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Integral-type operators on some analytic function spaces

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Abstract. In this paper, we study boundedness and compactness for the products of integral-type operators and composition operators between (α,β)-Bloch spaces of analytic functions in the unit disk Δ.

1. Introduction

Let $\Delta = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, and $H(\Delta)$ be the class of all analytic functions on Δ. An analytic function f on Δ is said to belong to the α-Bloch space $\mathcal{B}_\alpha = \mathcal{B}_\alpha(\Delta)(\alpha > 0)$, if

$$\mathcal{B}_\alpha(f) = \sup_{z\in\Delta}(1-|z|^2)^\alpha |f'(z)| < \infty. \tag{1.1}$$

The expression $\mathcal{B}_\alpha(f)$ defines a seminorm while the natural norm is given by $\|f\|_{\mathcal{B}_\alpha}(f) = |f(0)| + \mathcal{B}_\alpha(f)$. When $\alpha = 1$, $\mathcal{B}_1 = \mathcal{B}$ is the well-known Bloch space (see for example [**7**] and [**10**]). Let $\mathcal{B}_{\alpha,0}$ denote the subspace of $\mathcal{B}_\alpha$ consisting of all $f \in \mathcal{B}_\alpha$ for which

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |f'(z)| = 0.$$

This space is called the little α-Bloch space.
The (α,β)-Bloch space $\mathcal{B}_{\alpha,\beta}(\Delta) = \mathcal{B}_{\alpha,\beta}$ (see [**1**]) is defined by

$$\mathcal{B}_{\alpha,\beta}(f) = \sup_{a,z\in\Delta} \frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^\beta}|f'(z)| < \infty. \tag{1.2}$$

The expression $\mathcal{B}_{\alpha,\beta}(f)$ defines a seminorm while the natural norm is given by $\|f\|_{\alpha,\beta}(f) = |f(0)| + \mathcal{B}_{\alpha,\beta}(f)$. When $\beta = 0$, then we will get the well known α-Bloch space. If $\alpha = 1$ and $\beta = 0$; then we will get the Bloch space. The little

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(α,β)-Bloch space $\mathcal{B}_{\alpha,\beta,0}$ is a subspace of $\mathcal{B}_{\alpha,\beta}$ consisting of all $f\in\mathcal{B}_{\alpha,\beta}$ such that

$$\lim_{|z|\to 1^-}\lim_{|a|\to 1^-}\frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^{\beta}}|f'(z)|=0.$$

Let $\mathcal{A}^1$ denote the Bergman space, that is, the space of all $f\in H(\Delta)$ such that

$$\int_\Delta |f(z)|dm(z)<\infty,$$

where $dm(z)=\frac{1}{\pi}rdrd\theta$ is the normalized area measure on Δ.
Let $L: X\to Y$ be a linear operator, where X and Y are Banach spaces. The operator L is said to be compact if for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X, the sequence $(L(x_n))_{n\in\mathbb{N}}$ has a convergent subsequence. The operator L is said to be weakly compact if for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X, $(L(x_n))_{n\in\mathbb{N}}$ has a weakly convergent subsequence, i.e., there is a subsequence $(x_{n_m})_{m\in\mathbb{N}}$ such that for every $\Lambda\in Y^*$, $\Lambda(L(x_{n_m}))_{m\in\mathbb{N}}$ converges. A useful characterization for an operator to be weakly compact is the following Gantmacher's theorem:
L is weakly compact if and only if $L^{**}(X^{**})\subset Y$, where L^{**} is the second adjoint of L and Y is identified with its image under the natural embedding into its second dual Y^{**} (see [**4**]).
Let φ be an analytic self-map of $\mathbb{D}$. Associated with φ, the composition operator C_φ is defined by $C_\varphi f=f\circ\varphi$ for $f\in H(\mathbb{D})$. It is interesting to provide a function theoretic characterization when φ induces a bounded or compact composition operator on various spaces (see, for example, [**3**]).
Now, suppose that $\mathrm{g}:\Delta\to\mathbb{C}^1$ is a holomorphic map and $f\in H(\Delta)$. The integral-type operator J_{g} is defined by

$$J_{\mathrm{g}}f(z)=\int_0^z f(\xi)\mathrm{g}'(\xi)d\xi,\ \ z\in\Delta.$$

Another integral-type operator I_{g} is defined by

$$I_{\mathrm{g}}f(z)=\int_0^z f'(\xi)\mathrm{g}(\xi)d\xi,\ \ z\in\Delta.$$

The importance of the operators J_{g} and I_{g} comes from the fact that

$$J_{\mathrm{g}}f+I_{\mathrm{g}}f=M_{\mathrm{g}}-f(0)\mathrm{g}(0),$$

where the multiplication operator M_{g} is defined by $(M_{\mathrm{g}}f)(z)=\mathrm{g}(z)f(z)$. In [**8**] Pommerenke introduced the operator J_{g} and showed that J_{g} is a bounded operator on the Hardy space H^2 if and only if $\mathrm{g}\in BMOA$. In this paper, we consider the products of composition operator and integral-type operators, which are defined by (see [**6**])

$$C_\varphi J_{\mathrm{g}}(f)(z)=\int_0^{\varphi(z)} f(\xi)\mathrm{g}'(\xi)d\xi,\ \ C_\varphi I_{\mathrm{g}}(f)(z)=\int_0^{\varphi(z)} f'(\xi)\mathrm{g}(\xi)d\xi, \tag{1.3}$$

and also (see [**6**])

$$J_{\mathrm{g}}C_\varphi(f)(z)=\int_0^z (f\circ\varphi)(\xi)\mathrm{g}'(\xi)d\xi,\ \ I_{\mathrm{g}}C_\varphi(f)(z)=\int_0^z (f\circ\varphi)'(\xi)\mathrm{g}(\xi)d\xi. \tag{1.4}$$

The boundedness and compactness of operators (1.3) and (1.4) between (α,β)-Bloch-type spaces and/or little (α,β)-Bloch-type spaces are studied. The study of these operators naturally comes from the isometry of some function spaces. Namely, it was shown in [**5**] that an operator T is a surjective isometry of the Dirichlet space

$$\mathcal{D}^p = \left\{ f \in H(\Delta) \middle| \|f\|^p_{\mathcal{D}^p} |f(0)|^p + \int_\Delta |f'(z)|^p dm(z) < \infty \right\},$$

where $p \neq 2$, if and only if there is an automorphism ϕ of Δ and constants λ_1 and λ_2 such that

$$(1.5) \qquad (Tf)(z) = \lambda_1 f(0) + \lambda_2 \int_0^z (\phi'(\xi))^{2/p} f'(\phi(\xi)) d\xi$$

for every $f \in \mathcal{D}^p$. Let S^p be the space of all analytic functions f on Δ such that $f' \in H^p$. An operator T is a surjective isometry of S^p with respect to the norm $\|f\|^p_{S^p} = |f(0)|^p + \|f'\|^p_{H^p}$ if and only if there is an automorphism ϕ of Δ and constants λ_1 and λ_2 such that

$$(1.6) \qquad (Tf)(z) = \lambda_1 f(0) + \lambda_2 \int_0^z (\phi'(\xi))^{1/p} f'(\phi(\xi)) d\xi$$

for every $f \in S^p$. Note that the operators in (1.5) and (1.6) are of type in (1.4). Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \approx B$ means that there is a positive constant C such that $C^{-1}B \leqslant A \leqslant CB$.

2. Auxiliary results

In this section, we give some auxiliary results which are incorporated in the following lemmas.

LEMMA 2.1. *Let $f \in \mathcal{B}_{\alpha,\beta}(f)$, then*

$$|f(z)| \leqslant C \begin{cases} \|f\|_{\mathcal{B}_{\alpha,\beta}}\ , & \alpha,\beta \in (0,1), \alpha+\beta \neq 1; \\ \left(\frac{2}{1-|z|^2} + \ln \frac{4}{1-|z|^2}\right)\|f\|_{\mathcal{B}_{\alpha,\beta}}\ , & \alpha = \beta = 1, \\ \frac{\|f\|_{\mathcal{B}_{\alpha,\beta}}}{(1-|z|)^{\alpha+\beta-1}}\ , & \alpha,\beta > 1. \end{cases}$$

for some $C > 0$ independent of f.

PROOF. Suppose $f \in \mathcal{B}_{\alpha,\beta},\ 0 \leqslant t < 1$ and $z \in \Delta$,

$$\begin{aligned}
|f(z)-f(tz)| &= |z\int_t^1 f'(tz)dt| \leqslant \|f\|_{\mathcal{B}_{\alpha,\beta}} \int_t^1 \frac{|z|(1-|\varphi_a(tz)|^2)^\beta}{(1-|tz|^2)^{\alpha+\beta}}dt \\
&= \|f\|_{\mathcal{B}_{\alpha,\beta}} \int_t^1 \frac{|z|(1-|a|^2)^\beta(1-|tz|^2)^\beta}{(1-|tz|^2)^{\alpha+\beta}|1-\overline{a}tz|^{2\beta}}dt \\
&\leqslant \|f\|_{\mathcal{B}_{\alpha,\beta}} \int_t^1 \frac{|z|(1-|a|^2)^\beta}{(1-|tz|^2)^{\alpha}(1-|a|)^\beta(1-|tz|)^\beta}dt \\
&\leqslant \|f\|_{\mathcal{B}_{\alpha,\beta}} \int_t^1 \frac{|z|(2)^{2\beta}}{(1-|tz|^2)^{\alpha+\beta}}dt \\
&\leqslant (2)^{2\beta}\|f\|_{\mathcal{B}_{\alpha,\beta}} \int_{t|z|}^{|z|} \frac{dx}{(1-x^2)^{\alpha+\beta}}.
\end{aligned}$$

Let $I_{\alpha,\beta} = \int_{t|z|}^{|z|} \frac{dx}{(1-x^2)^{\alpha+\beta}}$, and $t|z| = 0$. If $\alpha, \beta \in (0,1)$, and $\alpha+\beta \neq 1$ then

$$I_{\alpha,\beta} \leqslant \int_0^{|z|} \frac{dx}{(1-x)^{\alpha+\beta}} = \frac{1}{1-(\alpha+\beta)}[1-(1-|z|)^{1-(\alpha+\beta)}] \leqslant \frac{1}{1-(\alpha+\beta)}$$

If $\alpha = \beta = 1$, then

$$\begin{aligned}
I_{1,1} = \int_0^{|z|} \frac{dx}{(1-x^2)^2} &= \int_0^{|z|} \frac{1}{4}\left(\frac{1}{(1-x)} + \frac{1}{(1-x)^2} + \frac{1}{(1+x)} + \frac{1}{(1+x)^2}\right)dx \\
&= C\left(\ln\frac{1+|z|}{1-|z|} + \frac{2|z|}{1-|z|^2}\right) \\
&\leqslant C(\frac{2}{1-|z|^2} + \ln\frac{4}{1-|z|^2})
\end{aligned}$$

Finally, if $\alpha, \beta > 1$, then

$$\begin{aligned}
I_{\alpha,\beta} &\leqslant \int_0^{|z|} \frac{dx}{(1-x)^{\alpha+\beta}} = \frac{1}{\alpha+\beta-1}\left(\frac{1}{(1-|z|)^{\alpha+\beta-1}} - 1\right) \\
&\leqslant \frac{C}{(\alpha+\beta-1)(1-|z|)^{\alpha+\beta-1}}.
\end{aligned}$$

From all of the above, we have

$$|f(z)| \leqslant C \begin{cases} \|f\|_{\mathcal{B}_{\alpha,\beta}}\ , & \alpha, \beta \in (0,1), \alpha+\beta \neq 1; \\ (\frac{2}{1-|z|^2} + \ln\frac{4}{1-|z|^2})\|f\|_{\mathcal{B}_{\alpha,\beta}}\ , & \alpha = \beta = 1, \\ \frac{\|f\|_{\mathcal{B}_{\alpha,\beta}}}{(1-|z|)^{\alpha+\beta-1}}\ , & \alpha, \beta > 1. \end{cases}$$

□

LEMMA 2.2. *Suppose that $\alpha, \beta \in (0,\infty)$. Then, the following statements are true.*
(a) $(\mathcal{B}_{\alpha,\beta,0})^* = \mathcal{A}^1$.

(*b*) $(\mathcal{A}^1)^* = \mathcal{B}_{\alpha,\beta}$.
(*c*) *The second dual of* $\mathcal{B}_{\alpha,\beta,0}$ *is* $\mathcal{B}_{\alpha,\beta}$.

Proof. The proof is much akin to the corresponding result in [**2**], so it will be omitted. □

Lemma 2.3. *Suppose that* $\alpha, \beta \in (0,\infty) \setminus \{1\}$. *Then there are two holomorphic maps* $f_1,, f_2 \in B_{\alpha,\beta}$ *with*

$$\sup_{z,a\in\Delta} \frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^{\beta}}(|f_1'(z)|+|f_2'(z)|) < \infty \tag{2.1}$$

and

$$\inf_{z,a\in\Delta} \frac{(1-|z|^2)^{\beta+\alpha-1}}{(1-|\varphi_a(z)|^2)^{\beta}}(|f_1'(z)|+|f_2'(z)|) > 0. \tag{2.2}$$

Proof. The proof is very similar to the corresponding result in [**9**] with simple modifications, so it will be omitted. □

Lemma 2.4. *Let* $f(z) = \sum_{n=0}^{\infty} b_n z^n$ *be holomorphic in* Δ. *If* $f \in \mathcal{B}_{\alpha,\beta}$ ($f \in \mathcal{B}_{\alpha,\beta,0}$, *respectively*) *for* $\alpha, \beta > 0$, *then*

$$\limsup_{n\to\infty} |b_n| n^{1-\alpha-\beta} < \infty \ \ (\lim_{n\to\infty} |b_n| n^{1-\alpha-\beta} = 0, \ resp).$$

Proof. For the proof of Lemma we first note that $(1-n^{-1})^{1-n} \to e$ as $n \to \infty$. Assume that $f \in \mathcal{B}_{\alpha,\beta}$. By the Cauchy formula one obtains for $n \geqslant 1$,

$$\begin{aligned}
|b_n| &= \left|(2\pi i n)^{-1}\int_0^{2\pi} f'(re^{i\theta}) r^{1-n} e^{i(1-n)\theta} d\theta\right| \\
&\leqslant (2\pi n)^{-1}\int_0^{2\pi}\left|f'(re^{i\theta})r^{1-n}\right|\frac{(1-r^2)^{\alpha+\beta}(1-|\varphi_a(re^{i\theta})|^2)^{\beta}}{(1-r^2)^{\alpha+\beta}(1-|\varphi_a(re^{i\theta})|^2)^{\beta}}d\theta \\
&= (2\pi n)^{-1}\|f\|_{\mathcal{B}_{\alpha,\beta}} r^{1-n}\int_0^{2\pi}\frac{(1-|\varphi_a(re^{i\theta})|^2)^{\beta}}{(1-r^2)^{\alpha+\beta}}d\theta \\
&\leqslant (2\pi n)^{-1}\|f\|_{\mathcal{B}_{\alpha,\beta}} r^{1-n}\int_0^{2\pi}\frac{(1-|a|^2)^{\beta}(1-r^2)^{\beta}}{(1-r^2)^{\alpha+\beta}(1-|a|)^{\beta}(1-r)^{\beta}}d\theta \\
&\leqslant Cn^{-1}(1-r)^{-\alpha-\beta}r^{1-n}
\end{aligned}$$

for all $0 < r < 1$; hereafter C denote positive constants. For $n > 1$ and for $r = 1 - n^{-1}$ we thus obtain

$$|b_n| \leqslant C_1 n^{\alpha+\beta-1}(1-n^{-1})^{1-n},$$

whence

$$\limsup_{n\to\infty} |b_n| n^{1-\alpha-\beta} < \infty.$$

The proof for the case $f \in \mathcal{B}_{\alpha,\beta,0}$ is similar to the above with a few modifications. □

LEMMA 2.5. *Let f be holomorphic function in Δ with the gap series expansion*

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}, \quad z \in \Delta$$

where for a constant $q > 1$ the natural numbers n_k, $k \geqslant 1$, satisfy $n_{k+1}/n_k \geqslant q$. Then for $\alpha, \beta > 0, \alpha + \beta \geqslant 1$, $f \in \mathcal{B}_{\alpha,\beta}$ if and only if

$$\limsup_{k\to\infty} |a_k| n_k^{1-\alpha-\beta} < \infty. \tag{2.3}$$

PROOF. First of all we notice by

$$\frac{(1-|\varphi_a(z)|^2)^{\beta}}{(1-|z|)^{2+\alpha+\beta}} \quad = \quad C(1-|\varphi_a(z)|^2)^{\beta} \sum_{n=0}^{\infty} A_n |z|^n \sum_{n=0}^{\infty} B_n |z|^n,$$

where $A_n \backsim \Gamma(1+\alpha)^{-1} n^{\alpha}$, $B_n \backsim \Gamma(1+\beta)^{-1} n^{\beta}$, that

$$\sum_{n=0}^{\infty} (n+1)^{\alpha} |z|^n \sum_{n=0}^{\infty} (n+1)^{\beta} |z|^n \leqslant \frac{C(1-|\varphi_a(z)|^2)^{\beta}}{(1-|z|)^{1+\alpha}(1-|z|)^{1+\beta}}, \quad z \in \Delta. \tag{2.4}$$

It then follows from (2.3) that

$$|zf'(z)| = \left| \sum_{k=1}^{\infty} a_k n_k z^{n_k} \right| \leqslant C \sum_{k=1}^{\infty} n_k^{\alpha+\beta} |z|^{n_k}, \tag{2.5}$$

whence,on making use of the Cauchy product, one obtains

$$\frac{|zf'(z)|}{(1-|z|)^2} \leqslant C \sum_{n=1}^{\infty} \left(\sum_{n_k \leqslant n} n_k^{\alpha+\beta} \right) |z|^n \leqslant C \sum_{n=1}^{\infty} \left(\sum_{n_k \leqslant n} n_k^{\alpha+\beta} \right) |z|^{2n}.$$

Let $k = \max\{k : n_k \leqslant n\}$. Then,

$$n^{-\alpha-\beta} \sum_{n_k \leqslant n} n_k^{\alpha+\beta} = \left(\frac{n_k}{n} \right)^{\alpha+\beta} \left[1 + \left(\frac{n_{k-1}}{n_k} \right)^{\alpha+\beta} + \ldots + \left(\frac{n_1}{n_k} \right)^{\alpha+\beta} \right] \tag{2.6}$$

$$\leqslant 1 + q^{-\alpha-\beta} + q^{-2(\alpha+\beta)} + \ldots = \frac{q^{\alpha+\beta}}{q^{\alpha+\beta}-1} = C.$$

Therefore,

$$\begin{aligned} \frac{|zf'(z)|}{(1-|z|)^2} \quad &\leqslant \quad C \sum_{n=1}^{\infty} n^{\alpha+\beta} |z|^{2n} = C|z| \sum_{n=0}^{\infty} (n+1)^{\alpha+\beta} |z|^{2n} \\ &\leqslant \quad \frac{C|z|(1-|\varphi_a(z)|^2)^{\beta}}{(1-|z|)^{\alpha+1}(1-|z|)^{\beta+1}} \quad for \ \ z \in \Delta, \end{aligned}$$

by (2.4), whence $f \in \mathcal{B}_{\alpha,\beta}$. □

LEMMA 2.6. *Let $\alpha, \beta \in (0, \infty), \alpha+\beta \geqslant 1$ and let $f \in H$ with $f(z) = \sum_{j=1}^{\infty} a_j z^{n_j}$, where for some constant $\lambda > 1$, the natural numbers n_j satisfy $n_{j+1}/n_j \geqslant \lambda$, $j \geqslant 1$. Then $\frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^{\beta}} |f'(z)| \lesssim 1$ for all $z, a \in \Delta$ if and only if $|a_j| n_j^{1-\alpha-\beta} \lesssim 1$ for all $j = 1, 2, \ldots$*

PROOF. The proof follows from lemma 2.4 and lemma 2.5. □

LEMMA 2.7. *Suppose that* $\alpha, \beta \in (0, \infty)$. *Then there exist two holomorphic maps* $f_1,, f_2 \in B_{\alpha,\beta}$ *such that*

$$\frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^\beta}(|f_1'(z)|+|f_2'(z)|) \approx 1, \tag{2.7}$$

for all $z, a \in \Delta$.

PROOF. Suppose $f_1, f_2 : \Delta \to \mathbb{C}$ such that

$$\frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^\beta}(|f_1'(z)|+|f_2'(z)|) \approx 1, \;\; for\ all\ z \in \Delta. \tag{2.8}$$

For a large natural number N (which is determined later on) choose a gap series:

$$f_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} N^{j(\alpha+\beta-1)} z^{N^j}, \;\; for\ \ all\ z \in \Delta.$$

Then, apply Lemma 2.6 with $a_j = N^{\alpha+\beta-1}$ and $n_j = N^j$ to infer that

$$\frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^\beta}|f'_{\alpha,\beta}(z)| \lesssim 1$$

holds for all $z \in \Delta$. Furthermore, let us verify the inequality:

$$\frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^\beta}|f'_{\alpha,\beta}(z)| \gtrsim 1, \;\; 1-N^{-k} \leqslant |z| \leqslant 1-N^{-(k+1/2)}, \;\; k=1,2,\ldots \tag{2.9}$$

Observe that for any $z \in \Delta$,

$$\begin{aligned} |f'_{\alpha,\beta}| &= \sum_{j=0}^{\infty} N^{j(\alpha+\beta)}|z|^{N^j-1} \geqslant h^{k(\alpha+\beta)}|z|^{N^k} - \sum_{j=0}^{k-1} N^{j(\alpha+\beta)}|z|^{N^j} - \sum_{j=k+1}^{\infty} N^{j(\alpha+\beta)}|z|^{N^j} \\ &= T_1 - T_2 - T_3. \end{aligned}$$

And then, fix a z in (2.9) and put $x = |z|^{N^k}$. Thus

$$[1-N^{-k}]^{N^k} \leqslant x \leqslant [(1-N^{-(k+1/2)})^{N^{k+1/2}}]^{N^{-1/2}}.$$

If k is large enough, then for $k \geqslant 1$ one has:

$$\frac{1}{3} \leqslant x \leqslant (\frac{1}{2})^{N^{-1/2}}, \tag{2.10}$$

and hence $T_1 \geqslant N^{k(\alpha+\beta)}/3$. Since it is easy to establish

$$T_2 \leqslant \sum_{j=0}^{k-1} N^{j(\alpha+\beta)} \leqslant \frac{N^{k(\alpha+\beta)}}{N^{(\alpha+\beta)}-1},$$

it remains to deal with T_3. Noting that

$$|z|^{N^n(N-1)} \leqslant |z|^{N^{k+1}(N-1)}, \;\; n \geqslant k+1,$$

namely , in T_3 the quotient of two successive terms is not greater than the ratio of the first two terms, one finds that the series of T_3 is controlled by the geometric series having the same first two terms. Accordingly (2.10) is applied to produce:

$$\begin{aligned} T_3 &\leqslant N^{(k+1)(\alpha+\beta)}|z|^{N^{k+1}}\sum_{j=0}^{\infty}\left(N^{(\alpha+\beta)}|z|^{(N^{k+2}-N^{k+1})}\right)^j = \frac{N^{(k+1)(\alpha+\beta)}|z|^{N^{k+1}}}{1-N^{\alpha+\beta}|z|^{(N^{k+2}-N^{k+1})}} \\ &= \frac{N^{k(\alpha+\beta)}N^{(\alpha+\beta)}x^N}{1-N^{(\alpha+\beta)}x^{(N^2-N)}} \leqslant \frac{N^{k(\alpha+\beta)}N^{(\alpha+\beta)}2^{-N^{1/2}}}{1-N^{(\alpha+\beta)}2^{-(N^{3/2}-N^{1/2})}}. \end{aligned}$$

The preceding estimates for $T_1,\ T_2$ and T_3 imply that for N large enough and the ranges of k and z specified in (2.9),

$$|f'_{\alpha,\beta}(z)| \geqslant \frac{N^{k(\alpha+\beta)}}{4} = \frac{(N^{(\alpha+\beta)})^{k+1/2}}{4N^{(\alpha+\beta)/2}} \geqslant \frac{(1-|\varphi_a(z)|^2)^{\beta}}{4N^{(\alpha+\beta)/2}(1-|z|^2)^{\alpha+\beta}},$$

reaching (2.9). In a similar manner, if

$$\mathrm{g}_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} N^{(j+1/2)(\alpha+\beta-1)}z^{N^j},\ \ for\ all\ z\in\Delta,$$

then $\frac{(1-|z|^2)^{\alpha+\beta}}{(1-|\varphi_a(z)|^2)^{\beta}}|\mathrm{g}'_{\alpha,\beta}(z)| \lesssim 1$ for all $z\in\Delta$ (owing to Lemma 2.6) and one can prove that if N is a large natural number, for example $N=m^2$ where m is a large natural number, then

$$\frac{(1-|z|^2)^{\alpha+\beta}|\mathrm{g}'_{\alpha,\beta}(z)|}{(1-|\varphi_a(z)|^2)^{\beta}} \gtrsim 1,\ \ 1-N^{-(k+1/2)} \leqslant |z| \leqslant 1-N^{-(k+1)},\ \ k=1,2,... \tag{2.11}$$

Of course, (2.9) and (2.11) yield (2.8) unless $f'_{\alpha,\beta}$ and $\mathrm{g}'_{\alpha,\beta}$ share a zero in $\{z\in\Delta : |z|<1-N^{-1}\}$, in which case one can replace $\mathrm{g}'_{\alpha,\beta}$ by $\mathrm{g}'_{\alpha,\beta}(\zeta z)$ for an appropriate ζ on the boundary of Δ (since $f'_{\alpha,\beta}(0)=1$). This completes the proof. □

Now, we prove the following lemma.

Lemma 2.8. *A closed set K in $\mathcal{B}_{\alpha_1,\beta_1,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\lim_{|a|\to 1}\sup_{f\in K}\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|f'(z)| = 0. \tag{2.12}$$

Proof. Suppose K is compact. If $\varepsilon>0$, then the balls centered at the elements of K with radii $\varepsilon/2$ cover K, so by compactness there exist $f_1,...,f_n\in K$ such that for every $f\in K$ we have $\|f-f_j\|_{\mathcal{B}_{\alpha_1,\beta_1}}<\varepsilon/2$ for some $1\leqslant j\leqslant n$, and consequently

$$\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|f'(z)| \leqslant \frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|f'_j(z)| + \varepsilon/2,$$

for all $z,a\in\Delta$. For each j, there exists an $r_j\in(0,1)$ such that

$$\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|f'_j(z)| \leqslant \varepsilon/2$$

whenever $r_j < |z| < 1$. Setting $r = \max\{r_1, ..., r_n\}$ we have

$$\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|f'(z)| \leqslant \varepsilon$$

whenever $r < |z| < 1$, and $f \in K$. So (2.12) holds.

Now suppose that $K \subset \mathcal{B}_{\alpha_1,\beta_1,0}$ is closed, bounded and satisfies (2.12). Then K is a normal family. If (f_n) is a sequence in K, by passing to a subsequence (which we do not relabel) we may assume that $f_n \to f$ uniformly on compact subsets of Δ. We show that $f_n \to f$ in $\mathcal{B}_{\alpha_1,\beta_1,0}$. Let $\varepsilon > 0$ be given. By (2.12) there exists an $r \in (0,1)$ such that

$$\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|\mathbf{g}'(z)| \leqslant \varepsilon/2$$

for all $r < |z| < 1$. and all $g \in K$. Since $f_n' \to f'$ uniformly on compact subsets of Δ, it follows that $f_n' \to f'$ pointwise on Δ, and thus

$$\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|f'(z)| \leqslant \varepsilon/2,$$

for all $r < |z| < 1$. Hence

$$\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|f_n'(z) - f'(z)| \leqslant \varepsilon,$$

for all $r < |z| < 1$. Since $f_n' \to f'$ uniformly on $r\overline{\Delta}$ (the closure of Δ), there exists an N_1 such that $|f_n'(z) - f'(z)| \leqslant \varepsilon$ for all $|z| \leqslant r$ and $n \geqslant N_1$. It follows that

$$\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|f_n'(z) - f'(z)| \leqslant \varepsilon,$$

for all $z, a \in \Delta$ and all $n \geqslant N_1$. Thus $f_n \to f$ in $\mathcal{B}_{\alpha_1,\beta_1,0}$. Since K is closed, it follows that $f \in K$. This prove that the set K is compact. □

The next lemma characterizes the compactness of the operators in (1.3) and (1.4) in an usable way.

Lemma 2.9. *The operator $C_\varphi J_{\mathrm{g}}$ (respect $C_\varphi I_{\mathrm{g}}; I_{\mathrm{g}} C_\varphi; J_{\mathrm{g}} C_\varphi$) $: \mathcal{B}_{\alpha,\beta} \to \mathcal{B}_{\alpha_1,\beta_1}$ is compact if and only if $C_\varphi J_{\mathrm{g}}$ (respect.$C_\varphi I_{\mathrm{g}}; I_{\mathrm{g}} C_\varphi; J_{\mathrm{g}} C_\varphi$):$\mathcal{B}_{\alpha,\beta} \to \mathcal{B}_{\alpha_1,\beta_1}$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{B}_{\alpha,\beta}$ which converges to zero uniformly on compact subsets of Δ, $C_\varphi J_{\mathrm{g}} f_k \to 0$ (respect.$C_\varphi I_{\mathrm{g}} f_k; I_{\mathrm{g}} C_\varphi f_k; J_{\mathrm{g}} C_\varphi f_k \to 0$) in $\mathcal{B}_{\alpha_1,\beta_1}$ as $k \to \infty$.*

Proof. Assume that the operator $C_\varphi J_{\mathrm{g}} : \mathcal{B}_{\alpha,\beta} \to \mathcal{B}_{\alpha_1,\beta_1}$ is compact and that $(f_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{B}_{\alpha_1,\beta_1}$ such that $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{B}_{\alpha,\beta}} < \infty$ and $f_k \to 0$ uniformly on compact subsets of Δ, as $k \to \infty$. By the compactness of $C_\varphi J_{\mathrm{g}} : \mathcal{B}_{\alpha,\beta} \to \mathcal{B}_{\alpha_1,\beta_1}$ it follows that $C_\varphi J_{\mathrm{g}} : \mathcal{B}_{\alpha,\beta} \to \mathcal{B}_{\alpha_1,\beta_1}$ is bounded and we have that the sequence $(C_\varphi J_{\mathrm{g}}(f_k))_{k\in\mathbb{N}}$ has a subsequence $(C_\varphi J_{\mathrm{g}}(f_{k_m}))_{m\in\mathbb{N}}$ which converges in $\mathcal{B}_{\alpha_1,\beta_1}$, say, to f. In view of Lemma 2.1, it is clear that for any compact $K \subset D$, there is a positive constant C_k such that

$$|C_\varphi J_{\mathrm{g}}(f_{k_m})(z) - f(z)| \leqslant C_k \|C_\varphi J_{\mathrm{g}}(f_{k_m}) - f\|_{\mathcal{B}_{\alpha_1,\beta_1}}, \quad for\, all\, z \in K.$$

This implies that $C_\varphi J_{\mathrm{g}}(f_{k_m})(z) - f(z) \to 0$ uniformly on compact subsets of Δ, as $m \to \infty$. Since $f_{k_m} \to 0$ on compact subsets of Δ, and by the following estimate

$$|C_\varphi J_{\mathrm{g}}(f_{k_m})(z)| = \left| \int_0^{\varphi(z)} f_{k_m}(\xi)\mathrm{g}'(\xi)d\xi \right| \leqslant \max_{|\xi| \leqslant |\varphi(\xi)|} |f_{k_m}(\xi)| \max_{|\xi| \leqslant |\varphi(\xi)|} |\mathrm{g}'(\xi)|$$

it is clear that for each $z \in \Delta$, $\lim_{m\to\infty} C_\varphi J_{\mathrm{g}}(f_{k_m})(z) = 0$. Hence the limit function f is equal to 0. Since it holds for every subsequence of $(f_k)_{k\in\mathbb{N}}$ the implication follows.

Conversely, let $(h_k)_{k\in\mathbb{N}}$ be any sequence in the ball $B_M = B_{\mathcal{B}_{\alpha,\beta}}(0, M)$ of $\mathcal{B}_{\alpha,\beta}$. From the fact $\sup_{k\in\mathbb{N}} \|h_k\|_{\mathcal{B}_{\alpha,\beta}} \leqslant M < \infty$, we have that the sequence $(h_k)_{k\in\mathbb{N}}$. is uniformly bounded on compact subsets of Δ and consequently normal by Montel's theorem. Hence we may extract a subsequence $(h_{k_j})_{j\in\mathbb{N}}$, which converges uniformly on compact subsets of Δ to some $h \in H(\Delta)$, moreover $h \in \mathcal{B}_{\alpha,\beta}$ and $\|h\|_{\mathcal{B}_{\alpha,\beta}} \leqslant M$. Thus, the sequence $(h_{k_j} - h)_{j\in\mathbb{N}}$ is such that $\|h_{k_j} - h\|_{\mathcal{B}_{\alpha,\beta}} \leqslant 2M < \infty$, and converges to 0 on compact subsets of Δ as $j \to \infty$. By the hypothesis we have that $C_\varphi J_{\mathrm{g}}(h_{k_j}) \to C_\varphi J_{\mathrm{g}}(h)$ in $\mathcal{B}_{\alpha,\beta}$. Thus the set $C_\varphi J_{\mathrm{g}}(B_M)$ is relatively compact, finishing the proof of the lemma for this case. The proofs in other cases are similar and are omitted. □

LEMMA 2.10. *Assume that* $\alpha, \beta \in (0, 1)$. *Then the operator* $C_\varphi J_{\mathrm{g}}$ *(respect* $C_\varphi I_{\mathrm{g}}; I_{\mathrm{g}} C_\varphi; J_{\mathrm{g}} C_\varphi$*)* $: \mathcal{B}_{\alpha,\beta} \to \mathcal{B}_{\alpha_1,\beta_1}$ *is compact if and only if*
$C_\varphi J_{\mathrm{g}}$ *(respect* $C_\varphi I_{\mathrm{g}}; I_{\mathrm{g}} C_\varphi; J_{\mathrm{g}} C_\varphi$*)* $: \mathcal{B}_{\alpha,\beta} \to \mathcal{B}_{\alpha_1,\beta_1}$ *is bounded and for any bounded sequence* $(f_k)_{k\in\mathbb{N}}$ *in* $\mathcal{B}_{\alpha,\beta}$ *which converges to zero uniformly on* $\overline{\Delta}$, $C_\varphi I_{\mathrm{g}} f_k \to 0$ *(respect.* $C_\varphi I_{\mathrm{g}} f_k; I_{\mathrm{g}} C_\varphi f_k; J_{\mathrm{g}} C_\varphi f_k \to 0$*) in* $\mathcal{B}_{\alpha_1,\beta_1}$ *as* $k \to \infty$.

PROOF. The proof is similar to the corresponding result in [**7**]. □

LEMMA 2.11. *Assume that* $h \in H(\Delta)$, $f \in \mathcal{B}_{\alpha,\beta}$, *for some* $\alpha, \beta > 0$, *and that* $z_0 \in \Delta$ *is fixed. Then, the following statements are true.*
(*a*) *There is a positive constant* C *independent of* f *such that*

$$\left| \int_0^{z_0} f(\xi)h(\xi)d\xi \right| \leqslant C\|f\|_{\mathcal{B}_{\alpha,\beta}} \max_{|\xi|\leqslant|z_0|} |h(\xi)|.$$

(*b*) *There is a positive constant* C *independent of* f *such that*

$$\left| \int_0^{z_0} f'(\xi)h(\xi)d\xi \right| \leqslant C\|f\|_{\mathcal{B}_{\alpha,\beta}} \max_{|\xi|\leqslant|z_0|} |h(\xi)|.$$

PROOF. (a) We have

$$\left|\int_0^{z_0} f(\xi)h(\xi)d\xi\right| \leqslant \max_{|\xi|\leqslant|z_0|}|f(\xi)| \max_{|\xi|\leqslant|z_0|}|h(\xi)| = \max_{|\xi|\leqslant|z_0|}\left|\int_0^{\xi} f'(u)du + f(0)\right| \max_{|\xi|\leqslant|z_0|}|h(\xi)|$$
$$\leqslant \left(|f(0)| + |z_0| \max_{|\xi|\leqslant|z_0|}|f'(\xi)|\right) \max_{|\xi|\leqslant|z_0|}|h(\xi)|$$
$$= \left(|f(0)| + \frac{|z_0|(1-|\varphi_a(z_0)|^2)^\beta}{(1-|z_0|^2)^{\beta+\alpha}} \max_{|\xi|\leqslant|z_0|}\frac{(1-|z_0|^2)^{\beta+\alpha}}{(1-|\varphi_a(z_0)|^2)^\beta}|f'(\xi)|\right) \max_{|\xi|\leqslant|z_0|}|h(\xi)|$$
$$\leqslant \max\left\{1, \frac{|z_0|(2)^{2\beta}}{(1-|z_0|^2)^{\beta+\alpha}}\right\}\|f\|_{\mathcal{B}_{\alpha,\beta}} \max_{|\xi|\leqslant|z_0|}|h(\xi)|,$$

this completes the proof of part (a).
(b) We have

$$\left|\int_0^{z_0} f'(\xi)h(\xi)d\xi\right| \leqslant |z_0| \max_{|\xi|\leqslant|z_0|}|f'(\xi)| \max_{|\xi|\leqslant|z_0|}|h(\xi)|$$
$$= \frac{|z_0|(1-|\varphi_a(z_0)|^2)^\beta}{(1-|z_0|^2)^{\beta+\alpha}} \max_{|\xi|=|z_0|}\frac{(1-|z_0|^2)^{\beta+\alpha}}{(1-|\varphi_a(z_0)|^2)^\beta}|f'(\xi)| \max_{|\xi|\leqslant|z_0|}|h(\xi)|$$
$$\leqslant \frac{|z_0|(2)^{2\beta}}{(1-|z_0|^2)^{\beta+\alpha}}\|f\|_{\mathcal{B}_{\alpha,\beta}} \max_{|\xi|\leqslant|z_0|}|h(\xi)|,$$

finishing the proof of the lemma. □

LEMMA 2.12. *The following are equivalent.*
(i) $f_n \in \mathcal{B}_{\alpha,\beta,0},\ f \in \mathcal{B}_{\alpha,\beta}$ *and* $\|f_n - f\| \to 0$.
(ii) *The following properties hold:*
(a) $f_n(z) \to f(z)$ *as* $n \to \infty$ *locally uniformly in* Δ.
(a) $\frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^\beta}|f_n'(z)| = 0$ *as* $|z| \to 1$ *uniformly in* Δ.

PROOF. $(i) \Rightarrow (ii)$. Let $f_n \in \mathcal{B}_{\alpha,\beta,0},\ f \in \mathcal{B}_{\alpha,\beta}$ and $\|f_n - f\| \to 0$. Then,

$$\frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^\beta}|f_n'(z)| \leqslant \frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^\beta}|f_m'(z)| + \|f_n - f_m\|$$
$$< \frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^\beta}|f_m'(z)| + \varepsilon,$$

for $m, n > N(\varepsilon)$ constants and $|z| < 1$. For some $\delta < 1$, we have

$$\frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^\beta}|f_n'(z)| < 2\varepsilon \ \text{ for } n > N(\varepsilon) \ \text{ and } \ \delta < |z| < 1,$$

hence (b) holds. The assertion (a) follows from the convergence in the (α,β)-Bloch norm implies locally uniform convergence.
$(ii) \Rightarrow (i)$. $f_n \in \mathcal{B}_{\alpha,\beta,0}$ by (b). Also, $f_n'(z) \to f'(z)$ for each $z \in \Delta$ by (a). Thus $f \in \mathcal{B}_{\alpha,\beta,0}$. Therefore, choose $\delta < 1$ such that

$$\frac{(1-|z|^2)^{\beta+\alpha}}{(1-|\varphi_a(z)|^2)^\beta}|f_n'(z) - f'(z)| < \varepsilon \ \text{ for } \ n = 1, 2, \ldots \ \text{ and } \ \delta < |z| < 1.$$

Then, using (a) to estimate the difference $|f_n'(z) - f'(z)|$, which implies

$$\|f_n - f\| \to 0.$$

□

THEOREM 2.1. *The space $\mathcal{B}_{\alpha,\beta,0}$ is separable closed nowhere dense subspace of $\mathcal{B}_{\alpha,\beta}$ and is identical with the closure of the polynomials in the (α,β)-Bloch norm. Further, $f \in \mathcal{B}_{\alpha,\beta,0}$ if and only if*

$$\|f(z) - f(tz)\| \to 0 \ \text{ as } \ t \to 0, \ \text{ for } \ |t| \leqslant 1.$$

PROOF. From Lemma 2.12, $f_n \in \mathcal{B}_{\alpha,\beta,0}$, $\|f_n - f\| \to 0 \Rightarrow f \in \mathcal{B}_{\alpha,\beta,0}$. Thus, $\mathcal{B}_{\alpha,\beta,0}$ is closed. Since, every polynomials is in $\mathcal{B}_{\alpha,\beta,0}$, so is the closure of the polynomials. Further, if $f \in \mathcal{B}_{\alpha,\beta}$, then $f(tz) \in \mathcal{B}_{\alpha,\beta,0}$ for every $t \in \Delta$. Now, since $\mathcal{B}_{\alpha,\beta,0}$ is closed so, $f \in \mathcal{B}_{\alpha,\beta,0}$ because $\|f(z) - f(tz)\| \to 0$. □

3. The boundedness and compactness of $C_\varphi J_{\mathrm{g}}$

In this section, we characterize the boundedness and compactness of the operator $C_\varphi J_{\mathrm{g}} : \mathcal{B}_{\alpha,\beta}(or\ \mathcal{B}_{\alpha,\beta,0}) \to \mathcal{B}_{\alpha_1,\beta_1}(or\ \mathcal{B}_{\alpha_1,\beta_1,0})$.

THEOREM 3.1. *Let φ be an analytic self-map of the unit disk and $\mathrm{g} \in H(\Delta)$. If $\alpha, \beta \in (0,1)$, with $\alpha + \beta \neq 1$ then the following statements hold.*
(a) $C_\varphi J_{\mathrm{g}} : \mathcal{B}_{\alpha,\beta}(or\ \mathcal{B}_{\alpha,\beta,0}) \to \mathcal{B}_{\alpha_1,\beta_1}$ is bounded if and only if

$$M^* := \sup_{z,a\in\Delta} \frac{(1-|z|^2)^{\beta_1+\alpha_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}} |\mathrm{g}'(\varphi(z))||\varphi'(z)| < \infty. \tag{3.1}$$

(b) $C_\varphi J_{\mathrm{g}} : \mathcal{B}_{\alpha,\beta}(or\ \mathcal{B}_{\alpha,\beta,0}) \to \mathcal{B}_{\alpha_1,\beta_1,0}$ is bounded if and only if

$$\lim_{|z|\to 1} \lim_{|a|\to 1} \frac{(1-|z|^2)^{\beta_1+\alpha_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}} |\mathrm{g}'(\varphi(z))||\varphi'(z)| = 0. \tag{3.2}$$

PROOF. (a) Assume that $C_\varphi J_{\mathrm{g}} : \mathcal{B}_{\alpha,\beta}(or\ \mathcal{B}_{\alpha,\beta,0}) \to \mathcal{B}_{\alpha_1,\beta_1}$ is bounded. From (3) we see that

$$(C_\varphi J_{\mathrm{g}} f)'(z) = f(\varphi(z))\mathrm{g}'(\varphi(z))\varphi'(z). \tag{3.3}$$

Choose $f_0(z) \equiv 1$. It is clear that $f_0 \in \mathcal{B}_{\alpha,\beta,0}$ and that $\|f_0\|_{\mathcal{B}_{\alpha,\beta}} = 1$. The boundedness of the operator $C_\varphi J_{\mathrm{g}} : \mathcal{B}_{\alpha,\beta}(or\ \mathcal{B}_{\alpha,\beta,0}) \to \mathcal{B}_{\alpha_1,\beta_1}$ implies that

$$\frac{(1-|z|^2)^{\beta_1+\alpha_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}} |\mathrm{g}'(\varphi(z))||\varphi'(z)| = \frac{(1-|z|^2)^{\beta_1+\alpha_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}} |(C_\varphi J_{\mathrm{g}} f_0)'(z)|$$

$$\leqslant \ \|C_\varphi J_{\mathrm{g}}\| \|f_0\|_{\mathcal{B}_{\alpha_1,\beta_1}} = \|C_\varphi J_{\mathrm{g}}\| < \infty, \tag{3.4}$$

for any $z, a \in \Delta$. Therefore,we obtain (3.1), as desired.
Now assume that (3.1) holds. Then, by Lemma 2.1 and (3.2) we have

$$\frac{(1-|z|^2)^{\beta_1+\alpha_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}} |(C_\varphi J_{\mathrm{g}} f)'(z)| \leqslant C \frac{(1-|z|^2)^{\beta_1+\alpha_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}} |\mathrm{g}'(\varphi(z))||\varphi'(z)| \|f\|_{\mathcal{B}_{\alpha,\beta}}. \tag{3.5}$$

From Lemma 2.11 (a) with $h = g'$ and $z_0 = \varphi(0)$, we have that

$$\begin{aligned} |(C_\varphi J_g f)(0)| &= \left| \int_0^{\varphi(0)} f(\xi) g'(\xi) \right| \\ &\leqslant C\|f\|_{\mathcal{B}_{\alpha,\beta}} \max_{|\xi|\leqslant|\varphi(0)|} |g'(\xi)| \leqslant C\|f\|_{\mathcal{B}_{\alpha,\beta}} \max_{|\xi|\leqslant|\varphi(0)|} |g'(\xi)|. \end{aligned} \tag{3.6}$$

Since $|\varphi(0)| < 1$, it follows that $\max_{|\xi|\leqslant|\varphi(0)|} |g'(\xi)| < \infty$. From this and by taking the supremum in (3.4) over $z, a \in \Delta$, we obtain

$$\begin{aligned} &\|C_\varphi J_g(f)\|_{\mathcal{B}_{\alpha_1,\beta_1}} \\ &\leqslant C\left(\sup_{z,a\in\Delta} \frac{(1-|z|^2)^{\beta_1+\alpha_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}} |g'(\varphi(z))||\varphi'(z)| + \max_{|\xi|\leqslant|\varphi(0)|} |g'(\xi)| \right) \|f\|_{\mathcal{B}_{\alpha,\beta}}, \end{aligned}$$

which in view of (3.1) and (3.5) implies the boundedness of

$$C_\varphi J_g : \mathcal{B}_{\alpha,\beta}(or\ \mathcal{B}_{\alpha,\beta,0}) \to \mathcal{B}_{\alpha_1,\beta_1}.$$

(b) Assume that $C_\varphi J_g : \mathcal{B}_{\alpha,\beta}(or\ \mathcal{B}_{\alpha,\beta,0}) \to \mathcal{B}_{\alpha_1,\beta_1,0}$ is bounded. Let $f_0(z) \equiv 1$, then $C_\varphi J_g(f_0) \in \mathcal{B}_{\alpha_1,\beta_1,0}$, that is (3.2) holds, as desired.
Now, assume (3.2) holds. Let $f \in \mathcal{B}_{\alpha,\beta}$, then from (3.5) we see that (3.2) implies $C_\varphi J_g(f_0) \in \mathcal{B}_{\alpha_1,\beta_1,0}$, for each $f \in \mathcal{B}_{\alpha,\beta}$. Moreover, (3.2) implies (3.1), so by (a) the operator $C_\varphi J_g : \mathcal{B}_{\alpha,\beta} \to \mathcal{B}_{\alpha_1,\beta_1}$ is bounded. Therefore, $C_\varphi J_g : \mathcal{B}_{\alpha,\beta} \to \mathcal{B}_{\alpha_1,\beta_1,0}$ is bounded too. □

THEOREM 3.2. *Let φ be an analytic self-map of the unit disk and $g \in H(\Delta)$. If $\alpha, \beta \in (0, 1)$ with $\alpha + \beta \neq 1$, then*
(a) $C_\varphi J_g : \mathcal{B}_{\alpha,\beta}(or\ \mathcal{B}_{\alpha,\beta,0}) \to \mathcal{B}_{\alpha_1,\beta_1}$ is compact if and only if (3.1) holds.
Also, the following statements are equivalent:
(b) $C_\varphi J_g : \mathcal{B}_{\alpha,\beta,0} \to \mathcal{B}_{\alpha_1,\beta_1,0}$ is compact;
(c) $C_\varphi J_g : \mathcal{B}_{\alpha,\beta,0} \to \mathcal{B}_{\alpha_1,\beta_1,0}$ is weakly compact;
(d) condition (3.2) holds;
(e) $C_\varphi J_g : \mathcal{B}_{\alpha,\beta} \to \mathcal{B}_{\alpha_1,\beta_1,0}$ is compact.

PROOF. (a) Assume that $C_\varphi J_g : \mathcal{B}_{\alpha,\beta}(or\ \mathcal{B}_{\alpha,\beta,0}) \to \mathcal{B}_{\alpha_1,\beta_1}$ is compact, then it is bounded and by Theorem 3.1 it follows that condition (3.1) holds.
Conversely, suppose that (3.1) holds. By Theorem 3.1, we know that $C_\varphi J_g : \mathcal{B}_{\alpha,\beta}(or\ \mathcal{B}_{\alpha,\beta,0}) \to \mathcal{B}_{\alpha_1,\beta_1}$ is bounded. By Lemma 2.10 , we should prove that $\|C_\varphi J_g f_k\|_{\mathcal{B}_{\alpha_1,\beta_1}} \to 0$ as $k \to \infty$ for each sequence $(f_k)_{k\in\mathbb{N}} \subset \mathcal{B}_{\alpha,\beta}$ (or $\mathcal{B}_{\alpha,\beta,0}$)$\to 0$, such that $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{B}_{\alpha,\beta}} < \infty$ and which converges to zero uniformly on $\overline{\Delta}$. We have

$$\begin{aligned} &\lim_{k\to\infty} \sup_{z,a\in\overline{\Delta}} \frac{(1-|z|^2)^{\beta_1+\alpha_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}} |(C_\varphi J_g f_k)'(z)| \\ &= \lim_{k\to\infty} \sup_{z,a\in\overline{\Delta}} \frac{(1-|z|^2)^{\beta_1+\alpha_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}} |g'(\varphi(z))||\varphi'(z)||f_k(\varphi(z))| \\ &\leqslant \sup_{z,a\in\overline{\Delta}} \frac{(1-|z|^2)^{\beta_1+\alpha_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}} |g'(\varphi(z))||\varphi'(z)| \lim_{k\to\infty} \|f_k\|_\infty = 0. \end{aligned}$$

On the other hand, we have

$$|(C_\varphi J_{\mathrm{g}} f_k)(0)| = \left| \int_0^{\varphi(0)} f_k(\xi)\mathrm{g}'(\xi)d\xi \right| \leqslant \|f_k\|_\infty \max_{|\xi|\leqslant|\varphi(0)|} |\mathrm{g}'(\xi)| \to 0, \tag{3.7}$$

as $k \to \infty$. From last two estimates the compactness follows.

(b) $\Rightarrow$ (c). By the definition every compact operator is weakly compact.

(c) $\Rightarrow$ (d). It is obvious that $C_\varphi J_{\mathrm{g}} : \mathcal{B}_{\alpha,\beta,0} \to \mathcal{B}_{\alpha_1,\beta_1,0}$ is bounded. Since $f_0(z) \equiv 1$ belongs to $\mathcal{B}_{\alpha,\beta,0}$, we have that $C_\varphi J_{\mathrm{g}}(1) \in \mathcal{B}_{\alpha_1,\beta_1,0}$, that is, (3.2) holds.

(d) $\Rightarrow$ (e). Condition (3.2) implies (3.1). Hence the set $C_\varphi J_{\mathrm{g}}(f : \|f\|_{\mathcal{B}_{\alpha,\beta}} \leqslant 1)$ is bounded in $\mathcal{B}_{\alpha_1,\beta_1}$. Moreover, from (3.5) it follows that the set is bounded in $\mathcal{B}_{\alpha_1,\beta_1,0}$. Taking the supremum in inequality (3.5) over the unit ball in $\mathcal{B}_{\alpha,\beta}$, then letting $|z| \to 1$, applying (3.2) and Lemma 2.8, we obtain that the implication is true.

(e) $\Rightarrow$(b). This implication is obvious. □

Now, we consider the case of $\alpha = \beta = 1$.

THEOREM 3.3. *Let φ be an analytic self-map of the unit disk and $\mathrm{g} \in H(\Delta)$. Then the following statements hold.*

(a) $C_\varphi J_{\mathrm{g}} : \mathcal{B}_{1,1}$(or $\mathcal{B}_{1,1,0}$) $\to \mathcal{B}_{\alpha_1,\beta_1}$ is bounded if and only if

$$\sup_{z,a\in\Delta} \frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}} |\mathrm{g}'(\varphi(z))||\varphi'(z)| \left(\frac{2}{1-|\varphi(z)|^2} + \ln \frac{4}{1-|\varphi(z)|^2} \right) < \infty. \tag{3.8}$$

(b) $C_\varphi J_{\mathrm{g}} : \mathcal{B}_{1,1,0} \to \mathcal{B}_{\alpha_1,\beta_1,0}$ is bounded if and only if conditions (3.2) and (3.8) hold.

PROOF. (a) First, assume $C_\varphi J_{\mathrm{g}} : \mathcal{B}_{1,1}$(*or* $\mathcal{B}_{1,1,0}$) $\to \mathcal{B}_{\alpha_1,\beta_1}$ is bounded. For $w \in \Delta$, set

$$f_w(z) = (1-|w|^2)\left(\frac{2}{1-\overline{w}z} + \ln \frac{4}{1-\overline{w}z} \right).$$

It is easy to see that $f_w \in \mathcal{B}_{1,1,0}$ and

$$\begin{aligned}
\|f_w\|_{\mathcal{B}_{1,1}} &= \frac{(1-\overline{w}z)^2}{(1-|\varphi_w(z)|^2)} \left(\frac{|1-|w|^2||\overline{w}|}{|1-\overline{w}z|} + \frac{|1-|w|^2||\overline{w}|}{(1-\overline{w}z)^2} \right) \\
&= \frac{(1-\overline{w}z)^2|1-\overline{w}z|^2}{(1-|w|^2)(1-\overline{w}z)} \left(\frac{|1-|w|^2||\overline{w}|}{|1-\overline{w}z|} + \frac{|1-|w|^2||\overline{w}|}{(1-\overline{w}z)^2} \right) \\
&\leqslant 4+4 \leqslant 8
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{|\mathrm{g}'(\varphi(z))||\varphi'(z)|(1-|\varphi(z)|^2)(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}} \left(\frac{2}{1-|\varphi(z)|^2} + \ln \frac{4}{1-|\varphi(z)|^2} \right) \qquad (3.9) \\
= \;& \frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}} |(C_\varphi J_{\mathrm{g}} f_{\varphi(z)})'(z)| \leqslant \|C_\varphi J_{\mathrm{g}} f_{\varphi(z)}\|_{\mathcal{B}_{\alpha_1,\beta_1}} \\
\leqslant \;& \|C_\varphi J_{\mathrm{g}}\| \|f_{\varphi(z)}\|_{\mathcal{B}_{1,1}} < \infty.
\end{aligned}$$

Taking the supremum in (3.9) over $z, a \in \Delta$, we obtain (3.8). Conversely, assume that (3.8) holds. By Lemma 2.1 and (3.3), we obtain

$$\begin{aligned}&\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|(C_\varphi J_{\rm g} f)'(z)|\\ (3.10)\quad &\leqslant C\|f\|_{\mathcal{B}_{1,1}}\frac{|{\rm g}'(\varphi(z))||\varphi'(z)|(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}\Big(\frac{2}{1-|\varphi(z)|^2}+\ln\frac{4}{1-|\varphi(z)|^2}\Big).\end{aligned}$$

From (3.10) and (3.8) with $\alpha = \beta = 1$, the boundedness of $C_\varphi J_{\rm g} : \mathcal{B}_{1,1}(\textit{or } \mathcal{B}_{1,1,0}) \to \mathcal{B}_{\alpha_1,\beta_1}$ follows.

(b) If $C_\varphi J_{\rm g} : \mathcal{B}_{1,1,0} \to \mathcal{B}_{\alpha_1,\beta_1,0}$ is bounded, then by (a) we see that (3.8) holds. By taking the function given by $f(z) \equiv 1$, we obtain (3.2). Now, suppose that (3.2) and (3.8) hold. Then for each polynomial p the following inequality holds

$$\begin{aligned}\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|(C_\varphi J_{\rm g} p)'(z)| &= \frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|{\rm g}'(\varphi(z))||\varphi'(z)||p(\varphi(z))|\\ &\leqslant \|p\|_\infty\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|{\rm g}'(\varphi(z))||\varphi'(z)|.\end{aligned}$$

From this and (3.2), we obtain that for each polynomial p, $C_\varphi J_{\rm g}(p) \in \mathcal{B}_{\alpha_1,\beta_1,0}$, the set of all polynomials is dense in $\mathcal{B}_{1,1,0}$, thus for every $f \in \mathcal{B}_{1,1,0}$ there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that

$$\|p_k - f\|_{\mathcal{B}_{1,1}} \to 0 \quad \text{as} \quad k \to \infty.$$

Hence,

$$\|C_\varphi J_{\rm g} p_k - C_\varphi J_{\rm g} f\|_{\mathcal{B}_{\alpha_1,\beta_1}} \leqslant \|C_\varphi J_{\rm g}\|\|p_k - f\|_{\mathcal{B}_{1,1}} \to 0 \ \text{ as } \ k \to \infty,$$

since, as we have already proved, the operator $C_\varphi J_{\rm g} : \mathcal{B}_{1,1,0} \to \mathcal{B}_{\alpha_1,\beta_1}$ is bounded. Hence $C_\varphi J_{\rm g}(\mathcal{B}_{1,1,0}) \subset \mathcal{B}_{\alpha_1,\beta_1,0}$. Since $\mathcal{B}_{\alpha_1,\beta_1,0}$ is closed subset of $\mathcal{B}_{\alpha_1,\beta_1}$, then $C_\varphi J_{\rm g}(\mathcal{B}_{1,1,0}) \to \mathcal{B}_{\alpha_1,\beta_1,0}$ is bounded. □

THEOREM 3.4. *Assume that φ is an analytic self-map of the unit disk and ${\rm g} \in H(\Delta)$. Then the following statements are equivalent:*
(a) $C_\varphi J_{\rm g} : \mathcal{B}_{1,1} \to \mathcal{B}_{\alpha_1,\beta_1}$ is compact and condition (3.2) holds;
(b) $C_\varphi J_{\rm g} : \mathcal{B}_{1,1,0} \to \mathcal{B}_{\alpha_1,\beta_1,0}$ is compact;
(c) $C_\varphi J_{\rm g} : \mathcal{B}_{1,1,0} \to \mathcal{B}_{\alpha_1,\beta_1,0}$ is weakly compact;
(d) Condition (3.2) holds and

$$(3.11)\quad \lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|{\rm g}'(\varphi(z))||\varphi'(z)|\Big(\frac{2}{1-|\varphi(z)|^2}+\ln\frac{4}{1-|\varphi(z)|^2}\Big) = 0,$$

(e) $C_\varphi J_{\rm g} : \mathcal{B}_{1,1} \to \mathcal{B}_{\alpha_1,\beta_1,0}$ is compact;
(f) $C_\varphi J_{\rm g} : \mathcal{B}_{1,1} \to \mathcal{B}_{\alpha_1,\beta_1,0}$ is bounded;

PROOF. (d)⇒ (a). Clearly (3.2) implies (3.1). From (3.11) we see that there is an $r_0 \in (0, 1)$ such that

$$\frac{(1-|z|^2)^{\beta_1+\alpha_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|{\rm g}'(\varphi(z))||\varphi'(z)|\left(\frac{2}{1-|\varphi(z)|^2}+\ln\frac{4}{1-|\varphi(z)|^2}\right) < \varepsilon$$

for every $|\varphi(z)| > r_0$. Let $(f_k)_{k\in\mathbb{N}}$ be a norm bounded sequence in $\mathcal{B}_{1,1}$ such that $f_k \to 0$ on compact subset of Δ as $k \to \infty$. By Lemma 2.1, we obtain

$$\begin{aligned}(3.12)\quad &\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|(C_\varphi J_{\mathrm{g}} f_k)'(z)|\\ &= |f_k(\varphi(z))|\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|\mathrm{g}'(\varphi(z))||\varphi'(z)|\\ &\leqslant \sup_{|\varphi(z)|\leqslant r_0}|f_k(\varphi(z))|\sup_{|\varphi(z)|\leqslant r_0}\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|\mathrm{g}'(\varphi(z))||\varphi'(z)|\\ &+C\|f_k\|_{\mathcal{B}_{1,1}}\sup_{|\varphi(z)|>r_0}|\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|\mathrm{g}'(\varphi(z))||\varphi'(z)|(\frac{2}{1-|\varphi(z)|^2}+\ln\frac{4}{1-|\varphi(z)|^2})\\ &\leqslant M\sup_{|\xi|\leqslant r_0}|f_k(\xi)| + \varepsilon\, C\|f_k\|_{\mathcal{B}_{1,1}}.\end{aligned}$$

We also have that

$$(3.13)\quad |(C_\varphi J_{\mathrm{g}} f_k)(0)| = |\int_0^{\varphi(0)} f_k(\xi)\mathrm{g}'(\xi)d\xi| \leqslant \max_{|\xi|\leqslant|\varphi(0)|}|f_k(\xi)|\max_{|\xi|\leqslant|\varphi(0)|}|\mathrm{g}'(\xi)| \to 0$$

as $k \to \infty$. Taking the supremum over $z, a \in \Delta$ and letting $k \to \infty$ in (3.12) and(3.13), we obtain that $\|C_\varphi J_{\mathrm{g}} f_k\|_{\mathcal{B}_{\alpha_1,\beta_1}} \to 0$ as $k \to \infty$. Hence, the operator $C_\varphi J_{\mathrm{g}} : \mathcal{B}_{1,1} \to \mathcal{B}_{\alpha_1,\beta_1}$ is compact.

(a) $\Rightarrow$ (b). Assume that $C_\varphi J_{\mathrm{g}} : \mathcal{B}_{1,1} \to \mathcal{B}_{\alpha_1,\beta_1}$ is compact and (3.2) holds. As in Theorem 3.3, for each polynomial p we have that $C_\varphi J_{\mathrm{g}}(p) \in \mathcal{B}_{\alpha_1,\beta_1,0}$. Because the polynomials are dense in $\mathcal{B}_{1,1,0}$ and $\mathcal{B}^{**}_{1,1,0} = \mathcal{B}_{1,1}$, it follows that the polynomials are w^*-dense in $\mathcal{B}_{1,1}$. Thus, for each $f \in \mathcal{B}_{1,1}$ there is a sequence of polynomials $(p_m)_{m\in\mathbb{N}}$, such that $\sup_{m\in\mathbb{N}}\|p_m\|_{\mathcal{B}_{1,1}} < \infty$ and $p_m \to f$ uniformly on compact subsets of Δ as $m \to \infty$. By the compactness, we have that there is a subsequence $(p_{m_k})_{k\in\mathbb{N}}$ such that

$$(3.14)\qquad \lim_{k\to\infty}\|C_\varphi J_{\mathrm{g}}(p_{m_k}) - C_\varphi J_{\mathrm{g}}(f)\|_{\mathcal{B}_{\alpha_1,\beta_1}} = 0,$$

which implies that $C_\varphi J_{\mathrm{g}}(\mathcal{B}_{1,1}) \subset \mathcal{B}_{\alpha_1,\beta_1,0}$. Hence, the image of the unit ball of $\mathcal{B}_{1,1}$ under the operator $C_\varphi J_{\mathrm{g}}$ is relatively compact in $\mathcal{B}_{\alpha_1,\beta_1,0}$, which implies that $C_\varphi J_{\mathrm{g}} : \mathcal{B}_{1,1,0} \to \mathcal{B}_{\alpha_1,\beta_1,0}$ is compact.

(b) $\Rightarrow$ (c).This implication is clear.

(c) $\Rightarrow$ (d). By putting $f(z) \equiv 1$, (3.2) follows. By Lemma 2.2 we know that $(\mathcal{B}_{\alpha,\beta,0})^{**} = \mathcal{B}_{\alpha,\beta}$. Since $C_\varphi J_{\mathrm{g}} : \mathcal{B}_{\alpha,\beta,0} \to \mathcal{B}_{\alpha_1,\beta_1,0}$ and $(\mathcal{B}_{\alpha_1,\beta_1,0})^* = (\mathcal{B}_{\alpha,\beta,0})^* = \mathcal{A}^1$, we have that $(C_\varphi J_{\mathrm{g}})^* : \mathcal{A}^1 \to \mathcal{A}^1$. Hence every bounded linear functional $\mathcal{L}$ on $\mathcal{B}_{\alpha_1,\beta_1,0}$ can be identified by a function $h \in \mathcal{A}^1$, so that for every $f \in \mathcal{B}_{1,1,0}$ and $h \in \mathcal{A}^1$, we have

$$\langle C_\varphi J_{\mathrm{g}}(f), h\rangle = \langle f, (C_\varphi J_{\mathrm{g}})^*(h)\rangle.$$

On the other hand, by Lemma 2.2 we have $(\mathcal{A}^1)^* = \mathcal{B}_{\beta,\alpha}$, which implies that $(C_\varphi J_{\mathrm{g}})^{**} : \mathcal{B}_{\alpha,\beta} \to \mathcal{B}_{\alpha_1,\beta_1}$. Hence every $f \in \mathcal{B}_{\alpha,\beta,0}$ can be viewed as an element of the space $(\mathcal{A}^1)^*$ and

$$\langle f, (C_\varphi J_{\mathrm{g}})^*(h)\rangle = \langle (C_\varphi J_{\mathrm{g}})^{**}(f), h\rangle.$$

From these two equalities, we have that

$$\langle C_\varphi J_{\mathrm{g}}(f), h\rangle = \langle (C_\varphi J_{\mathrm{g}})^{**}(f), h\rangle.$$

for every $h \in \mathcal{A}^1$. By a known consequence of Hann-Banach theorem we obtain $(C_\varphi J_{\mathrm{g}})^{**}(f) = (C_\varphi J_{\mathrm{g}})(f)$ for every $f \in \mathcal{B}_{\alpha,\beta,0}$. Since $\mathcal{B}_{\alpha,\beta,0}$. is w^*-dense in $\mathcal{B}_{\alpha,\beta}$, it follows that $(C_\varphi J_{\mathrm{g}})^{**}(f) = (C_\varphi J_{\mathrm{g}})(f)$ for every $f \in \mathcal{B}_{\alpha,\beta}$. From this and by Gantmacher's theorem we have that $C_\varphi J_{\mathrm{g}}(\mathcal{B}_{\alpha,\beta}) \subset (\mathcal{B}_{\alpha_1,\beta_1,0})$.
Now assume that the condition (3.11) does not hold. If it were, then it would exist an $\varepsilon_0 > 0$ and a sequence $(\varphi(z_k))_{k\in\mathbb{N}} \subset \Delta$, such that $\lim_{k\to\infty} |\varphi(z_k)| = 1$, and

$$\frac{(1-|z_k|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z_k)|^2)^{\beta_1}}|\mathrm{g}'(\varphi(z))||\varphi'(z)|\left(\frac{2}{1-|\varphi(z)|^2} + \ln\frac{4}{1-|\varphi(z)|^2}\right) \geqslant \varepsilon_0 > 0$$

for sufficiently large k. We may also assume that

$$\frac{1-|\varphi(z_{k-1})|}{2} > 1-|\varphi(z_k)|, \ \ k\in\mathbb{N}.$$

Then, for every non-negative integer s there is at most one $\varphi(z_k)$ such that

$$1-\frac{1}{2^s} \leqslant |\varphi(z_k)| < 1-\frac{1}{2^{s+1}}.$$

Hence, there is $M_2 \in \mathbb{N}$ such that for any Carleson window

$$Q = \{re^{i\theta} | 0 < 1-r < l(Q), |\theta-\theta_0| < l(Q)\}$$

and $s \in \mathbb{N}$, there is at most M elements in the following set

$$\{\varphi(z_k) \in Q | 2^{-(s+1)} l(Q) < 1-|\varphi(z_k)| < 2^{-s} l(Q)\}.$$

Therefore, $(\varphi(z_k))_{k\in\mathbb{N}}$ is an interpolating sequence for $\mathcal{B}_{1,1}$. For a function $h \in \mathcal{B}_{1,1}$, let

$$h(\varphi(z_k)) = \left(\ln\frac{4}{1-|\varphi(z_k)|^2} + \frac{2}{1-|\varphi(z_k)|^2}\right) \ k\in\mathbb{N}.$$

Then, we have

$$\frac{(1-|z_k|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z_k)|^2)^{\beta_1}}|(C_\varphi J_{\mathrm{g}} h)'(z_k)| = \frac{(1-|z_k|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z_k)|^2)^{\beta_1}}|\mathrm{g}'(\varphi(z_k))||\varphi'(z_k)||h(\varphi(z_k))|$$
$$= \frac{(1-|z_k|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z_k)|^2)^{\beta_1}}|\mathrm{g}'(\varphi(z_k))||\varphi'(z_k)|\left(\frac{2}{1-|\varphi(z)|^2} + \ln\frac{4}{1-|\varphi(z_k)|^2}\right) \geqslant \varepsilon_0.$$

Thus, $C_\varphi J_{\mathrm{g}}(h) \notin \mathcal{B}_{\alpha_1,\beta_1,0}$ which is a contradiction. (e) $\Rightarrow$ (b). This implication is obvious. (d) $\Rightarrow$ (e). Suppose that (3.2) and (3.11) hold. By (3.11), we have that for every $\varepsilon > 0$, there exists an $r \in (0,1)$, such that

$$\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|\mathrm{g}'(\varphi(z))||\varphi'(z)|\left(\frac{2}{1-|\varphi(z)|^2} + \ln\frac{4}{1-|\varphi(z)|^2}\right) < \varepsilon$$

when $r < |\varphi(z)| < 1$. By (3.2), there exists a $\sigma \in (0,1)$ such that

$$\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|g'(\varphi(z))||\varphi'(z)| < \frac{\varepsilon}{\left(\frac{2}{1-r^2}+\ln\frac{4}{1-r^2}\right)}$$

when $\sigma < |z| < 1$. Therefore, when $\sigma < |z| < 1$ and $r < |\varphi(z)| < 1$, we have

$$\frac{(1-|z|^2)^{\alpha_1,\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|g'(\varphi(z))||\varphi'(z)|\left(\frac{2}{1-|\varphi(z)|^2}+\ln\frac{4}{1-|\varphi(z)|^2}\right) < \varepsilon. \tag{3.15}$$

On the other hand, if $|\varphi(z)| \leqslant r$ and $\sigma < |z| < 1$, we have

$$\begin{aligned}&\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|g'(\varphi(z))||\varphi'(z)|\left(\frac{2}{1-|\varphi(z)|^2}+\ln\frac{4}{1-|\varphi(z)|^2}\right)\\ &< \frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|g'(\varphi(z))||\varphi'(z)|\left(\frac{2}{1-r^2}+\ln\frac{4}{1-r^2}\right) < \varepsilon.\end{aligned} \tag{3.16}$$

Combining (3.15) with (3.16), we obtain

$$\lim_{|z|\to 1}\lim_{|a|\to 1}\frac{|g'(\varphi(z))||\varphi'(z)|(1-|z|^2)^{\alpha_1,\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}\left(\frac{2}{1-|\varphi(z)|^2}+\ln\frac{4}{1-|\varphi(z)|^2}\right) = 0. \tag{3.17}$$

By Lemma 2.1, we have

$$\begin{aligned}&\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|(C_\varphi J_g f)'(z)|\\ &\leqslant C\|f\|_{\mathcal{B}_{1,1}}\frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}}|g'(\varphi(z))||\varphi'(z)|\left(\frac{2}{1-|\varphi(z)|^2}+\ln\frac{4}{1-|\varphi(z)|^2}\right).\end{aligned} \tag{3.18}$$

From (3.18), condition (3.8) follows. Hence the set $C_\varphi J_g(\{f : \|f\|_{\mathcal{B}_{1,1}} \leqslant 1\})$ is bounded in $\mathcal{B}_{\alpha_1,\beta_1}$. Moreover, from (3.18) it follows that the set is bounded in $\mathcal{B}_{\alpha_1,\beta_1,0}$. Taking the supremum in (3.18) over all $f \in \mathcal{B}_{1,1}$ such that $\|f\|_{\mathcal{B}_{1,1}} \leqslant 1$, then letting $|z| \to 1$, $|a| \to 1$ employing (3.18) and Lemma 2.8, we obtain the desired result. Finally note that the implication (e) ⇒(f) is obvious, and that (f) ⇒ (d) follows from the proof of (c) ⇒ (d). □

THEOREM 3.5. *Assume that φ is an analytic self-map of the unit disk and $g \in H(\Delta)$. Then the operator $C_\varphi J_g : \mathcal{B}_{1,1} \Rightarrow \mathcal{B}_{\alpha_1,\beta_1}$ is compact if and only if it is bounded and condition (29) holds.*

PROOF. Sufficiency. Since $C_\varphi J_g : \mathcal{B}_{1,1} \Rightarrow \mathcal{B}_{\alpha_1,\beta_1}$ is bounded, by taking $f_0(z) \equiv 1$, we see that (3.1) holds. The rest of the proof is the same as the proof of Theorem 3.4 (d)⇒(a) and is omitted.

Necessity. Assume that $(z_k)_{z\in\mathbb{N}}$ is a sequence in Δ such that $\lim_{k\to\infty}|\varphi(z_z)| = 1$ (if such a sequence does not exist then (3.11) is vacuously satisfied). Let

$$f_k(z) = (1-|\varphi(z_k)|^2)\frac{\left(\frac{2}{1-\overline{\varphi(z_k)}z}+\ln\frac{4}{1-\overline{\varphi(z_k)}z}\right)^2}{\frac{2}{1-|\varphi(z_k)|^2}+\ln\frac{4}{1-|\varphi(z_k)|^2}}, k \in \mathbb{N}.$$

By some simple calculation, we find that $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{B}_{1,1}} \leqslant C$. Moreover $f_k \to 0$ uniformly on compact subsets of Δ as $k \to \infty$. Since $C_\varphi J_{\mathrm{g}} : \mathcal{B}_{1,1} \Rightarrow \mathcal{B}_{\alpha_1,\beta_1}$ is compact, by Lemma 2.9, we have $\lim_{k\to\infty} \|C_\varphi J_{\mathrm{g}} f_k\|_{\mathcal{B}_{\alpha_1,\beta_1}} = 0$. From this and since

$$\|C_\varphi J_{\mathrm{g}} f_k\|_{\mathcal{B}_{\alpha_1,\beta_1}} \geqslant \sup_{z,a\in\Delta} \frac{(1-|z|^2)^{\alpha_1+\beta_1}}{(1-|\varphi_a(z)|^2)^{\beta_1}} |(C_\varphi J_{\mathrm{g}} f_z)'(z)|$$

$$\geqslant \frac{(1-|z_k|^2)^{\alpha_1+\beta_1}}{(1-|\varphi(z_k)|^2)^{\beta_1-1}} |\mathrm{g}'(\varphi(z_z))||\varphi'(z_z)| \left(\frac{2}{1-|\varphi(z_k)|^2} + \ln \frac{4}{1-|\varphi(z_k)|^2} \right),$$

we have that

$$\lim_{k\to\infty} \frac{(1-|z_k|^2)^{\alpha_1+\beta_1}}{(1-|\varphi(z_k)|^2)^{\beta_1-1}} |\mathrm{g}'(\varphi(z_z))||\varphi'(z_z)| \left(\ln \frac{4}{1-|\varphi(z_k)|^2} + \frac{2}{1-|\varphi(z_k)|^2} \right) = 0,$$

which implies that (3.11) holds. □

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