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ON THE ANTIINVERSE AND COREGULAR SEMIGROUPS AND SOME THEIR APPLICATIONS

Kalčo Todorov

ABSTRACT. In the present survey are considered some basic results, conserning the two – sided identical semigroups, their subclasses of antiinverse and coregular semigroups, as well as some application of this investigations towards problem of the Hadamard matrices.

1. Introduction and preliminaries

An element a of a semigroup S we call *two – sided identical* if there exists an element $b \in S$ (which we call neutral to a) such that

$$a = bab \tag{1}$$

A semigroup S we call *two – sided identical* if every element of S is two-sided identical.

For elements a and b of a semigroup S, we say that they are *mutually antiinverse* if the following conditions hold

$$aba = b \quad \text{and} \quad bab = a. \tag{2}$$

A semigroup S is *antiinverse* if for every $a \in S$ there exists its antiinverse element $b \in S$.

An element a of a semigroup S we call *coregular* and $b \in S$ its *coinverse*, if

$$a = aba = bab. \tag{3}$$

A semigroup S we call *coregular*, if every element of S is coregular.

It is evident that every coregular semigroup is two – sided identical. Moreover, every coregular semigroup is simultaneously regular and antiinverse, but the converse is not true.

The first systematic investigation of the above shown classes semigroups are the ones of the antiinverse semigroups and are included in a sequence of papers of S. Bogdanović and other as well as in the paper [21] of Sharp. Almost simultaneously with the investigations of antiinverse semigroups have appeared also the ones of the coregular semigroups (see Bijev, Todorov [2]).

Somewhat later have been introduced and investigated by the author the two – sided identical semigroups (see K. Todorov [22]).

The class of the two – sided identical semigroups is a quite large class of semigroups. To it belong all the semigroups, possesing two – sided identity element (in particular the groups). A lot of examples indicate however the presence of two – sided identical semigroups without two – sided identity element. To the class of two – sided identical semigroups belong some of the regular semigroups – such as for example any antiinverse semigroup. This two classes does not cover each other, as shown in the coming two examples.

A more complete idea about semigroups, belonging to the intersection of the classes of two – sided identical, regular and coregular semigroups can be obtain from the paper [3] by Bijev, Todorov , where has been delivered a complete classification of the abstract semigroups, included in the symmetric semigroup T_3 of degree 3, whereby for each one of them is remarked, whether it is reguilar, two – sided identical or coregular.

From the results known for the two – sided identical semigroups I'll cite the following statements.

Theorem 1. *Let* $\mathbf{E}(S)$ *be the set of the idempotents of the semigroup* S, *let* $N(+)$ *be the additive semigroup of natural numbers and let for the elements* $a,\ b \in S,\ e \in \mathbf{E}(S)$ *and* $m \in N(+)$ *we have*

$$a = bab \quad \text{and} \quad b^m = e. \tag{4}$$

Then

a) $a^2 = (ab^i)^2 = (b^i a)^2, \qquad i = 1, 2, ..., m;$

b) $a^{2k} = b^p a^{2k} b^{m-p};\ a^{2k-1} = b^p a^{2k-1} b^p,\ \ p = 1, 2, ..., m,\ k = 1, 2, ...\ .$

c) Any element of the semigroup $< a, b >$ *can be represented as the type* $a^i b^j$, *where* $i = 0, 1, ...;\ j = 0, 1, ..., m-1$ *and* $i + j > 0$.

Both the antiinverse and coregular semigroups may be considered as a subclasses of the class of the two – sided identical.

Coregular and antiinverse form subclasses of the class of regular semigroups. Although coregular semigroups form a subclass of the class of antiinverse semigroups, strictly containing the class of commutative antiinverse semigroups.

From the theory of the antiinverse semigroups I schall cite the following statements

Theorem 2. *Let S be a semigroup. Then the semigroup S is a antiinverse iff*

$$(\forall a \in S)(\exists b \in S)(a^2 = b^2,\ ba = a^3 b,\ a^5 = a).$$

Theorem 3. *Let G be a group. Then G is antiinverse iff G is a union of subgroups which belong to the class of the trivial group, of the cyclic group of order 2 and of the quaternion group.*

Theorem 4. (*Bogdanović, [7] Theorem 4.1*) *Let S be a semigroup. Then all proper subsemigroups of S are antiinverse iff one of the following conditions hold:*

1. $(\forall a \in S)(a = a^3)$,
2. *S is a cyclic group of prime-power order* (> 1),
3. *S is the cyclic group of order* 4,
4. *S is $M(2,1)$ semigroup, i.e.* $S = < a >: a^{2+1} = a^2$,
5. *S is $M(2,2)$ semigroup.*

Define a relation ρ on a semigroup S as follows: $a\rho b \Leftrightarrow a$ and b are antiinverse in S. Let

$$\mathbf{S}[a] = \{x \in S \mid x\rho a\},$$

for all $a \in S$.

Theorem 5. *If S is a commutative semigroup, the following are equivalent:*
(i) *S is antiinverse.*
(ii) *ρ is a congruence on S.*
(iii) *$\mathbf{S}[a]$ is a subsemigroup of S, for all a in S.*
(iv) *$a^3 = a$, for all a in S.*

Coregular in the multiplicative matrix semigroup $\mathbf{M}_2(\mathbf{R})$ of real matrix is the matrix

$$\begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}, \quad 0 \le \beta < 2\pi,\ \beta \ne \pi,$$

defining the axial symmetries of the figures of a given plane.

In the class of coregular semigroups are valid the following statements:

Theorem 6. *Let the elements a and b of a semigroup S satisfy condition (e). Then:*
a) $a^3 = a$.
b) $a^2 b^2 a^2 = a^2 = b^2 a^2 b^2$.

c) $a^2b = ab^2 = a$.
d) $b^2 = b$ *implies* $a^2 = a$.
e) $a^2 = b^2$ *implies* $a = b^3$.

Theorem 7. *For a semigroup S the following conditions are equivalent:*
a) S is coregular.
b) $a^3 = a$ for every element a of S.
c) S is a union of disjoint groups, the elements of which are of order ≤ 2.

Some interesting continuations of the already shown classes of semigroups are containted further in the papers of Bijev [1] and Chvalina and Matouškova [10].

Accordingh to Bijev [1] any representation of the element x of an arbitrary semigroup S in the form

$$x = ab, \quad a, b \in S \text{ for } a^3 = a \text{ and } b^3 = b$$

is called coregular. A good motivation for the examination of this representations is the fact, that in the multiplicative semigroup of all ortogonal matrices of rank 2

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \quad \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}, \quad 0 \leq \beta < 2\pi,$$

the matrix of the second type are coregular, and the matrix of the first type are coregular representative.

Theorem 8. *Let $(\mathbf{A}, f)$ be a unar. Then the semigroup End $(\mathbf{A}, f)$ is coregular iff it is antiinverse.*

Theorem 9. *Let T_X be the symmetric semigroup on the set $\mathbf{X}$. The following conditions are equivalent:*
1) *T_X is coregular.* 2) *T_X is antiinverse.* 3) *card* $\mathbf{X} \leq 2$.

Theorem 10. *Let $\mathbf{X}$ be an infinite set. There exists a coregular commutative subsemigroup H_X of the symmetric semigroup T_X (not generated by idempotent elements only) such that card$\mathbf{H}_X$ = card $\mathbf{X}$.*

A quite striking application of the two – sided identical semigroups turned out to be the one bound to the problem of the Hadamard matrices.

The quaternion group Q is an origin in the study of the antiinverse semigroup, which as well the coregular semigroups, present subclasses of the class of two-sided identical semigroups. In this respect the quaternion group Q as a two – sided identical semigroup admits further generalizations (see Magnus and al.[18] and Neuman [19]) about its genetic code $Q = \langle i, j : i = jij, \ j = iji \rangle$

The following statements hold:

Theorem 11. *Let S be the semigroup generated by the elements of a countable set $\mathrm{M} = \{a_1, a_2, ...\}$ subject only to the relations*

$$a_1 = a_2 a_1 a_2, \ a_i = a_j a_i a_j = a_{i+1} a_i a_{i+1} \tag{5}$$

for each two elements $a_i, a_j \in \mathrm{M}$ with $1 \leq j < i$. Then:

1) $a_i^2 = a_j^2 = (a_k a_l)^2$ *for $a_i, a_j, a_l, a_l \in \mathrm{M}$ with $k \neq 1$. Further is putting $a_i^2 = -1$.*

2) $a_i a_j = -a_j a_i$ *for $a_i, a_j \in \mathrm{M}$ with $i \neq j$.*

3) $a_i a_j a_i = a_j$ *for every two elements $a_i, a_j \in \mathrm{M}$ with $i \neq j$.*

4) *S is a group.*

5) *Let $a \in \mathrm{M}$ and let l denote the number of the factors of the element $c = c_1 c_2 ... c_k$, $(c_i \in \mathrm{M}, i = 1, 2, ..., k)$ equal to the element a. Then $ca = (-1)^{k-l} ac$.*

6) $a^4 = 1$ *for every element $a \in S$.*

7) $|\mathbf{C}(S)| = 2$.

8) *The group S contains only two conjugate singleton classes and each one of its other conjugate classes is a two – element one.*

9) *Any subgroup of S is a normal subgroup of S iff it contains at least one conjugate class distinct from the class of* 1.

10) *The centralizer $\mathbf{C}_S(c)$ of every element $c \in S$ is infinite.*

11) $\langle a_i, a_j \rangle \cong Q$ *for $a_i, a_j \in \mathrm{M}$ with $i \neq j$.*

12) *S is a locally finite group.*

13) *The subsemigroups of S generated by the subsets $\mathbf{U}$ of M with $n = |\mathbf{U}| > 1$ are groups having properties* 1)–6), 9), 11).

14) *Let $\mathbf{C}(S_n)$ denote the center of the semigroup $S_n = \langle a_1, ..., a_n \rangle$ then*

$$\mathbf{C}(S_n) = \begin{cases} \{\pm 1, \pm a_1 a_2 ... a_n\} & \textit{is odd;} \\ \{\pm 1\} & \textit{is even;} \end{cases}.$$

15) *Each element $c \in S_n$ may be written in the form*

$$c = (-1)^{u_o} a_1^{u_1} a_2^{u_2} ... a_n^{u_n},$$

where $u_i \in \{0; 1\}, i = 0, 1, ..., n$.

16) *Let $a \in \mathrm{M}_n = \{a_1, ..., a_n\}$ and let l denote the number of the factors of the element $c = c_1 c_2 ... c_k, (c_i \in \mathrm{M}_n, i = 1, 2, ..., k)$, coinciding with the element a. Then $ca = (-1)^{k-l} ac$.*

17) *The conjugate class $K_a = \{bab^{-1},\ b \in S_n\}$ for every element $a \in \mathrm{S}_n \backslash \{1, -1\}$ is two-element and coincides with the set $\{\pm a\}$.*

18) $|\mathrm{S}_n| = 2^{n+1}$.

Corollary 1. *Every element c of the semigroup S of order 4 (defined in theorem 8) generates a normal subgroup of S.*

The properties 1)–18) of the semigroup S in Theorem 11 strongly depend on its genetic code (5). Adding a new relation give a new semigroup. The following Theorem 12 is a special case of Theorem 11.

Theorem 12. *Let S be a semigroup generated by the elements of the countable set $\mathrm{M} = \{a_1, a_2, ...\}$ subject only to the relations (5) and $a_k = a_k^2$ for some $a_k \in \mathrm{M}$. Then S is a commutative group with identity element $1 = a_k$ in which each non-identity element is an element of order 2.*

The groups $S_n = \langle a_1, a_2, ..., a_n\rangle$ in Theorem 11 we shall call n – *generated q. groups.*

There exists a close connection between the so constructed n–generated q. groups and the Hadamard matrices.

As is well known, in 1893 Hadamard proved that if $X = (x_{ij})$ is a square matrix of order n then holds the inequality

$$|detX|^2 \leq \prod_{i}^{n} \sum_{j=1}^{n} |x_{ij}|^2,$$

where the equality

$$|detX|^2 = \prod_{i}^{n} \sum_{j=1}^{n} |x_{ij}|^2$$

holds iff

$$\sum_{k=1}^{k=n} x_{ik}x_{jk} = 0, \; i \neq j, \; i,j = 1, ..., n \quad or \quad x_{ij} = 0 \quad for\ some\ i.$$

By definition, a square matrix H of order n whose entries are $+1$ and -1 is called a Hadamard matrix of order n provided that its rows are pairwise orthogonal, in other words

$$\mathbf{HH'} = \mathbf{H'H} = n\mathbf{E},$$

where $\mathbf{H'}$ is the transposed matrix of $\mathbf{H}$.

It is known (see Hedayat and Wallis [15]) that there exist Hadamard matrices of orders 1 and 2, but it can be shown that every other Hadamard matrix has order $4t$ for some positive integer t. The question: "How many different Hadamard matrices of a given order might exist?" is a very difficult question to answer and researchers' interest in it varies for almost a whole century.

Hadamard matrices of infinitely many orders have been constructed, and it has been conjectured that one exists for every t, but no general proof is available, and the number of unsettled orders is infinite.

From the above described basic properties of the n–generated quaternion group one can construct Hadamard matrices by fixing a representative from every conjugate class K_a, $a \notin \{1, -1\}$ and from the set $\{1, -1\}$. Taken in a given order the so fixed elements of the n–generated quaternion group S_n, together with the rows and columns corresponding to them comprise a subtable of the multiplicative table of the semigroup S_n. The signs of the elements of this subtable, taken in the same order, form the Hadamard matrix. In this way we obtain from the q. group Q the following Hadamard matrix:

$$\begin{array}{c|cccc} 1\backslash & 1 & i & j & k \\ i & & -1 & k & -j \\ j & & -k & -1 & i \\ k & & j & -i & -1 \end{array} \quad \Rightarrow \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

These relations are described by the author in details in [24].

What are the further possibilities for obtaining of the Hadamard matrices on the basis of the considered n–generated q. groups? As the calculations show with the multiplication table of the integral quaternions

$$\pm 1,\ \pm i,\ \pm j,\ \pm k, \quad \frac{\pm 1\ \pm i\ \pm j\ \pm k}{2}$$

is associated Hadamard matrices of order 12 of the type

$$\begin{pmatrix} A_1 & A_2 & A_3 \\ -B_1 & B_2 & B_3 \\ C_1 & -C_2 & -B_2 \end{pmatrix}$$

where $\mathbf{A}_i$, $\mathbf{B}_j$, $\mathbf{C}_k$ are Hadamard matrices of order 4.

All this is a good motivation for the further investigations of the corresponding algebras of the n-generated q. group. For the present because of the principal difficulties as compared to the already known results a more thorough investigation is made on algebra of the 3–generated q. group. The received basic results can be formulated as follows:

Lemma 1. *The (group) algebra H_3 (of the quaternion group S_3 in Theorem 11) over the field $\mathbf{R}$ of the real number is a 8–dimensional (associative) algebra, containing the quaternion algebra H.*

Every element $a = \sum_{i=0}^{i=7} a_i e_i \in H_3$ *is a divisor of zero iff*

$$a_7 = \varepsilon a_0,\ a_6 = -\varepsilon a_1,\ a_5 = \varepsilon a_2,\ a_4 = -\varepsilon a_3,$$

where $\varepsilon = \pm 1$. *If*

$$a = a_0(1 + \varepsilon e_7) + a_1(e_1 - \varepsilon e_6) + a_2(e_2 + \varepsilon e_5) + a_3(e_3 - \varepsilon e_4)$$

and

$$a^* = b_0(1 - \varepsilon e_7) + b_1(e_1 + \varepsilon e_6) + b_2(e_2 - \varepsilon e_5) + b_3(e_3 + \varepsilon e_4).$$

then

$$aa^* = a^* a = 0.$$

Theorem 13. *Let us put*

$$I^- = \{q^- = a_0(e_0 - e_7) + a_1(e_1 + e_6) + a_2(e_2 - e_5) + a_3(e_3 + e_4)\},$$

$$I^+ = \{q^+ = b_0(e_0 + e_7) + b_1(e_1 - e_6) + b_2(e_2 + e_5) + b_3(e_3 - e_4)\},$$

where $a_i, b_j \in \mathbf{R}$. $i, j = 0, 1, .., 7$. *Then:*

1) $I^- \cap I^+ = \{0\}$,

2) I^- *and* I^+ *are the uniquely non-trivial* minimal *ideals of the algebra* H_3,

3) $I^- \cong H \cong I^+$, *where* H *is the quaternion algebra,*

4) $H_3 = I^- + I^+$.

Let us put

$$2f_i = \begin{cases} e_i - e_{7-i}, & \text{if } i = 0, 2; \\ e_i + e_{7-i}, & \text{if } i = 1, 3, 5, 7; \\ e_{7-i} - e_i, & \text{if } i = 4, 6. \end{cases}$$

Here, to every element $a \in H_3$, the norm $N(a) \in \mathbf{R}$ and minimal polynomial are associated. When determining these characteristics is used the regular representation of the quaternions of the algebra H_3 by means of finite matrices.

Further, for every element $a \in H_3$ are determined the essential properties of its conjugate element $\overline{a} \in H_3$, as well as the integral elements of the algebra H_3.

By formulating the propositions and their proofs equally both the basis f and the basis e of the algebra H_3 is used.

2. Norm and minimal polynomial

Let us recall that if

$$q = a_0 + a_1 i + a_2 j + a_3 k \tag{6}$$

is an arbitrary element of the quaternion algebra H, then:

1) the positive number $N(q) = \sum_{i=0}^{i=3} a_i^2$ is the norm of the quaternion q;
2) $h(x) = x^2 - 2a_0 x + N(q)$ is the minimal polynomial, satisfied by the quaternion q;
3) $\bar{q} = a_0 - a_1 i - a_2 j - a_3 k$ is the conjugate to a quaternion, as $q\bar{q} = \bar{q}q = N(q)$;
4) by the regular representation of the algebra H (over the field of real numbers $\mathbf{R}$) by Skornjakov [28] and Deuring [13] to the quaternion q in (6) corresponds the matrix

$$\mathbf{A} = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix},$$

where $N(q) = \sqrt{|A|}$ and $|A|$ is the determinant of the matrix A.

Remark 1. Here and on, $(a)^{1/2k}$ for $a \geq 0$ and k - a natural number means the arithmetical root of the number a, i. e. the module of the complex number $(a)^{1/2k}$.

Let $q = \sum_{i=0}^{i=7} c_i f_i = q_1 + q_2$ be an arbitrary element of the quaternion algebra H_3, where

$$q_1 = \sum_{i=0}^{i=3} c_i f_i \in I^-, \qquad q_2 = \sum_{i=4}^{i=7} c_i f_i \in I^+,$$

i.e.

$$q_1 = (c_0, c_1, c_2, c_3, 0, 0, 0, 0), \qquad q_2 = (0, 0, 0, 0, c_4, c_5, c_6, c_7) \in H_3.$$

The positive number $N(q)$:

$$N(q) = N(q_i) \in \mathbf{R}, \qquad \textit{if } q = q_i,\ i = 1, 2$$

or

$$N(q) = (N(q_1)N(q_2))^{1/2} \in \mathbf{R}, \qquad \textit{if } q \neq q_i, i = 1, 2$$

we shall call the *norm* of the quaternion q.

For the elements of the quaternion algebra H_3 a describtion of the above characteristics is given in the following.

Theorem 14. *Let $q = \sum_{i=0}^{i=7} c_i f_i$ be an arbitrary element of the quaternion algebra H_3 and let*

$$q_1 = \sum_{i=0}^{i=3} c_i f_i \in I^-, \ q_2 = \sum_{i=4}^{i=7} c_i f_i \in I^+, \ q \neq q_i, \ i = 1, 2.$$

Then:

1) *of the basis f by a regular representation of H_3, the matrix*

$$A_f = \begin{pmatrix} B_f & 0 \\ 0 & C_f \end{pmatrix},$$

corresponds to the quaternion q, where

$$B_f = \begin{pmatrix} c_0 & -c_1 & -c_2 & -c_3 \\ c_1 & c_0 & -c_3 & c_2 \\ c_2 & c_3 & c_0 & -c_1 \\ c_3 & -c_2 & c_1 & c_0 \end{pmatrix}, \quad C_f = \begin{pmatrix} c_7 & -c_6 & c_5 & c_4 \\ c_6 & c_7 & -c_4 & c_5 \\ -c_5 & c_4 & c_7 & c_6 \\ -c_4 & -c_5 & -c_6 & c_7 \end{pmatrix}.$$

2) $N(q) = \sqrt[4]{|A_f|}1 = ((\sum_{i=0}^{i=3} c_i^2)(\sum_{i=4}^{i=7} c_i^2))^{1/2} = (N(q_1)N(q_2))^{1/2}$ *is the norm of the quaternion q (see Remark 1).*

3) *the quaternion $q = \sum_{i=0}^{i=7} c_i f_i$ satisfies the minimal polynomial*

$$h(x) = x^4 - 2(c_0 + c_7)x^3 + (N(q_1) + N(q_2) + 4c_0c_7)x^2 - \\ -2(c_7 N(q_1) + c_0 N(q_2))x + N^2(q).$$

Corollary 2. *Let $q = \sum_{i=0}^{i=7} c_i f_i$ be an arbitrary element of the quaternion algebra H_3 and let*

$$q_1 = \sum_{i=0}^{i=3} c_i f_i \in I^-, \ q_2 = \sum_{i=4}^{i=7} c_i f_i \in I^+.$$

The norm $N(q)$ of every element $q \in H_3$ has the following basic properties:

1. $N(q) \geq 0$, *and* $N(q) = 0$ *iff* $q = 0$.

2. $N(\alpha q) = \alpha^2 N(q)$ *for* $\alpha \in \mathbf{R}$ *and* $q \in H_3$.

3. $N(xy) = N(x)N(y) = N(yx), \ x, y \in H_3$.

4. Let $q = \sum_{i=0}^{i=7} a_i e_i$,

$$s = \sum_{i=0}^{i=7} a^2 \quad \text{and} \quad t = 2(-a_0a_7 + a_1a_6 - a_2a_5 + a_3a_4), \tag{7}$$

then

i) $N^2(q) = s^2 - t^2 = (s+t)(s-t) =$
$= [(a_0 - a_7)^2 + (a_1 + a_6)^2 + (a_2 - a_5)^2 + (a_3 + a_4)^2]x$
$[(a_0 + a_7)^2 + (a_1 - a_6)^2 + (a_2 + a_5)^2 + (a_3 - a_4)^2]$.

ii) $h(x) = x^4 - 4a_0x^3 + 2[2a_0^2 + s - 2a_7^2]x^2 - 4(a_0 s + a_7 t)x + s^2 - t^2$.

3. Conjugate elements

Let

$$q = \sum_{i=0}^{i=7} c_i f_i = q_1 + q_2,$$

where

$$q_1 = \sum_{i=0}^{i=3} c_i f_i \in I^-, \quad q_2 = \sum_{i=4}^{i=7} c_i f_i.$$

The quaternion $\bar{q} \in H_3$:

$$\bar{q} = \bar{q}_i, \; if \; q = q_i, \; i = 1,2$$

or

$$\bar{q} = (c_0 - \sum_{i=0}^{i=3} c_i f_i)(N^{-1}(q_1)N(q_2))^{1/2} +$$

$$+(c_7 f_7 - \sum_{i=4}^{i=6} c_i f_i)(N(q_1)N^{-1}(q_2))^{1/2},$$

if $q \neq 0$ and $q \neq q_i$, $i = 1,2$, we shall call *conjugate* of the quaternion $q \in H_3$.

Theorem 15. *For every quaternion q of H_3 and its conjugate $\bar{q}$ there hold:*

1) $q\bar{q} = \bar{q}q = N(q)$.
2) $N(q) = N(\bar{q})$.
3) $\bar{\alpha}\bar{q} = \alpha\bar{q}$ *for every* $\alpha \in \mathbf{R}$.
4) $\bar{\bar{q}} = q$.
5) $\bar{x}\bar{y} = \bar{y}\bar{x}$ *for* $x, y \in H_3$.

4. Integral elements

The integral quaternions of the algebra H have been determined for a first time by Hurwitz in [16] (see Hurvitz [17]). The method which we use in the present paragraph is more sophisticated than the one given by Dickson in [14].

The integral quaternions of the algebra H (see Dickson p. 148, Theorem 1) are given by

$$q = a_0\rho + a_1 i + a_2 j + a_3 k, \; \rho = \frac{1 + i + j_k}{2} \tag{8}$$

for integral values of a_0, a_1, a_2, a_3. Expressed otherwise, they are the quaternions whose four coordinates are either all integers or all halves of odd integers.

Proposition 1. *(see Dickson, p. 157) The first components of the elements of any (maximal) set of integral elements, with properties* **R**, **C**, **U** *of a direct sum* $B + C$ *constitute a (maximal) set of integral elements of the first component algebra* B*, and similarly for the second components. Conversely, given a (maximal) set* **b** *of integral elements* b *of a rational algebra* B *and a (maximal) set* **c** *of integral elements* c *of another rational algebra* C*, such that* B *and* C *have module* β *and* γ *and have a direct sum, then if we add every* b *to every* c *we obtain sums forming a (maximal) set of integral elements of the direct sum* $B + C$.

The next theorem is also immediate from the foregoing Proposition and (8).

Theorem 16. *The integral quaternions of the algebra* H_3 *are given by*

$$q = a_0\rho + a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4 + a_5 f_5 + a_6 f_6 + a_7\tau,$$

where

$$\rho = \frac{f_0 + f_1 + f_2 + f_3}{2}, \qquad \tau = \frac{f_4 + f_5 + f_6 + f_7}{2},$$

for integral values of $a_0,\ a_1,\ \ldots,\ a_7$.

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Bulgarian Academy of Sciences, Institute of Mathematics, POBox 373, 1090 Sofia, Bulgaria

E-mail address: kjt@ bgearn.acad.bg