FILOMAT (Niš) 9:3 (1995), 931–940
Algebra, Logic & Discrete Mathematics
Niš, April 14–16, 1995.

DETERMINANTAL REPRSENTATION OF GENERALIZED INVERSES OVER INTEGRAL DOMAINS

Predrag Stanimirović and Miomir Stanković

ABSTRACT. In this paper we introduce a general form of determinantal representation of generalized inverses, for matrices which admit rank factorizations over an integral domain. We investage necessary and sufficient conditions for existence of generalized inverses. Finally, we examine correlations between the minors of generalized inverses and minors of the source matrix.

1. Introduction and preliminaries

We consider an integral domain $\mathbb{I}$ with an involution $\lambda : a \mapsto \overline{a}$. For an $m \times n$ matrix A let $\alpha = \{\alpha_1, \dots, \alpha_r\}$ and $\beta = \{\beta_1, \dots, \beta_r\}$ be the subsets of $\{1, \dots, m\}$ and $\{1, \dots, n\}$, respectively. Then $A\begin{pmatrix} \alpha_1 \dots \alpha_r \\ \beta_1 \dots \beta_r \end{pmatrix} = |A^\alpha_\beta|$ denotes the minor of A determined by the rows indexed by α and the columns indexed by β. If $\alpha = \{1, \dots, m\}$, then $|A^\alpha_\beta|$ can be simply denoted $|A_\beta|$, and similarly if $\beta = \{1, \dots, n\}$, then $|A^\alpha_\beta|$ can be denoted by $|A^\alpha|$. Also, the algebraic complement of $|A^\alpha_\beta|$ is defined by

$$\frac{\partial}{\partial a_{ij}} |A^\alpha_\beta| = A_{ij}\begin{pmatrix} \alpha_1 \dots \alpha_{p-1}\ i\ \alpha_{p+1} \dots \alpha_r \\ \beta_1 \dots \beta_{q-1}\ j\ \beta_{q+1} \dots \beta_r \end{pmatrix} = (-1)^{p+q} A\begin{pmatrix} \alpha_1 \dots \alpha_{p-1}\ \alpha_{p+1} \dots \alpha_r \\ \beta_1 \dots \beta_{q-1}\ \beta_{q+1} \dots \beta_r \end{pmatrix}.$$

The r-th compound matrix $C_r(A)$ of A is a matrix of order $\binom{m}{r} \times \binom{n}{r}$ defined on the following way. The rows of $C_r(A)$ are indexed by the r-element subsets of $\{1, \dots, m\}$; the columns are indexed by the r-element subsets of $\{1, \dots, n\}$, and the (α, β) entry of $C_r(A)$ is defined as $|A^\alpha_\beta|$.

For $A \in \mathbb{C}^{m\times n}$ consider the following Penrose [13] equations in X (where $A^* = (\overline{A})^T$):

(1) $AXA = A$ (2) $XAX = X$ (3) $(AX)^* = AX$ (4) $(XA)^* = XA$.

If $m = n$ we add

$$(5) \qquad AX = XA\,.$$

For a subset $\mathcal{S}$ of $\{1, 2, 3, 4, 5\}$ the set of matrices G obeying the conditions represented in $\mathcal{S}$ will be denoted by $A\{\mathcal{S}\}$. A matrix $G \in A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and is denoted by $A^{(\mathcal{S})}$.

The starting point of the investigations of this paper is the determinantal representation of generalized inverses of complex matrices [1, 4, 7, 15, 16]. Also, we use the determinantal representation of the *Moore-Penrose inverse*, the *weighted Moore-Penrose inverse* and the *group inverse* over integral domain ([2], [11] and [12]). Let us recall

Theorem 1.1. [2] *Let A be an $m \times n$ matrix of rank r over $\mathbb{I}$, and let $A = PQ$ be a rank factorization of A. Then the following conditions are equivalent;*

(i) *A has a Moore-Penrose inverse.*
(ii) *P^*P and QQ^* are invertible over $\mathbb{I}$.*
(iii) *$C_r(A)$ has a Moore-Penrose inverse.*
(iv) $\sum\limits_{\alpha,\beta} |\overline{A}_\beta^\alpha| \cdot |A_\beta^\alpha|$ *is invertible in $\mathbb{I}$, where α, β run over r-element subsets of $\{1, \dots, m\}$ and $\{1, \dots, n\}$ respectively.*

Furthermore, the Moore-Penrose inverse $G = (g_{ij})$, if it exists is given by $G = A^\dagger = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^*$ and

$$g_{ij} = \left(\sum_{\gamma,\delta} |\overline{A}_\delta^\gamma|\,|A_\delta^\gamma|\right)^{-1} \cdot \sum_{\alpha:j\in\alpha;\ \beta:i\in\beta} |\overline{A}_\beta^\alpha| \frac{\partial}{\partial a_{ji}} |A_\beta^\alpha|\,.$$

Similar results are obtained for the weighted Moore-Penrose inverse $A_{M,N}^\dagger$ [12], which satisfies equations (1), (2) and

$$(6) \quad (MAX)^* = MAX \qquad (7) \quad (NXA)^* = NXA.$$

Theorem 1.2. [12] *Let A be an $m \times n$ matrix of rank r over $\mathbb{I}$, and let $A = PQ$ be a rank factorization of A. Then the following conditions are equivalent:*

(i) *A has a weighted Moore-Penrose inverse with respect to M and N.*
(ii) *P^*MP and $QN^{-1}Q^*$ are invertible over $\mathbb{I}$.*
(iii) *$C_r(A)$ has a weighted Moore-Penrose inverse with respect to $C_r(M)$ and $C_r(N)$.*

(iv) $\sum_{\alpha,\beta} |(N^{-1}A^*M)^{\beta}_{\alpha}||A^{\alpha}_{\beta}| = \sum_{\alpha,\beta} |(\overline{MAN^{-1}})^{\alpha}_{\beta}||A^{\alpha}_{\beta}|$ *is invertible in* $\mathbb{I}$.

Determinantal representation of the group inverse over an integral domain $\mathbb{I}$ is introduced in [11]:

Theorem 1.3. [11] *Let A be an $m \times n$ matrix of rank r over $\mathbb{I}$, and let $A = PQ$ be a rank factorization of A. Then the following conditions are equivalent:*

(i) *A has a group inverse.*
(ii) *$C_r(A)$ has a group inverse.*
(iii) $\sum_{\gamma} |A^{\gamma}_{\gamma}|$ *is invertible in* $\mathbb{I}$.

Furthermore, the group inverse $G = (g_{ij})$, if it exists, is given by

$$g_{ij} = \left(\sum_{\gamma} |A^{\gamma}_{\gamma}|\right)^{-2} \cdot \sum_{\alpha:j\in\alpha;\,\beta:i\in\beta} |(A^T)^{\alpha}_{\beta}| \frac{\partial}{\partial a_{ji}} |A^{\alpha}_{\beta}|.$$

The main results of this paper are:

(1) Generalization of the *algebraic complement* and determinant, and obtain general form of the *determinantal representation* for different classes of generalized inverses: the Moore-Penrose inverse, the weighted Moore-Penrose inverse, the group inverse and the left(right) inverses. In this way, we generalize the results obtained in [16].

(2) Necessary and sufficient conditions for existence of the *general determinantal representation*, and partially, existence of the *Moore-Penrose* inverse, the *weighted Moore-Penrose* inverse and the group inverse.

(3) Correlations between the minors of diferent classes of generalized inverses and minors of the given matrix.

2. Determinantal representations of generalized inverses

First we generalize the concepts of determinant and *algebraic complement* (see [1, 2, 4, 7, 8, 11, 12, 15, 16, 17]).

Definition 2.1. The generalized determinant of an $m \times n$ matrix A of rank r, denoted by $N_{(R,r)}(A)$, is defined by

$$N_{(R,r)}(A) = \sum_{\alpha,\beta} |\overline{R}^{\alpha}_{\beta}||A^{\alpha}_{\beta}|, \tag{2.1}$$

where R is an $m \times n$ matrix satisfying condition

$$rank(AR^*) = rank(R^*A) = rank(A). \tag{2.2}$$

Note that (2.2) is satisfied if and only if $rank(R) \geq rank(A) = r$.

Definition 2.2. Let A, R be $m \times n$ matrices over $\mathbb{I}$ and let R satisfies (2.2). The *generalized algebraic complement* of A corresponding to a_{ij} is defined by

$$A_{ij}^{(\dagger,R)} = \sum_{\alpha:j\in\alpha;\beta:i\in\beta} |\overline{R}_{\beta}^{\alpha}| \frac{\partial}{\partial a_{ji}} |A_{\beta}^{\alpha}|. \tag{2.3}$$

In a similar way can be generalized the notion of *adjoint matrix*.

Definition 2.3. Matrix whose elements are equal to $A_{ij}^{(\dagger,R)}$ we denote by $adj^{(\dagger,R)}(A)$, and we write it as the generalized adjoint matrix of A, corresponding to R.

Finally, in the following definition we introduce the *general determinantal representation* of generalized inverses over an integral domain.

Definition 2.4. Let given an $m\times n$ matrix A of rank r over $\mathbb{I}$ and $m\times n$ matrix R which satisfies condition (2.2). General determinantal representation of generalized inverses of A is defined by

$$A^{(\dagger,R)} = \left(N_{(R,r)}(A)\right)^{-1} \cdot adj^{(\dagger,R)}(A). \tag{2.4}$$

For two full-rank matrices A and R we have:

Lemma 2.1. *If A is an $m \times n$ matrix of full-rank and matrix R has the same dimensions and rank, then:*

(i) $N_{(R,r)}(A) = \begin{cases} |AR^*|, & r = m \\ |R^*A|, & r = n. \end{cases}$

(ii) $A_{ij}^{(\dagger,R)} = \begin{cases} (R^*adj(AR^*))_{ij}, & r = m \\ (adj(R^*A)R^*)_{ij}, & r = n. \end{cases}$

(iii) $A^{(\dagger,R)} = \begin{cases} R^*(AR^*)^{-1}, & r = m \\ (R^*A)^{-1}R^*, & r = n. \end{cases}$

Proof. (i) Follows from the Cauchy-Binet Theorem.

(ii) The relation $(A^*adj(AA^*))_{ij} = \sum_{\beta:i\in\beta} |\overline{A}_{\beta}| \frac{\partial}{\partial a_{ji}} |A_{\beta}|$ is obtained in [1], [6]. The result (*ii*) can be obtained in a similar way, substituting the matrix A^* by the matrix R^*.

(*iii*) It is implied by (*i*) and (*ii*). □

Now we investage main properties of the *generalized adjoint matrix*, *generalized algebraic complement* and *generalized determinant*.

Lemma 2.2. *Let $A = PQ$ be a full-rank factorization of an $m \times n$ matrix A of rank r, R_1 be an $r \times n$ matrix of rank r and R_2 be an $m \times r$ matrix of rank r. Generalized adjoint matrix satisfies the following:*

$$adj^{(\dagger,R_1)}(Q) \cdot adj^{(\dagger,R_2)}(P) = adj^{(\dagger,R_2R_1)}(A).$$

Proof. For $1 \le i \le n$, $1 \le j \le m$ we obtain the following representation for (i,j)-th element of the matrix product $adj^{(\dagger,R_1)}(Q) \cdot adj^{(\dagger,R_2)}(P)$:

$$\left(adj^{(\dagger,R_1)}(Q) \cdot adj^{(\dagger,R_2)}(P)\right)_{ij} = \sum_{k=1}^{r} Q_{ik}^{(\dagger,R_1)} \cdot P_{kj}^{(\dagger,R_2)} =$$

$$= \sum_{k=1}^{r} \sum_{\beta:i\in\beta} |(\overline{R_1})_\beta| \frac{\partial}{\partial q_{ki}} |Q_\beta| \cdot \sum_{\alpha:j\in\alpha} |(\overline{R_2})^\alpha| \frac{\partial}{\partial p_{jk}} |P^\alpha| =$$

$$= \sum_{\alpha:j\in\alpha;\beta:i\in\beta} |(\overline{R_2R_1})_\beta^\alpha| \cdot \sum_{k=1}^{r} \frac{\partial}{\partial p_{jk}} |P^\alpha| \frac{\partial}{\partial q_{ki}} |Q_\beta| .$$

Using the Cauchy-Binet formula we get

$$\sum_{k=1}^{r} \frac{\partial}{\partial p_{jk}} |P^\alpha| \frac{\partial}{\partial q_{ki}} |Q_\beta| = \frac{\partial}{\partial a_{ji}} |A_\beta^\alpha|,$$

which implies

$$\left(adj^{(\dagger,R_1)}(Q) \cdot adj^{(\dagger,R_2)}(P)\right)_{ij} = \left(adj^{(\dagger,R_2R_1)}(A)\right)_{ij}. \quad \square$$

Similarly, the following lemma can be proved:

Lemma 2.3. *If $A = PQ$ is a full-rank factorization of an $m \times n$ matrix A of rank r, R_1 and R_2 are matrices of apropriate sizes, and satisfy $rank(QR_1) = rank(R_2P) = r$, then the generalized determinant satisfies*

$$N_{(R_1,r)}(Q) \cdot N_{(R_2,r)}(P) = N_{(R_2R_1,r)}(A).$$

Proof. From Lemma 2.1 and the Cauchy-Binet theorem we obtain:

$$N_{(R_1,r)}(Q) \cdot N_{(R_2,r)}(P) = |QR_1^*| \cdot |R_2^*P| =$$

$$= \sum_{\beta} |(\overline{R_1})_\beta| |Q_\beta| \cdot \sum_{\alpha} |(\overline{R_2})^\alpha| |P^\alpha| =$$

$$= \sum_{\alpha,\beta} |(\overline{R_2R_1})_\beta^\alpha| |A_\beta^\alpha| = N_{(R_2R_1,r)}(A). \quad \square$$

From Lemma 2.2 and Lemma 2.3 we have.

Corollary 2.1. *If $A = PQ$ is a full-rank factorization of an $m \times n$ matrix A of rank r, R_1 and R_2 satisfy conditions from Lemma 2.2 and Lemma 2.3, then the generalized determinantal representation satisfies*

$$Q^{(\dagger,R_1)} \cdot P^{(\dagger,R_2)} = A^{(\dagger,R_2R_1)}.$$

The following theorem showes the properties of determinantal representation of generalized inverses.

Theorem 2.1. *Let A, R be $m \times n$ matrices of rank r and $A = PQ$ be a full-rank factorization of A. Then:*

(i) $A^\dagger = Q^{(\dagger,Q)}P^{(\dagger,P)} = A^{(\dagger,A)}$;

(ii) $A^\dagger_{M,N} = Q^{(\dagger,QN^{-1})}P^{(\dagger,MP)} = A^{(\dagger,MAN^{-1})}$;

(iii) $A^\# = Q^{(\dagger,Q^*)}P^{(\dagger,P^*)} = A^{(\dagger,A^*)}$;

(iv) $A^{-1} = A^{(\dagger,R)}$, *for arbitrary regular R and regular A;*

(v) $A^{(\dagger,R)}$ *represents the left(right) inverses, for a full-rank matrix A.*

Proof. (i) Follows from $A^\dagger = Q^\dagger P^\dagger$ and Lemma 2.1.

(ii) It is implied by $A^\dagger_{M,N} = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((MP)^*P)^{-1}(MP)^*$ [12], [17] and Lemma 2.1. Furthermore, from Definition 2.4, we obtain the following determinantal representation for $A^\dagger_{M,N}$, (see [17]) and [12, Theorem 8]:

$$\left(A^\dagger_{M,N}\right)_{ij} = \left(\sum_{\gamma,\delta} |\,(\overline{MAN^{-1}})^\gamma_\delta\,|\,|\,A^\gamma_\delta\,|\right)^{-1} \cdot \sum_{\alpha,\beta} |\,(\overline{MAN^{-1}})^\alpha_\beta\,|\,\frac{\partial}{\partial a_{ji}}\,|\,A^\alpha_\beta\,|.$$

(iii) Follows from Theorem 1.3 and

$$\left(\sum_\gamma |\,A^\gamma_\gamma\,|\right)^{-2} \cdot \sum_{\alpha:j\in\alpha\,;\,\beta:i\in\beta} |\,(A^T)^\alpha_\beta\,|\,\frac{\partial}{\partial a_{ji}}\,|\,A^\alpha_\beta\,| =$$

$$= \left(\sum_{\gamma,\delta} |\,(A^T)^\gamma_\delta\,|\,|\,A^\gamma_\delta\,|\right)^{-1} \cdot \sum_{\alpha:j\in\alpha\,;\,\beta:i\in\beta} |\,(A^T)^\alpha_\beta\,|\,\frac{\partial}{\partial a_{ji}}\,|\,A^\alpha_\beta\,|.$$

(v) For example, suppose $r = m$. Using the Laplace's development for the square minors A_β we get

$$N_{(R,m)}(A) = \sum_\beta \overline{R}_\beta \left[\sum_{k=1}^{r} a_{ij_k}\frac{\partial}{\partial a_{ij_k}}\,|\,A_\beta\,|\right] =$$

$$= \sum_{l=1}^{n} a_{il}\left[\sum_{\beta,l\in\beta} \overline{R}_\beta \frac{\partial}{\partial a_{il}}\,|\,A_\beta\,|\right] = \sum_{l=1}^{n} a_{il}A^{(R,m)}_{li}.$$

For $p \neq q$, $1 \leq p,q \leq m$, substituting in the minors of A, the q-th row by the p-th row, and using $N_{(R,m)}(A) = \sum_{\beta} \overline{R}_\beta A_\beta = 0$, in the same way we prove $\sum_{l=1}^{n} a_{pl} A_{lq}^{(R,m)} = 0$. Hence, $g_{ij} = \delta_{ij} N_{(R,m)}(A)$, and consequently $A \cdot A^{(\dagger,R)} = I_m$, for arbitrary R. This means that $A^{(\dagger,R)}$ represents the class of *right inverses* of the full-rank matrix A in the case $r = m \leq n$.

On the other hand, it can be proved that $A^{(\dagger,R)}$ represents the class of *left inverses* of A, in the case $r = n \leq m$. □

In the following theorem we examine existence of the general determinantal representation.

Theorem 2.2. *Let A, R be $m \times n$ matrices of rank r over $\mathbb{I}$, $A = PQ$ be a full-rank factorization of A and $R = ST$ be a full-rank factorization of R. Then the following conditions are equivalent:*

(i) *$A^{(\dagger,R)}$ exists.*
(ii) *QT^* and S^*P are invertible matrices in $\mathbb{I}$.*
(iii) *$N_{(T,r)}(Q)$ and $N_{(S,r)}(P)$ are invertible in $\mathbb{I}$.*
(iv) *$N_{(R,r)}(A)$ is invertible in $\mathbb{I}$.*

Proof. (i)⇒(ii): If $A^{(\dagger,R)}$ exists, from Corollary 2.1 and Lemma 2.1, we get

$$A^{(\dagger,R)} = A^{(\dagger,ST)} = Q^{(\dagger,T)} \cdot P^{(\dagger,S)} = T^*(QT^*)^{-1}(S^*P)^{-1}S^*.$$

From $AA^{(\dagger,R)}A = A$ follows

$$QT^*(QT^*)^{-1}(S^*P)^{-1}S^*P = I,$$

which implies (ii).

(ii)⇒(i): If QS^* and T^*P are invertible, from Lemma 2.1 and Corollary 2.1, we conclude

$$T^*(QT^*)^{-1}(S^*P)^{-1}S^* = Q^{(\dagger,T)} \cdot P^{(\dagger,S)} = A^{(\dagger,ST)} = A^{(\dagger,R)}.$$

(ii)⇔(iii) A square matrix over a ring $\mathbb{I}$ is invertible if and only if its determinant is invertible in $\mathbb{I}$ [9], [10]. Hence, QT^* and S^*P are invertible matrices if and only if $|QT^*|$ and $|S^*P|$ are invertible in $\mathbb{I}$. Finally, from Lemma 2.1 we obtain

$$|QT^*| = N_{(T,r)}(Q), \qquad |S^*P| = N_{(S,r)}(P).$$

(iii)⇔(iv) An application of Lemma 2.3 implies

$$N_{(T,r)}(Q) \cdot N_{(S,r)}(P) = N_{(ST,r)}(A) = N_{(R,r)}(A).$$

Therefore, $N_{(R,r)}(A)$ is invertible if and only if both $N_{(S,r)}(P)$ and $N_{(T,r)}(Q)$ are invertible. □

Corollary 2.2. *Let A be an $m \times n$ matrix of rank r, and $A = PQ$ be its full-rank factorization. The following conditions are equivalent:*

(i) $A^{\dagger}$ *exists.*
(ii) QQ^* *and* P^*P *are invertible matrices in* $\mathbb{I}$.
(iii) $N_{(Q,r)}(Q)$ *and* $N_{(P,r)}(P)$ *are invertible in* $\mathbb{I}$.
(iv) $N_{(A,r)}(A)$ *is invertible in* $\mathbb{I}$.

Corollary 2.3. *For an $m \times n$ matrix A of rank r the following conditions are equivalent:*

(i) $A^{\dagger}_{M,N}$ *exists.*
(ii) P^*MP *and* $QN^{-1}Q^*P$ *are invertible matrices in* $\mathbb{I}$.
(iii) $N_{(QN^{-1},r)}(Q)$ *and* $N_{(PM,r)}(P)$ *are invertible in* $\mathbb{I}$.
(iv) $N_{(MAN^{-1},r)}(A)$ *is invertible in* $\mathbb{I}$.

Corollary 2.4. *Let A be a square matrix of order n, rank r and $rank(QP) = r$. Then the following conditions are equivalent:*

(i) $A^{\#}$ *exists.*
(ii) QP *is invertible matrix in* $\mathbb{I}$.
(iii) $N_{(P^*,r)}(Q)$ *and* $N_{(Q^*,r)}(P)$ *are invertible in* $\mathbb{I}$.
(iv) $N_{(A^*,r)}(A)$ *is invertible in* $\mathbb{I}$.
(v) $\sum\limits_{\gamma} | A^{\gamma}_{\gamma} |$ *is invertible in* $\mathbb{I}$.

Proof. Note that $(iv) \Leftrightarrow (v)$ follows from

$$\left(N_{(A^*,r)}(A)\right)^{-1} = \left(\sum_{\gamma} | A^{\gamma}_{\gamma} |\right)^{-2}. \quad \square$$

In the following part of this section we represent minors of generalized of A, in terms of minors of A and arbitrary matrix R, which satisfies the condition (2.2).

Theorem 2.3. *Let A, R be matrices of type $m \times n$ whose rank is r and let $A = PQ$ be a full-rank factorization of A. Then for all α, β we have*

$$\begin{aligned} | (A^{(\dagger,R)})^{\alpha}_{\beta} | &= \left(\sum_{\gamma,\delta} | A^{\gamma}_{\delta} | \, | (\overline{R})^{\gamma}_{\delta} |\right)^{-1} \cdot | (\overline{R})^{\alpha}_{\beta} | = \\ &= \left(N_{(R,r)}(A)\right)^{-1} | (\overline{R})^{\alpha}_{\beta} | . \end{aligned} \tag{2.5}$$

Proof. In ([2], Theorem 3.) is proved the following relation for the reflexive generalized inverses $G = (g_{ij})$ of A:

$$g_{ij} = \sum_{\alpha,\beta} | G^{\alpha}_{\beta} | \frac{\partial}{\partial a_{ji}} | A^{\alpha}_{\beta} | .$$

The proof can be completed using that $A^{(\dagger,R)}$ is a reflexive generalized inverse and

$$\left(A^{(\dagger,R)}\right)_{ij} = \left(\sum_{\gamma,\delta} | A^{\gamma}_{\delta} | | (\overline{R})^{\gamma}_{\delta} | \right)^{-1} \cdot \sum_{\alpha,j\in\alpha\ ,\delta,i\in\delta} | (\overline{R})^{\alpha}_{\beta} | \frac{\partial}{\partial a_{ji}} | A^{\alpha}_{\beta} | . \quad \square$$

In particular, the last theorem and Theorem 2.1 imply:

Corollary 2.5. *Let A, R be matrices of type $m \times n$ and rank r over $\mathbb{I}$. Then for all α, β is valid:*

$$| A^{\#}{}^{\alpha}_{\beta} | = \left(\sum_{\gamma} | A^{\gamma}_{\gamma} | \right)^{-2} | A^{\beta}_{\alpha} | = \left(N_{(A^*,r)}(A)\right)^{-1} | (A^T)^{\alpha}_{\beta} | ;$$

$$| A^{\dagger}{}^{\alpha}_{\beta} | = \left(\sum_{\gamma,\delta} | A^{\gamma}_{\delta} | | \overline{A}^{\gamma}_{\delta} | \right)^{-1} | \overline{A}^{\alpha}_{\beta} | = \left(N_{(A,r)}(A)\right)^{-1} | A^{\alpha}_{\beta} | ;$$

$$| (A^{\dagger}_{M,N})^{\alpha}_{\beta} | = \left(\sum_{\gamma,\delta} | A^{\gamma}_{\delta} | | (\overline{MAN^{-1}})^{\gamma}_{\delta} | \right)^{-1} | (\overline{MAN^{-1}})^{\alpha}_{\beta} | =$$

$$= \left(N_{(MAN^{-1},r)}(A)\right)^{-1} | (\overline{MAN^{-1}}{}^{\alpha}_{\beta} | .$$

Proof. If $m = n$, and $R = A^*$, in (2.5) we obtain $G = A^{\#}$. Similarly, for $R = A$ we obtain $G = A^{\dagger}$, and $G = A^{\dagger}_{M,N}$ is induced by $R = MAN^{-1}$. $\square$

Note that correlations between minors of A and corresponding minors of the Moore-Penrose and group inverse are proved (in another way) in [2] and [11], respectively.

References

[1] Arghiriade, E. e Dragomir, A., *Une nouvelle definition de l'inverse generalisee d'une matrice*, Rendiconti dei Lincei, serir XIII Sc. fis. mat. e nat. **35** (1963), 158–165.

[2] Bapat, R.B.; Bhaskara, K.P.S. and Manjunatha Prasad, K., *Generalized inverses over integral domains*, Linear Algebra Appl. **140** (1990), 181–196.

[3] Ben-Israel, A. ; Grevile, T.N.E., *Generalized Inverses: Theory and Applications*, Wiley-Interscience, New York, 1974.

[4] Ben-Israel, A., *Generalized inverses of matrices: a perspective of the work of Penrose*, Math. Proc. Camb. Phil. Soc. **100** (1986), 407–425.

[5] Bhaskara Rao, K.P.S., *On generalized inverses of matrices over integral domains*, Linear Algebra Appl. **49** (1983), 179–189.

[6] Gabriel, R., *Das verallgemeinerte inverse einer matrix ü ber einem beliebigen Körper - analytish betrachtet*, J. Rewie Ansew Math. **244(V)** (1968), 83–93.

[7] Gabriel, R., *Extinderea complementilor algebrici generalizati la matrici oarecare*, Studii si cercetari matematice **17 -Nr. 10** (1965), 1566–1581.

[8] Gabriel, R., *Das allgemeine element des verallgemeinerte inverse von Moore-Penrose*, Rev. Roum. Math. Pures et appl. **37 No 6** (1982), 689–702.

[9] Kesler, S.S., Krupnik N.J., *On invertibility matrices with elements from the ring*, Uchennie Zapiski, Kishinjev **91** (1967), 51–54. (in Russian)

[10] Krupnik N.J., *On normal solvability problem and singular and integral equations index*, Uchennie Zapiski, Kishinjev **82** (1965), 3–7. (in Russian)

[11] Prasad, K.M.; Bhaskara Rao, K.P.S. and Bapat, R.B., *Generalized inverses over integral domains. II. Group inverses and Drazin inverses*, Linear Algebra Appl. **146** (1991), 31–47.

[12] Prasad, K.M. and Bapat, R.B., *The Generalized Moore-Penrose inverse*, Linear Algebra Appl. **165** (1992), 59–69.

[13] Penrose, R., *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. **51** (1955), 406–413.

[14] Radić, M., *Some contributions to the inversions of rectangular matrices*, Glasnik Matematički **1 (21) -No. 1** (1966), 23–37.

[15] Stanimirović, P. and Stanković, M., *Generalized algebraic complement and Moore-Penrose inverse*, FILOMAT **8** (1994), 57–64.

[16] Stanimirović, P., *Computing pseudoinverses using minors of an arbitrary matrix*, FILOMAT (1994).

[17] Stanimirović, P. and Stanković, M., *Determinantal representation of weighted Moore -Penrose inverse*, Matematički Vesnik **46** (1994), 41–50.

University of Niš, Philosophical Faculty, Department of Mathematics, Ćirila i Metodija 2, 18000 Niš, Yugoslavia

University of Niš, Faculty of Occupational Safety, Čarnojevića 10a, 18000 Niš, Yugoslavia