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A NONSTANDARD PROOF OF STEINHAUS'S THEOREM

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ABSTRACT. We give an intuitive and easy proof of the well known Steinhaus's theorem by use of nonstandard analysis.

In [5] Hugo Steinhaus proved the following very useful result.

Theorem 1. (Hugo Steinhaus, 1920) *Let A be a set with positive Lebesgue measure $\lambda(A) = m > 0$. Then, there is an interval $[a, b]$, such that $[a, b] \subseteq A - A = \{x - y | x, y \in A\}$.*

Before we prove the theorem we introduce some notation from nonstandard analysis and prove the lemma. We make use of Loeb measure [1].

We can suppose that $A \subseteq [0,1]$. Let $T_H = \{0, \frac{1}{H}, \frac{2}{H}, \dots, 1\}$ (for $H \in {}^*N \backslash N$) be a hyperfinite interval, ${}^*\mathcal{P}(T_H)$ a set of hyperfinite subsets of T_H, $\mu_H(A) = \frac{|A|}{H}$ (for $A \in {}^*\mathcal{P}(T_H)$) counting measure, $L(\mu_H)$ Loeb measure, $[a,b]_H = \{x \in T_H | a \leq x \leq b\}$ (for $a, b \in T_H$) and $st_H : T_H \to [0,1]$ standard part map. Let $\alpha * \beta = \frac{[\alpha, \beta H]}{H}$, where $[\alpha]$ is a integer part of $\alpha \in {}^*R_{\text{fin}}$.

Lemma. *Let A be a set with positive Lebesgue measure $\lambda(A) = m > 0$. Let B be a hyperfinite set, $B \subseteq st_H^{-1}(A)$ and $\mu_H(B) > \frac{3}{10}m$. Then, there are $a \in T_H$ and $n \in N$ such that*

$$(*) \qquad \mu_H\left(B \bigcap \left[a, a + \frac{1}{n}\right]_H\right) > \frac{3}{4n}$$

Proof. Suppose that there are not a and n such that $(*)$ holds.

The set $S = \left\{n \in {}^*N | (\forall a \in T_H)\left(\mu_H\left(B \cap \left[a, a + \frac{1}{n}\right]_H\right) \leq \frac{3}{4n}\right)\right\}$ is internal and $N \subseteq S$. By over shpil there is $K \in {}^*N \backslash N$ such that $K \in S$ and $\frac{K}{H} \approx 0$.

Let $B' = \{\frac{S}{K} | B \cap [\frac{S}{K}, \frac{S+1}{K})_H \neq \emptyset\}$. Then $st_K(B') \subseteq A$, $B' \subseteq st_K^{-1}(A)$ and $\mu_K(B') \leq L(\mu_K)(st_K^{-1}(A)) = m$. According to the fact that k $\in S$ we have

$$|B| = \sum_{B\cap[\frac{S}{K},\frac{S+1}{K})_H \neq \emptyset} \left| B \cap \left[\frac{S}{K}, \frac{S+1}{K}\right)_H \right|$$

$$\leq |B'| \max_S \left| B \cap \left[\frac{S}{K}, \frac{S+1}{K}\right)_H \right| \leq |B'| \frac{3}{4}\frac{H}{K}.$$

It follows then that

$$\frac{9}{10} m \leq \mu_H(B) \leq \frac{|B'|\frac{3}{4}\frac{H}{K}}{H} = \frac{3}{4}\frac{|B'|}{K} = \frac{3}{4}\mu_K(B') \leq \frac{3}{4} m$$

a contradiction □

Proof of the theorem. Let a and n be as in lemma above. Let $I = \left[-\frac{1}{4n}, \frac{1}{4n}\right]_H$ and $C = B \cap \left[a, a + \frac{1}{n}\right]_H$. Then $I \subseteq C - C$. Otherwise, $C \cap (C + x) = \emptyset$ for some $x \in I$ and $\frac{5}{4n} \geq \mu_H(C) + \mu_H(C + x) \geq \frac{6}{4n}$. Contradiction. Finally, $st_H I \subseteq st_H(C - C) = st_H(C) - st_H(C) \subseteq A - A$. □

Now, we shall give an application of Steinhaus's theorem and method from [3] and [4].

First, we give the following definitions and theorems.

Definition 1. A function $f: R \to R$ is *measurable on* $A \subseteq R$ iff for each $r \in R \cup \{\infty\}$ a set $\{x \in R | f(x) \leq r\} \cap A$ is a measurable.

Let $L(\mu)$ be the Loeb measure obtained from counting measure μ on

$$T_{K,H} = \left\{-K, -K + \frac{1}{H}, \ldots, K - \frac{1}{H}, K\right\}$$

(for $k \in N$).

Let f be a map from $[-K, K]$ into R and let F be a interval map from $T_{K,H}$ into $^*R$.

Definition 2. The function F is a lifting of the function f if and only if

$$L(\mu)\{x \in T_{K,H} | st_H(F(x)) \neq f(st_H(x))\} = 0$$

Definition 3. The function F is a uniform lifting of the function f if and only if $st_H(F(x)) = f(st_H(x))$ for each $x \in T_{K,H}$.

The following two theorems are of great importance for example in probability theory (see [1]).

Theorem 2. (see [1]) *The function f is Lebesgue measurable if and only if it has a lifting function F.*

Theorem 3. (see [1]) *The function f is continuous if and only if it has a uniform lifting function F.*

The proofs of theorems can be found in [1].

Theorem 4. *Let $f^i(x-y) = g_i(f^1(x), f^1(y), \dots f^m(x), f^m(y), x, y)$ $(i = 1, \dots, m)$ be a system of functional equations, such that $g_i \colon R^{2m+2} \to R$ (for $i = 1, \dots, m$) are continuous functions. Let $A \subseteq R$ be a set of positive Lebesgue measure. Then, if all solutions f^i are measurable in A if follows that its are continuous at zero.*

Proof. Let B, n and H be as in proof of the last theorem. Let $h_i(x) = \begin{cases} f^i(x) & x \in A \\ 0 & x \in R \setminus A \end{cases}$ and let H_i be a lifting of h_i (Theorem 2). Then, using Theorem 1, we can define an improved lifting function, such that for each $x \in \left[-\frac{1}{4n}, \frac{1}{4n}\right]_H \subseteq B - B$

$$F^i(x) = \min\{{}^*g(H_1(y), H_1(z), \dots, H_m(y), H_m(z), y, z) | x = y - z, x, y \in B\}.$$

Then, for some $y_0, z_0 \in B$

$$\begin{aligned} stF^i(x) &= st^*g_i(H_1(y_0), H_1(z_0), \dots, H_m(y_0), H_m(z_0), y_0, z_0) \\ &= g_i(stH_1(y_0), stH_1(z_0), \dots, stH_m(y_0), stH_m(z_0), sty_0, stz_0) \\ &= g_i(h_1(sty_0), h_1(stz_0), \dots, h_m(sty_0), h_m(stz_0), sty_0, stz_0) \\ &= f^i(sty_0 - stz_0) = f^i(st(y_0 - z_0)) = f^i(stx) \end{aligned}$$

(where, we write st instead of st_H).

Hence F^i is a uniform lifting function for f^i on the interval $\left[-\frac{1}{4n}, \frac{1}{4n}\right]$ and by Theorem 3 f^i is continuous on the same interval. □

References

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