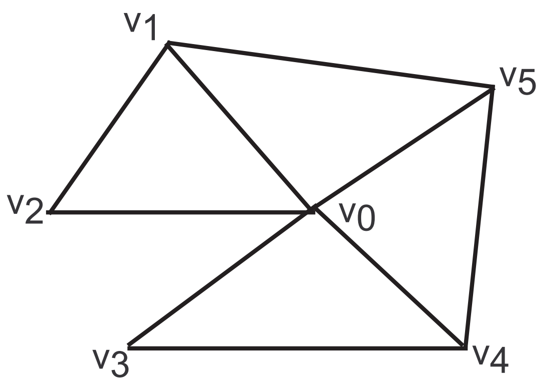

FIGURE 3

$\{v_0\}$ is a mpc-set of G. But there are two paths namely $\{v_2, v_0, v_5\}, \{v_2, v_1, v_5\}$ between the pair v_2, v_5 of non-adjacent vertices in G.

THEOREM 3.4. *G is a semi-complete graph such that* $|mpcs(G)| = 2$. *Then* $\gamma(G) = 2$.

PROOF. Under the given hypothesis and Theorem.(3.3), it follows that $\gamma(G) \geqslant 2$. By Th.(3.2) follows that there is a dominating set with 2 elements; so $\gamma(G) \leqslant 2$. Hence $\gamma(G) = 2$. □

The converse of the above theorem is false in view of the following:

EXAMPLE 3.4. For the graph given in Remark (3.1), $\{v_2, v_6\}$ is a minimum dominating set and so $\gamma(G) = 2 \neq 3 = |mpcs(G)|$.

THEOREM 3.5. *G is a semi-complete graph such that the triangles formed by the edges in G have a common edge, say uv iff* $\{u\}$ *and* $\{v\}$ *are mpc-sets in G(*$\Rightarrow$ *independent sets in G).*

PROOF. Under the given hypothesis ,let $S = \{u\}$.Let x, y be non-adjacent vertices in G.
$\Rightarrow \{x, y\} \neq \{u, v\}$. So x, y lie on different triangles of G
$\Rightarrow$ Since uv is a common edge of the triangles, follows that $\{x, u, y\}$ and $\{x, v, y\}$ are shortest $x - y$ paths in G
$\Rightarrow \{u\}, \{v\}$ are mpc-sets in G.
Conversely, assume that the vertices u, v of G are such that $\{u\}, \{v\}$ are mpc-sets in G.
$\Rightarrow \gamma(G) = 1$
$\Rightarrow$ any vertex $x \notin \{u, v\}$ of G is adjacent with both u and v. Further u and v must be adjacent in G ; otherwise we get a contradiction to the hypothesis. Thus all the triangles have a common edge uv.
This completes the proof of the theorem. □

THEOREM 3.6. *G be a semi-complete graph which has a cut-vertex,say* v_0. *Then* $\{v_0\}$ *is a mpc-set in G(*$\Rightarrow |mpcs(G)| = 1$ *).*

PROOF. By hypothesis follows that v_0 is adjacent to all the other vertices in G
$\Rightarrow \{v_0\}$ is a pc-set in G
$\Rightarrow$ it is an mpc-set in $G(|mpcs(G)| = 1)$. □

The converse of Theorem.(3.6) is false in view of the following example:

EXAMPLE 3.5. Consider the following graph G in Figure 4
$\{v_0\}$ is a mpc-set in G with $|\{v_0\}| = 1$; but v_0 is not a cut-vertex of G.

Now, we switch on to Edge path connector sets.

4. EDGE PATH CONNECTOR SET

Definitions 4.1. (i) Let $G = (V, E)$ be a graph. Then $E' \subseteq E$ is said to be an edge path connector set(Ed.pc-set) for G iff for every pair of non-adjacent vertices u, v in G there is a shortest $u - v$ path whose edges are from E'.
(ii) A Ed.pc-set with minimum cardinality is said to be a mEd.pc-set for G.

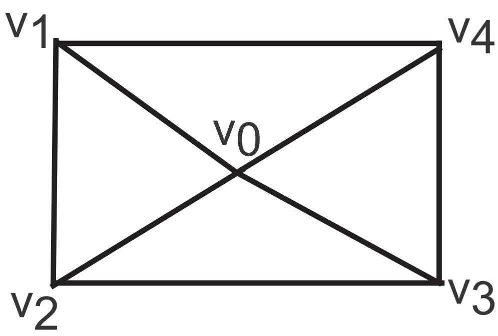

FIGURE 4

Observation 4.1. G admits an Ed.pc-set $\Rightarrow G$ is non empty connected. In that case E is itself an Ed.pc-set. So we are interested in minimum Ed.pc-sets only.

EXAMPLE 4.1. For the graph given in Figure 5

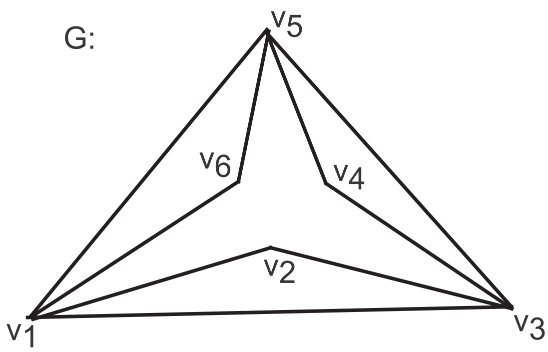

FIGURE 5

$\{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1, v_1v_3, v_3v_5\}$ is an Ed.pc-set for G. We observe that this is a mEd.pc-set.

Result 4.1. Any Ed.pc-set in a (connected) graph is an edge dominating set.
Proof. Let $G = (V, E)$ be a nonempty, connected graph and E' be an Ed.pc-set for G. If $E' = E$ then the result is trivial.

Otherwise, let $e \in E - E'$. Take any $e' \in E'$.If $e\&e'$ are adjacent in G, then e' dominates e. Otherwise, let u be an end of e and u' be an end of e'. Since $u\&u'$

are non-adjacent vertices in G, there is a shortest $u - u'$ path whose edges are from E'.
$\Rightarrow$ there is an edge $f \in E'$ such that $e\&f$ are adjacent(having the common end u) in G. Hence E' is an edge dominating set in G.

The converse of the above result is false in view of the following:

Example 4.2. For the graph given in Example(3.2),$E' = \{v_4v_6, v_2v_6, v_2v_4\}$ is an edge dominating set in G; it is not an Ed.pc-set for G, since there is no shortest $v_3 - v_5$ path whose edges are from E'.

Theorem 4.1. *(Characterization Result) G is a purely semi-complete graph with edge set E. Then $E^{'} \subseteq E$ is an Ed.pc set in G iff for every pair of distinct non-adjacent vertices u, v in G, there are adjacent edges e, f in $E^{'}$ such that e incident with u and f incident with v.*

Proof. Under the given hypothesis, let $E^{'}$ be an Ed.pcs(G). Let u, v be two non-adjacent vertices in G. Now follows that any shortest path between u and v is of length 2. So by the definition of $E^{'}$ follows the necessary part.

Conversely if $E^{'}$ has the property stated, clearly $E^{'}$ is an Ed.pc set in G. □

Result 4.2. G is a purely semi-complete graph and S is a pc-set. Then the set of all edges which are incident with the vertices of S is an Ed.pc-set for G.
Proof: Under the given hypothesis, let $E' = \{e \in E : e \textit{ is incident with an element of } S\}$.
Let v_1, v_2 be two non-adjacent vertices in G. Since G is semi-complete and S is a pc-set for G follows that there is $v_3 \in S$ such that $\{v_1, v_3, v_2\}$ is a path in G.
$\Rightarrow v_1v_3, v_3v_2 \in E'$
$\Rightarrow$ A shortest path from v_1 to v_2 has edges from E'
$\Rightarrow E'$ is an Ed.pc-set in G.

Observation 4.2. The converse of the above result is false in view of the following:

Example 4.3. Consider the graph G in Figure 6:
$S = \{v_5\}$ is a pc set for G and $E^{'} = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1\}$ is an Ed.pc-set for G. But, except the edges v_4v_5 and v_5v_1 no other edges of $E^{'}$ is incident with v_5.

Result 4.3. G is a purely semi-complete graph. If $|mpcs(G)| \neq 1$ then any Ed.pc-set for G is an edge cover for G.
Proof: Let E' be an Ed.pc-set for G. Since $|mpcs(G)| \neq 1$ follows that $\gamma(G) \neq 1$. Hence for every $u \in V(G)$ there is a $v \in V$ such that u is not adjacent to v in G. Since G is semi-complete any shortest $u - v$ path in G has length 2. Since E' is an Ed.pc-set for G there is a $w \in V$ such that $uw, vw \in E'$
$\Rightarrow$ every vertex of G lies on a an edge of E'.
$\Rightarrow E'$ is an edge cover for G.

The converse is false in view of the following:

Example 4.4. Consider the graph G in Figure 7:

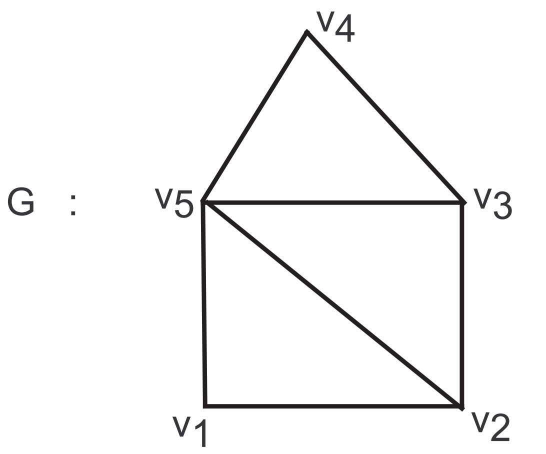

FIGURE 6

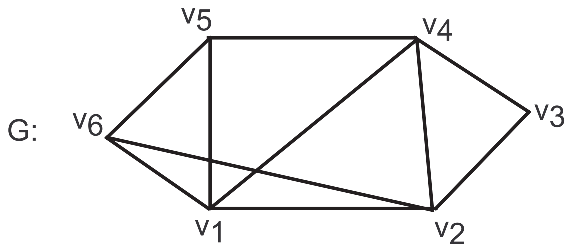

FIGURE 7

$\{v_1v_2, v_3v_4, v_5v_6\}$ is an edge cover for G; but it is not an Ed.pc-set for G, since there is no shortest $v_1 - v_3$ path. Further $|mpcs(G)| \neq 1$.

Result 4.4. $G = (V, E)$ is a purely semi-complete graph having a unique path of length 2 between any pair of non-adjacent vertices. Then G has
(i) Unique mpc set with single element,say v_0.
(ii)Unique mEd.pc set $E^{'}$ given by $\{v_0v : v \in V - \{v_0\}\}$.
Proof: By hypothesis, in virtue of Theorems (2.2) and (2.3), it follows that E is a union of edge disjoint triangles having a common vertex say v_0. Now follows that v_0 is the only vertex which is adjacent with all other vertices of G. Hence follows that $\{v_0\}$ is the only mpc set in G. This proves (i).

Consider $E^{'} = \{v_0v : v \in V - \{v_0\}\}$.
Let v_1 and v_2 be any two non-adjacent vertices in G, then clearly $v_1 \neq v_0 \neq v_2$ and

$\{v_1, v_0, v_2\}$ is a(the) shortest $v_1 - v_2$ path, where $v_0v_1, v_0v_2 \in E^{'}$. Hence $E^{'}$ is an Ed.pc set of G.

If $'n'$ is the number of edge disjoint triangles in G, then follows that $|E^{'}| = 2n$.

If $E^{''} \subseteq E$ with $|E^{''}| < 2n$ then there is atleast one $v \in V - \{v_0\}$ such that $v_0v \notin E^{''}$. Since G has atleast four vertices there is a vertex $v^{'}$ which is non-adjacent with v. Now there is no path of length 2 between v and $v^{'}$ with edges from $E^{''} \Rightarrow E^{''}$ is not an Ed.pc set for G. Hence $E^{'}$ is a mEd.pc set for G. Clearly $E^{'}$ is unique. This proves (ii).

Thus the proof of the theorem is complete.

Note 4.1. If the edge set of G is a union of $'n'$ edge disjoint triangles then, we observe that $|mEd.pcs(G)| = 2n$. The converse of this is false in view of:

EXAMPLE 4.5. Consider the following graph in Figure 8:

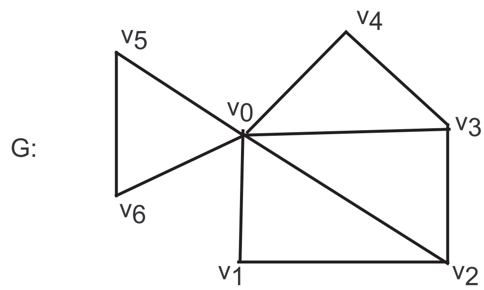

FIGURE 8

$|mEd.pcs(G)| = |\{v_0v_1, v_0v_2, v_0v_3, v_0v_4, v_0v_5, v_0v_6\}| = 6 = 2(3) = 2$(Number of edge disjoint triangles).

But the edge set of G is not a union of '3' edge disjoint triangles.

Result 4.5. G is a purely semi-complete graph which is a union of $'n'$ triangles having a common edge. Then $|mEd.pcs(G)| = n$.

Proof: Under the given hypothesis follows that there are $(n + 2)$ vertices in G. Let uv be the common edge of all the $'n'$ triangles. Now follows that $\{u\}$ and $\{v\}$ are mpc-sets for G.Now by Result(4.7),$\{uw : w \in V(G)\}$ is an Ed.pc-set for G.Since $u\&v$ are adjacent with all the remaining $(n + 1)$ vertices it follows that $E' = \{uw : w \in V(G)\} - \{uv\}$ is an Ed.pc-set for G with $|E'| = (deg_G(u)) - 1 = n$ Similarly $E'' = \{vw : w \in V(G)\} - \{uv\}$ is an Ed.pc-set for G with $E'' = n$.

Let $E_0 \subseteq E$ with $|E_0| < n$

$\Rightarrow$ there is a $w \in V(G) - \{u, v\}$ such that uw and vw are not in E_0. Let w' be any non-adjacent vertex with w. Now there is no shortest $w - w'$ path with edges from E_0

$\Rightarrow |mEPCS(G)| = |E'| = |E''| = n.$

Result 4.6. G is a purely semi-complete graph with edge set E and S is a pc-set for G. Let $F = \{e \in E : e\ is\ incident\ with\ S\}$; then $H = G(F)$ is connected.
Proof: Under the given hypothesis $G[S] \subseteq H$. Let $v_1, v_2 \in V(H)$.
Now either none of v_1, v_2 are in S or atleast one of v_1, v_2 is in S.
Case:1 $v_1, v_2 \notin S$.
Now there exists $v_3, v_4 \in S$ such that $v_1v_3, v_2v_4 \in F$. Since $v_3, v_4 \in S$ and $G[S]$ is connected, there is a $v_3 - v_4$ path in $G[S]$
$\Rightarrow v_1, v_2$ are connected in F.
Case:2 Only one of $v_1, v_2 \notin S$.
w.l.g we can suppose that $v_2 \notin S \Rightarrow v_1 \in S$. Now there is a $v_3 \in S$ such that $v_2v_3 \in F$. Since there is a $v_1 - v_3$ path in S follows that v_1, v_2 are connected in F.
Thus F is a connected graph.

Result 4.7. G is a purely semi-complete graph with $(n+1)$ vertices and having a unique mpc-set and $|mpcs(G)| = 1$. If $|mEd.pcs(G)| < n$, then G is strong semi-complete . **Proof:** Under the given hypothesis there exists a subgraph G' of G such that the edge set of G' is a union of disjoint triangles having common vertex $\Rightarrow |mEd.pcs(G^{'})| = n$.
Since $|mEd.pcs(G)| < n \Rightarrow \exists\ vertices\ v_1, v_2$ on different triangles that are adjacent in G.
$\Rightarrow \exists$ an edge between two vertices lying on different triangles having a common vertex. Hence by a Theorem.(2.3) G is strong semi-complete.
Now,we consider path- critical edges.

5. ON PATH-CRITICAL EDGES

Definitions 5.1. (i) An edge e in a nonempty,connected graph is said to be a path-critical edge w.r.t a mpc-set S in G iff $|mpcs(G - e)| > |mpcs(G)|$.
(ii) G is said to be path-critical edge free w.r.t. S iff no edge of G is a path-critical edge w.r.t. S.

Example 5.1. (i) In the following graph in Figure 9
$S = \{v_2\}$ is the only mpc-set in G. The edge v_2v_5 is a path-critical edge(w.r.t. S) in G, since $\{v_2, v_5\}, \{v_2, v_4\}$ are mpc-sets in $G - v_2v_5$.
So $|mpcs(G - v_2v_5)| = 2 > 1 = |mpcs(G)|$.
(ii) In the graph given in Remark(3.1), $S = \{v_2, v_4, v_6\}$ is the only mpc-set in G. For any edge e of G,$mpcs(G - e) = \{v_2, v_4, v_6\} = mpcs(G)$.
So G has no path-critical edges (w.r.t S). Thus G is a path-critical edge free w.r.t S.

Theorem 5.1. *(Characterization Result) G is a purely semi-complete graph and S is a mpc-set of G. Then the edge $e = uv$ of G is a path-critical edge in G w.r.t S iff u and v do not a common neighbour from S ($\Rightarrow N(u) \bigcap N(v) \bigcap S = \Phi$).*

Proof. Under the given hypothesis, let $e = uv$ be a path-critical edge in G w.r.t S. Suppose u and v have a common neighbour from S. Let x,y be any two non-adjacent vertices in $G - e$.If $\{x, y\} = \{u, v\}$, then by our supposition there is a w in S such that $\{w, x, y\}$ is a (minimum) path in $(G - e)$.

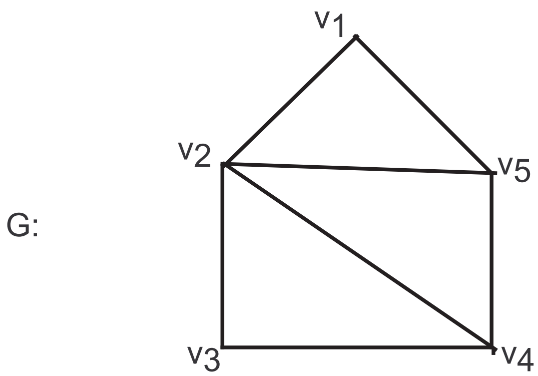

FIGURE 9

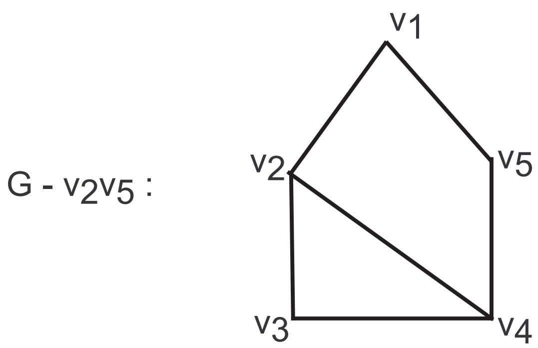

FIGURE 10

If $\{x, y\} \neq \{u, v\}$ then x and y are non-adjacent vertices in G as well . Since G is semi-complete, by the definition of S there is a w_0 in S such that $\{x, w_0, y\}$ is a minimum path in $G - e$. Hence follows that S is also a mpc-set in $G - e$.
$\Rightarrow e$ is not a path-critical edge w.r.t. S in G and hence our supposition is false.

Conversely, assume that $e = uv$ is such that u and v do not have a common neighbour from S. Let

$$V_1 = \{w \in V(G) : \{u, w, v\} is\ a\ path\ in\ G\}.$$

Since G is semi-complete it follows that $V_1 \neq \Phi$.
By hypothesis $S \bigcap V_1 = \Phi \Rightarrow S$ is not a path connector set for $G - e$. Further $S' = S \bigcup \{w_0\}$, where $w_0 \in V_1$ is a pc-set in $G - e$. By the property of S, it follows that S' is a mpc-set for $G - e$.Hence

$$|mpcs(G - e)| = |mpcs(G)| + 1 > |mpcs(G)|$$

$\Rightarrow e$ is a path-critical edge w.r.t S in G.

This completes the proof of the theorem. □

Corollary 5.1. *G is a purely semi-complete graph and S is a mpc-set for G. Then G is path-critical edge free graph w.r.t S iff the ends of each edge of G has atleast one neighbour from S.*

Theorem 5.2. *G is a purely semi-complete graph whose edge set is a union of triangles having a common edge. Then there is exactly one path-critical edge w.r.t any mpc-set in G.*

Proof. Under the given hypothesis, let $e = uv$ be the common edge of the triangles. Now follows that $\{u\}$ and $\{v\}$ are the only mpc-sets in G. Clearly e is a path-critical edge w.r.t these mpc-sets. Further for any other edge $'f'$ of G, u and v are the only mpc-sets in $G - f$ also. So no other edge is path-critical w.r.t $\{u\}$ and $\{v\}$.

This completes the proof of the theorem. □

Theorem 5.3. *G is a purely semi-complete graph with $'n'$ vertices and $|mpcs(G)| = 1$. Then G has exactly $(n-1)$ path-critical edges w.r.t any mpc-set S of G.*

Proof. By hypothesis we can assume that, $S = \{v_0\}(v_0 \in V(\mathrm{G}))$ is a mpc-set for G. Then for any $v_1 \in V - S$, we have $v_1v_0 \in E(\mathrm{G})$.
$\Rightarrow |mpcs(\mathrm{G}{-}v_0v_1)| = 2 > 1 = |mpcs(\mathrm{G})|$
$\Rightarrow v_0v_1$ is a critical edge for G,w.r.t S.
Let $u, v \in E(\mathrm{G}) \ni u \neq v_0 \neq v$. Since G is semi-complete follows that $G - uv$ is connected and so $\{u, v_0, v\}$ is a minimum path in it $\Rightarrow uv$ is not a path-critical edge w.r.t. $S \Rightarrow G$ has exactly $(n-1)$ critical edges. □

Corollary 5.2. *G be a purely semi-complete graph which is path-critical edge free w.r.t a mpc-set S of G. Then $|mpcs(G)| > 1$.*

Proof. Under the given hypothesis, if $|mpcs(G)| = 1$; then by Theorem.(5.3) it follows that G has critical edges w.r.t the mpc-set, say S. This contradicts the hypothesis on S. Hence the result holds. □

Observation 5.1. The converse of the above corollary is false in view of the following example in Figure 11:

$S = \{v_1, v_3\}$ is a mpcs(G), but G is not critical edge free graph w.r.t S.

Finally, we end up by considering the neighbourhood sets.

6. ON NEIGHBOURHOOD SETS

Using the Result(2.1) and Corollary(5.1)we have the following characterization result for a pc-set in a purely semi-complete graph to be an n-set for G.

Theorem 6.1. *S is a pc-set in a purely semi-complete graph G whose vertex set is V. S is an n-set of G iff every edge in (the subgraph) $< V - S >$ is a non-critical edge in G w.r.t S.*

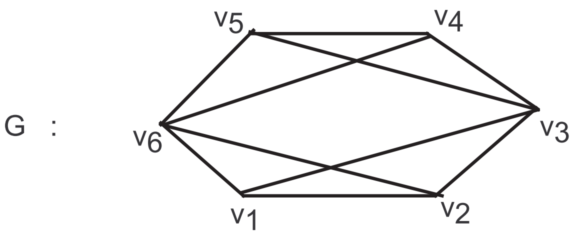

Figure 11

Observation 6.1. *S is an n- set of a purely semi-complete graph G. Then every edge of G need not be a non-critical edge for G w.r.t S.*

Example 6.1. Consider the graph G given in Figure 12

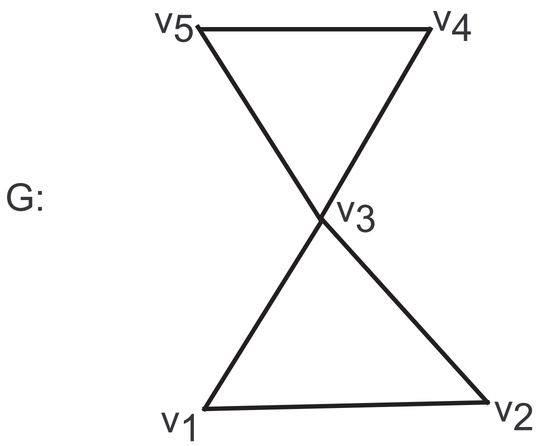

Figure 12

Clearly S = $\{v_3\}$ is an n-set of G. But the edges v_1v_2, v_3v_4 are critical w.r.t S.

Theorem 6.2. *S is an independent path connector set for the purely semi-complete graph G. Then S is an n-set for G.*

Proof. Under the given hypothesis, by observation.(3.2(ii)) it follows that $|S| = 1$. Let S=$\{v_0\}$. So follows that every vertex of G other than v_0 is adjacent with v_0. Hence follows that S is an n-set. □

Remark 6.1. The converse of the above theorem is false in view of the following:

EXAMPLE 6.2. Consider the graph G given in Example(4.1):
$S = \{v_1, v_3, v_5\}$ is an n-set for G. But this is not an independent set(Infact, any two of them are adjacent in G).

THEOREM 6.3. *G is a purely semi-complete G with vertex set V and $S \subseteq V$. If each triangle in G has atleast one vertex from S then S is an n-set of G.*

PROOF. Under the given hypothesis, consider any edge $e = pq$ of G. Since G is semi-complete there is an $r \in V$ such that $\{p, q, r\}$ is a path in G. Now $\{p, q, r, p\}$ is a triangle in G. By hypothesis either p or q or r is in $S \Rightarrow e \in < N[v] >$, where $v \in S$. Since e is arbitrary follows that $G = \bigcup_{v \in S} < N[v] >$.
Thus S is an n-set of G. □

Observation 6.2. The converse of Theorem.(6.2) is false in view of the following:

EXAMPLE 6.3. For the graph in Example(3.2),$S = \{v_1, v_3, v_5\}$ is an n-set for G. But the triangle $\{v_2, v_4, v_6\}$ has no vertex from S.

From Theorem.(6.2),we have the following:

COROLLARY 6.1. *G be a purely semi-complete graph, then $n(G) \leqslant s$, where s is the number of vertex disjoint triangles in G.*

THEOREM 6.4. *G is a purely semi-complete graph in which there is a unique path of length 2, between any pair of non-adjacent vertices in G. Then any pc-set is an n-set for G.*

PROOF. Under the given hypothesis,by Theorem.(2.1) and Theorem.(2.2) G is a union of edge disjoint triangles where all the triangles of G have a common vertex(say v_0). Then any non-trivial pc-set of G, contains v_0. By the property of v_0, in virtue of Theorem.(6.3) follows that any pc-set is an n-set for G. □

Finally we prove the following:

THEOREM 6.5. *G is a purely semi-complete graph with vertex set V and $S \subseteq V$ is an independent n-set of G. If $< S^c >$ is a clique, then S^c is a pc-set for G.*

PROOF. Under the given hypothesis, let u, v be any non-adjacent vertices in G. Since G is semi-complete there is $w \in V$ such that $\{u, w, v\}$ is a path in G.
Since S is an n-set follows that atleast one of u, v is in S.
Without loss of generality we can suppose that $u \in S$. Since w is adjacent with u in G and S is an independent set follows that $w \notin S \Rightarrow w \in S^c$. Since u, v are arbitrary non-adjacent vertices in G follows that S^c is a pc-set(Infact minimum pc-set) in G. □

Remark 6.2. The converse of the above Theorem is false in view of the following:

EXAMPLE 6.4. Consider the following graph G in Figure 13:
Let $S = \{v_2, v_3, v_5, v_6\}$. Now $S^c = \{v_1, v_4\}.< S^c >$ is a clique and S^c is a pc-set for G.
But S is not an independent n-set of G.

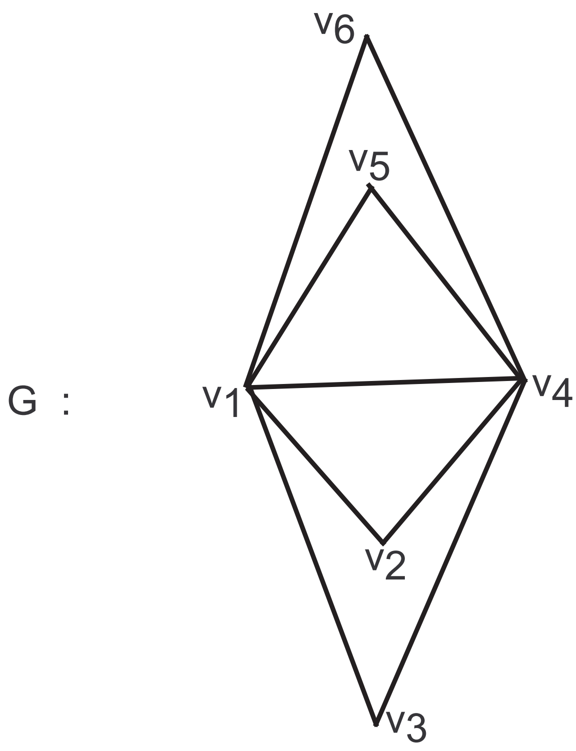

FIGURE 13

7. Conclusion

As semi-complete graphs play a vital role in tackling defence problems, a complete study of these graphs gives an overall view to apply them in our practical problems. Thus a continuous study about these graphs is made.

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