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SEMIDIRECT PRODUCTS OF SOME SEMIGROUPS

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ABSTRACT. Regular semidirect products of semigroups have been studied by G. B. Preston [5], orthodox by T. Saito [6] and E-inversive by F. Cabro and M. Micoli [3] e.t.c. In the present paper we study semidirect products belonging to one of the following classes of semigroups: π-regular, semilattices of Archimedean semigroups, Archimedean, left Archimedean, right Archimedean and other. At the end of the paper we give a new proof of the Wilkinsons theorem.

Let T and S be semigroups and let $\theta : S \to EndT$ be an antimorphism of S into the endomorphism semigroup of T. For $s \in S$, $t \in T$ we denote $t(s\theta)$ by t^s. If $t, t_1 \in T$, $s, s_1 \in S$ then $(tt_1)^s = t^s t_1^s$ and $(t^s)^{s_1} = t^{s_1 s}$. By semidirect product of T and S with structural mapping θ we mean the set $T \times S$ with the following multiplication:

$$(t,s)(t_1,s_1) = (tt_1^s, ss_1), \text{B. H. Neumann, [4]}.$$

This product will be denoted by $T_\theta \times S$.

By $\mathbf{Z}^+$ we denote the set of all positive integers. If $a, b \in S$, then $a \mid_l b$ if $xa = b$ for some $x \in S^1$. A semigroup S is π-regular if for every $a \in S$ there exist $n \in \mathbf{Z}^+$ such that $a^n \in a^n S a^n$. By $E(S)$ we denote the set of all idempotents of S. A semigroup S is Archimedean (left Archimedean, right Archimedean) if for all $a, b \in S$ there exists $n \in \mathbf{Z}^+$ such that $a^n \in SbS$ ($a^n \in Sb, a^n \in bS$). For undefined notions and notations we refer to [1].

Regular semidirect products of semigroups have been studied by G. B. Preston [5], orthodox by T. Saito [6] and E-inversive by F. Cabro and M. Micoli [3] e.t.c. In the present paper we study semidirect products belonging to one of the following classes of semigroups: π-regular, semilattices of Archimedean semigroups, Archimedean, left Archimedean, right

Archimedean and other. At the end of the paper we give a new proof of the Wilkinsons theorem.

Theorem 1. *The following conditions are equivalent on the semigroup* $U = T_\theta \times S$.

(i) U *is a* π*-regular,*

(ii) S *is a* π*-regular, and for all* $t \in T$ *and* $s \in S$ *there exist* $x \in T$, $y \in S$ *and* $m \in \mathbf{Z}^+$ *such that:*

$$(1)\qquad tt^s t^{s^2} \dots t^{S^{m-1}} x^{s^m} (tt^s t^{s^2} \dots t^{S^{m-1}})^{s_y^m} = tt^s t^{s^2} \dots t^{S^{m-1}},$$

(iii) S *is a* π*-regular, and for every* $s \in S$ *there exists* $y \in S$ *and* $m \in \mathbf{Z}^+$ *such that for* $e = s^m y \in E(S)$ *and for every* $t \in T$ *it holds:*

$$(2)\qquad tt^s t^{s^2} \dots t^{S^{m-1}} \in T(tt^s t^{s^2} \dots t^{S^{m-1}})^e$$

and

$$(3)\qquad (tt^s t^{s^2} \dots t^{s^{m-1}})^e \in (tt^s t^{s^2} \dots t^{S^{m-1}})^e t^e (tt^s t^{s^2} \dots t^{S^{m-1}})^e.$$

Proof. (i) $\Rightarrow$ (ii) Let U be π-regular. Then for $(t,s) \in U$ there exist $(x.y) \in U$ and $m \in \mathbf{Z}^+$ such that:

$$(4)\qquad (tt^s \dots t^{s-1} x^{s^m} (tt^s \dots t^{s^{m-1}})^{s^m y}, s^m y s^m) = (tt^s \dots t^{s^{m-1}}, s^m)$$

whence it follows (ii).

(ii) $\Rightarrow$ (iii) Let (ii) hold. By (1) it follows (2), since for $s \in T$ there exist $y \in S$ and $m \in \mathbf{Z}^+$ such that $s^m y s^m = s^m$ and $s^m y = e \in E(S)$. Moreover, acting on (1) with $e = s^m y$ gives

$$(tt^s \dots t^{S^{m-1}})^e = (tt^s \dots t^{s^{m-1}})^e (x^{s^m})^e (tt^s \dots t^{S^{m-1}})^e,$$

so (3) holds.

(iii) $\Rightarrow$ (i). Suppose that (iii) hold and let $(t,s) \in U$. Then there exist $y \in S$ and $m \in \mathbf{Z}^+$ such that $s^m y s^m = s^m$. Also by (3) there exists $u \in T$ such that:

$$(tt^s \dots t^{S^{m-1}})^e = (tt^s \dots t^{s^{m-1}})^e u^e (tt^s \dots t^{S^{m-1}})^e$$

and by (2) we obtain

$$(tt^s \dots t^{S^{m-1}})^e = v(tt^s \dots t^{s^{m-1}})^e$$

for some $v \in T$. Let $x = u^y$. Then

$$\begin{aligned} &tt^s \dots t^{s^{m-1}} x^{s^m} (tt^s \dots t^{s^{m-1}})^{s^m y} = \\ &v(tt^s \dots t^{s^{m-1}})^e (u^y)^{s^m} (tt^s \dots t^{s^{m-1}})^e = \\ &v(tt^s \dots t^{s^{m-1}})^e u^e (tt^s \dots t^{s^{m-1}})^e = \\ &v(tt^s t^{s^2} \dots t^{s^{m-1}})^e = \\ &tt^s \dots t^{s^{m-1}}. \end{aligned}$$

This proves (4) and completes th proof of π-regularity of U. $\square$

Note that for $m = 1$ the condition (2) becomes $t \in Tt^e$, and (3) means that T^e is a regular semigroup, so as a consequence we obtain Theorem 6 [5].

Corollary 1. (G. B. Preston [5]) *Let $U = T_\theta \times S$ be a semidirect product of semigroups. Then U is regular if and only if:*

(i) *S is regular, and*

(ii) *for all s in S there exists $y \in S$ such that for $e = sy \in E(S)$ T^e is regular and for every t in T, $t \in Tt^e$.*

Theorem 2. *Let $U = T_\theta \times S$ be semidirect product of semigroups. Then U is a semilattice of Archimedean semigroups if and only if next conditions holds:*

(i) *S is a semilattice of Archimedean semigroups and*

(ii) $(\forall s, s_1 \in S)(\forall t, t_1 \in T)(\exists u \in T)(\exists y, v \in S)(\exists n \in \mathbf{Z}^+)((ss_1)^n = ys^2v \Rightarrow (tt^s u^{s^2})^y \mid_l \prod_{i=0}^{n-1} (tt_1^s)^{(ss_1)^i}$.

Proof. Let U be semilattice of Archimedean semigroups. Then for all (t, s), $(t_1, s_1) \in U$ there exist $(x, y), (u, v) \in U$ and $n \in \mathbf{Z}^+$, such that:

$$((t, s)(t_1, s_1))^n = (x, y)(t, s)^2(u, v).$$

From this we can obtain (i) and (ii).

Conversely, suppose that conditions (i) and (ii) holds. Then by (i) and Theorem 1 from [2] it follows that for arbitrrary $s, s_1 \in S$ exist $y, v \in S$ and $n \in \mathbf{Z}^+$ such that

$$(ss_1)^n = ys^2v \tag{5}$$

From (ii) and (5) we can conclude tat there exists $x \in T$ such that:

$$tt_1^s(tt_1)^{ss_1} \dots (tt_1)^{(ss_1)^{n-1}} = x(tt^s u^{s^2})^y = x(tt^s)^y u^{ys^2}. \tag{6}$$

Consequently, for all $(t,s),(t_1,s_1) \in U$ there exist $(x,y),(u,v) \in T$ and $n \in \mathbf{Z}^+$ such that

$$\begin{aligned} ((t,s)(t_1,s_1))^n &= (tt_1^s, ss_1)^n \\ &= ((t,t_1^s)(tt_1^s)^{ss_1}\dots(tt_1^s)^{(ss_1)^{n-1}}, (ss_1)^n) \\ &= (x(tt^s)^y u^{ys^2}, ys^2v) \quad \text{from (5) and (6)} \\ &= (xy)(t,s)^2(u,v) \end{aligned}$$

which together with Theorem 1 from [2] implies that U is a semilattice of Archimedean semigroups. □

Theorem 3. *Let $U = T_\theta \times S$ semidirect product of semigroups. Then U is an Archimedean semigroup if and only if next conditions are fulfield:*

(i) *S is an Archimedean semigroup, and*
(ii) $(\forall t,t_1 \in T)(\forall s,s_1 \in S)(\exists u \in T)(\exists y,v \in S)(\exists k \in \mathbf{Z}^+)(s_1^k = ysv \Rightarrow (tu^s)^y \underset{l}{|} \prod_{i=o}^{k-1} t_1^{s_1^i})$.

Proof. Let U be an Archimedean semigroup. Then for all $(t,s),(t_1,s_1) \in U$ there exist $(x,y),(u,v) \in U$ and $k \in \mathbf{Z}^+$ such that:

$$(t_1,s_1)^k = (x,y)(t,s)(u,v).$$

From this immediately follows that conditions (i) and (ii) holds.

Conversely, suppose that conditions (i) and (ii) are fulfield. Then from (i) we obtain that for all $s,s_1 \in S$ there exist $y,v \in S$ and $k \in \mathbf{Z}^+$ such that

$$s_1^k = ysv, \tag{7}$$

hence, from (ii), we conclude that for all $t,t_1 \in T$ there exist $x,u \in T$ such that:

$$\prod_{i=0}^{k-1} t^{s_1^i} = t_1 t_1^{s_1} \dots t_1^{s_1^{m-1}} = x(tu^s)^y. \tag{8}$$

Consequently, for all $(t,s),(t_1,s_1) \in U$ there exist $(x,y),(u,v) \in U$ and $k \in \mathbf{Z}^+$ such that:

$$\begin{aligned} (t_1,s_1)^k &= (t_1 t_1^{s_1} \dots t_1^{s_1^{m-1}}, s_1^m) \\ &= (x(tu^s)^y, ysv) = (xt^y u^{ys}, ysv) \quad \text{by(7)and(8)} \\ &= (x,y)(t,s)(u,v), \end{aligned}$$

which means that U is an Archimedean semigroup. □

From Theorem 3, putting $k = 1$, we obtain the next corollary.

Corollary 2. *Semidirect product of semigroups $U_\theta \times S$ is a simple semigroup if and only if next conditions are fulfield:*

(i) *S is a simple semigroup, and*

(ii) $(\forall t, t_1 \in T)(\forall s, s_1 \in S)(\exists u \in T)(\exists y, v \in S)(s_1 = ysv \Rightarrow (tu^s)^y \underset{l}{|} t_1)$.

Theorem 4. *Semidirect product of semigroups $U = T_\theta \times S$ is left Archimedean semigroup if and only if next conditions are fulfield:*

(i) *S is a left Archimedean semigroup, and*

(ii) $(\forall t, t_1 \in T)(\forall s, s_1 \in S)(\exists u \in T)(\exists y, v \in S)(\exists k \in \mathbf{Z}^+)(s_1^k = ys \Rightarrow t^y \underset{l}{|} \prod_{i=0}^{k-1} t_1^{s_1^i})$.

Proof. By multiplying in U we can simply prove that U is a left Archimedean iff for all $t, t_1 \in T$, $s, s_1 \in s$ exist $x \in U$, $y \in S$ and $k \in \mathbf{Z}^+$ such that:

$$(t_1, s_1)^k = (t, t_1^{s_1} \dots t_1^{s_1^{k-1}}, s_1^k) = (xt^y, ys) = (x, y)(t, s) \tag{9}$$

holds. Suppose that U is a left Archimedean semigroup. Then from (9) we can obtain that conditions (i) and (ii) holds.

The converse of the theorem can be obtained immediately from (9). □

From Theorem 3, putting $k = 1$, we obtain next corollary.

Corollary 2. *Semidirect product of semigroups $U_\theta \times S$ is a left simple semigroup if and only if next conditions are fulfield:*

(i) *S is a left simple semigroup, and*

(ii) $(\forall t, t_1 \in T)(\forall s, s_1 \in S)(\exists y \in S)(s_1 = ys \Rightarrow t^y \underset{l}{|} t_1)$.

The proof for next result is similar as the proof of Theorem 4.

Theorem 5. *Semidirect product of semigroups $U = T_\theta \times S$ is a right Archimedean semigroup if and only if next conditions are fulfield:*

(i) *S is a right Archimedean semigroup, and*

(ii) $(\forall t, t_1 \in T)(\forall s, s_1 \in S)(\exists x \in T)(\exists k \in \mathbf{Z}^+)(t, t_1^{s_1} \dots t_1^{s_1^{k-1}} = tx^s)$.

Corollary 3. *Semidirect product of semigroups $U_\theta \times S$ is a right simple semigroup if and only if next conditions are fulfield:*

(i) *S is a right simple semigroup, and*

(ii) $(\forall t, t_1 \in T)(\forall s, s_1 \in S)(\exists x \in S)(t_1 = xt^s)$.

G. B. Preston in [5] gave the proof for Wilkinson's theorem, we shall give here one more proof for this theorem.

Theorem 6. *Semidirect product of semigroups $U_\theta \times S$ is a group if and only if T and S are groups and $S\theta \subseteq AutT$.*

Proof. Let U be group. Then by Corollaries 2 and 3 we obtain that S is a group. Let e be the identity element in group S. Then from (ii) of Corollary 2 we obtain:

$$(\forall t, t_1 \in T)(\exists x \in S)(t_1 = tx^e) \tag{10}$$

and

$$(\forall t, t_1 \in T)(\exists x \in S)(t_1 = xt^e). \tag{11}$$

Hence, from (9) we conclude that for every $u \in T$ there exists $x \in t$ such that $u = u^l x^l$ whence $u^l = (u^l x^l)^l = u^l x^l = u$. Consequently, $e\theta$ is an identity mapping. Now by (10) and (11) we obtain that equations $t_1 = tx$, $t_1 = xt$ have solutions in T, so T is a group.

Since for every $t \in T$ and $s \in S$

$$t = t^l = t^{ss^{-1}} = (t^{s^{-1}})^{s^{-1}},$$

we conclude that every mapping $s\theta$ has it's inverse mapping $s^{-1}\theta$, so $S\theta \subseteq AutT$.

Conversely, let S and T be groups and let e be identity in S, f identity in T. By streightforward verification we obtain that (f, e) is an identity of semigroup$T$ and that every $(t, s) \in U$ has it's inverse element $(t, s)^{-1} = (t^{-1}, s^{-1})$, so U is a group. □

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