FILOMAT (Niš) 9:3 (1995), 941–946

Algebra, Logic & Discrete Mathematics
Niš, April 14–16, 1995.

SEMI-FREDHOLM ELEMENTS IN BANACH ALGEBRAS

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ABSTRACT. In this paper we define the set of semi-Fredholm elements in a semisimple Banach algebra and we prove that the perturbation class of this set is a closed twosided ideal of this algebra.

1. Introduction

Let X be Banach space and let $B(X)$ be Banach space of all bounded linear transformations of X into X. For $T \in B(X)$ we let $N(T)$ denote the kernel of T, $N(T) = \{x \in X | T(x) = 0\}$ and we let $R(T)$ denote the range of T, $R(T) = \{y \in X | T(x) = y \text{ for some } x \in X\}$. If $T \in B(X)$ and $R(T)$ is closed, we say that T is a semi-Fredholm operator if either $dim(N(T)) < \infty$ or $dim(X/R(T)) < \infty$. We have two classes of semi-Fredholm operators,

$$\Phi_+(X) = \{T \in X | R(T) \text{ is closed and } dimN(T) < \infty\} \text{ and}$$

$$\Phi_-(X) = \{T \in B(X) | R(T) \text{ is closed and } dimX/R(T) < \infty\}.$$

We also set $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$ and call this set of Fredholm operators on X. It is known that $T \in B(X)$ is Fredholm if and only if $\pi(T)$ is invertible in the Calkin algebra $C(X)(C(X) = B(X)/K(X)$ where $K(X)$ is set of compact operators and π denote the natural homomorphism of $B(X)$ onto $C(X), \pi(T) = T + K(X)$. Index of T is defined by $i(T) = dim(N(T)) - dim(X/R(T))$. Set of finite dimensional operators is denoted by $F(X)$ and let $G_l(A)(G_r(A))$ be set left (right) invertible elements of algebra A.

If A is a semisimple Banach algebra, x is defined to be a Fredholm element of A if there exists a $y \in A$ such that $xy - 1, yx - 1 \in soc(A)$. Set of Fredholm elements is denoted by $\Phi(A)$.

2. Preliminaries

Let X be Banach space and let $T \in B(X)$. Next two theorems are proved by Yood in [8]:

Theorem 2.1. *$T \in \Phi_+(X)$ and $i(T) \leq 0$ if and only if there exists $T_0 \in B(X)$ and $K \in K(X)(K \in F(X))$ such that $T = T_0 + K$ where T_0 is bounded below.*

Theorem 2.2. *$T \in \Phi_-(X)$ and $i(T) \geq 0$ if and only if there exists $T_0 \in B(X)$ and $K \in K(X)(K \in F(X))$ such that $T = T_0 + K$ and $R(T_0) = X$.*

By [2, Theorem 57.19] $T \in B(X)$ is bounded below if and only if T is not left topological divisor of zero and operator $T \in B(X)$ is onto if and only if T is not right topological divisor of zero [2, Corollary 57.17]. Let A be a semisimple Banach algebra and let H_l and H_r be a sets defined by

$$H_l = \{x \in A | x \text{ is not left topological divisor of zero } \},$$

$$H_r = \{x \in A | x \text{ is not right topological divisor of zero } \}.$$

Sets H_l and H_r are open semigroups [5, p.21].

In [6] Rowell defined set of left-Fredholm elements

$$\Phi_l(A) = \{x \in A | \exists y \in A \text{ such that } yx - 1 \in soc(A)\},$$

$$\Phi_l^{\leq 0} = \{x \in \Phi_l(A) | ind(x) \leq 0\},$$

and proved [6, Theorem 5.3] that $x \in \Phi_l^{\leq 0}(A)$ if and only if there exists $u \in soc(A)$ and $a \in G_l(A)$ such that $x = a + u$.

3. Results

In this paper we define sets

$$\Phi_+^-(A) = \{x \in A | \exists a \in H_l \exists k \in soc(A) \text{ such that } x = a + k\},$$

$$\Phi_-^+(A) = \{x \in A | \exists a \in H_r \exists k \in soc(A) \text{ such that } x = a + k\}.$$

Remark. If we put $A = B(H), H$ is Hilbert space, then $\Phi_+^-(A) = \Phi_l^{\leq 0}$ because T is not left topological divisor of zero if and only if T is left invertible in $B(H)$ [2, Theorem 57.19].

In general case is $G_l(A) \subset H_l$ and $G_r(A) \subset H_r$ [5, p.20]. From this fact we get $\Phi_l^{\leq 0}(A) \subset \Phi_+^-(A)$. If X is Banach space then $\Phi_l(X) \subset \Phi_+(X)$.

Definition 3.1. Let A be semisimple Banach Algebra. We defined sets of semi-Fredholm elements Φ_+ and Φ_- in A by

$$\Phi_+(A) = \Phi_+^-(A) \cup \Phi(A) \text{ and } \Phi_-(A) = \Phi_-^+(A) \cup \Phi(A).$$

Lemma 3.2. (1) *If $x, y \in \Phi_+^-(A)$ then $xy \in \Phi_+^-(A)$.*
(2) *If $x \in \Phi_+^-(A)$ and $k \in soc(A)$ then $x + k \in \Phi_+^-(A)$.*
(3) *If $x \in \Phi_+^-(A)$ and $\lambda \in \mathbb{C}, \lambda \neq 0$, then $\lambda x \in \Phi_+^-(A)$.*

Proof. (1) Let $x = a_1 + k_1$ and $y = a_2 + k_2$ such that $a_1, a_2 \in H_l$ and $k_1, k_2 \in soc(A)$. Then we have

$$xy = (a_1 + k_1)(a_2 + k_2) = a_1a_2 + k_1a_2 + a_1k_2 + k_1k_2 = a_1a_2 + k,$$

where we put $k_1a_2 + a_1k_2 + k_1k_2 = k \in soc(A)$. From fact that H_l is semigroup we have $a_1a_2 \in H_l$ and from this $xy \in \Phi_+^-(A)$.
(2) and (3) is obvious. □

Let $Min(A)$ be a set of minimal idempotents of A and let e be a fixed minimal idempotent of A. We shall write $\widehat{x}$ to denote left regular representation of Banach algebra A on the Banach space Ae, that is $\widehat{x}(y) = xy$ for $y \in Ae$.

Theorem 3.3. *If $x \in \Phi_+^-(A)$ then $\widehat{x}$ is semi-Fredholm operator on Ae and $i(\widehat{x}) \leq 0$.*

Proof. Let $x \in \Phi_+^-(A)$. Then there exist $a \in H_l$ and $k \in soc(A)$ such that $x = a + k$. Let $ye \in Ae$ be arbitrary element. Then we have

$$\widehat{x}(ye) = xye = (a + k)ye = aye + kye = \widehat{a}(ye) + \widehat{k}(ye) = (\widehat{a} + \widehat{k})ye.$$

From this we get $\widehat{x} = \widehat{a} + \widehat{k}$ and $\widehat{k}$ is compact on Ae because $dim(\widehat{k}) < \infty$. As $\widehat{a}$ is not left topological divisor of zero [2, Theorem 57.4], we get from Theorem 2.1 that $\widehat{x}$ is semi-Fredholm operator on Ae and $i(\widehat{x}) \leq 0$. □

If $y \in \Phi(A)$ then $\widehat{y}$ is Fredholm operator on Ae [1, Theorem F.2.6]. From this fact and Theorem 3.3 we have that if $x \in \Phi_+(A)$ then $\widehat{x}$ is semi-Fredholm operator on Ae. From [1, Example F.4.2] and fact that algebra $A/K(X)$ is commutative because it is generated by $T + K(X)$ we get that the converse of Theorem 3.3 is false.

Lemma 3.4. *Set $\Phi_+^-(A)$ is open.*

Proof. It is known that H_l is open. Let $x \in \Phi_+^-(A)$. Then there exist $a \in H_l$ and $k \in soc(A)$ such that $x = a + k$, and there exists $\epsilon > 0$ such that for $u \in A, \|u\| < \epsilon$ implies $a - u \in H_l$. Let $y \in A$ and let $\|x - y\| < \epsilon$. Then we have

$$y = x - (x - y) = a - (x - y) + k,$$

and $a - (x - y) \in H_l$. That means that $y \in \Phi_+^-(A)$. □

Let S be a subset of Banach space A. Perturbation class of set S is

$$P(S) = \{a \in A | a + s \in S \text{ for all } s \in S\}.$$

Next two lemmas are valid with assumption $\lambda S \subset S$ for every $\lambda \neq 0$.

Lemma 3.5. [3, Lemma 5.5.3] *$P(S)$ is linear subspace of A. If S is open subset of A, then $P(S)$ is closed.*

Lemma 3.6. [3, Lemma 5.5.5] *Let A be a Banach algebra with unit and let G be a group invertible elements in A. If $GS \subset S$, then $P(S)$ is a left ideal; if $SG \subset S$, then $P(S)$ is a right ideal.*

Theorem 3.7. *$P(\Phi_+^-(A))$ is a closed two sided ideal of A.*

Proof. In Lemma 3.4 is shown that $\Phi_+^-(A)$ is a open set and from Lemma 3.5 it follows that $P(\Phi_+^-(A))$ is a closed set.

Let $b \in G(A)$ and $x = a + k, a \in H_l, k \in soc(A)$. Then we have

$$bx = ba + bk = ba + k_1,$$

where we put $bk = k_1 \in soc(A)$. Suppose that $ba \notin H_l$. Then there exists a sequence $\{y_n\}_{n=1}^{\infty}$ in A such that $bay_n \to 0, n \to \infty$. But in this case $b^{-1}bay_n = ay_n \to 0$,
$n \to \infty$, which is impossible. Thus $ba \in H_l$. From this it follows that $bx \in \Phi_+^-(A)$, so we have

$$G(A)\Phi_+^-(A) \subset \Phi_+^-(A). \tag{1}$$

From the other side

$$xb = ab + kb = ab + k_2,$$

where $k_2 \in soc(A)$ and $ab \in H_l$. (If $ab \notin H_l$ then there exists sequence $\{z_n\}_{n=1}^{\infty}$ such that $abz_n = a(bz_n) \to 0, n \to \infty$ and $a \notin H_l$ what is contradiction.) It follows that

$$\Phi_+^-(A)G(A) \subset \Phi_+^-(A). \tag{2}$$

From (1), (2) and Lemma 3.6 it follows that $P(\Phi_+^-(A))$ is two sided ideal of A. □

Let now suppose that A and B are Banach algebras with identity 1 and $T : A \to B$ is a homomorphism of Banach algebras. Suppose that T is bounded and $T(1) = 1$. In [4] Harte defined $a \in A$ as Fredholm element with respect to T if and only if $T(a) \in G(B)$. Analogously we define left and right Fredholm elements with respect to T as

$$\Phi_l(A) = \{a \in A | T(a) \in G_l(B)\},$$

$$\Phi_r(A) = \{a \in A | T(a) \in G_r(B)\}.$$

Next Lemma follows immediately:

Lemma 3.8. *(1) If $x, y \in \Phi_l(A)$ then $xy \in \Phi_l(A)$.*
(2) If $xy \in \Phi_l(A)$ then $y \in \Phi_l(A)$.
(3) If $x \in \Phi_l(A)$ and $u \in N(T)$ then $x + u \in \Phi_l(A)$.
(4) If $x \in \Phi_l(A)$ and $\lambda \in \mathbb{C}, \lambda \neq 0$ then $\lambda x \in \Phi_l(A)$.
(5) $\Phi_l(A)$ is open set.

Theorem 3.9. *$P(\Phi_l(A))$ is a closed two sided ideal of A.*

Proof. $\Phi_l(A)$ is open set, so $P(\Phi_l(A))$ is closed set of A.

Let $z \in G(A), y \in \Phi_l(A)$ and let $b \in B$ be a left inverse for $T(y)$. Now we have

$$bT(z^{-1})T(zy) = bT(z^{-1})T(z)T(y) = bT(y) = 1,$$

so $zy \in \Phi_l(A)$ and $G(A)\Phi_l(A) \subset \Phi_l(A)$.

Similarly $\Phi_l(A)G(A) \subset \Phi_l(A)$, and by Lemma 3.6 it follows that $P(\Phi_l(A))$ is a two sided ideal of A. □

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