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ON HARTE'S THEOREM FOR REGULAR BOUNDARY ELEMENTS

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ABSTRACT. This paper is a paraphrase and extension on my talk given at the conference *Algebra, Logic and Dicrete Mathematics*, Niš, April 14-16, 1995, and it is inspired by Harte's theorem (Proc. Amer. Math. Soc. 99(1987), 328-330). In this paper we would like to present some results and problems connected with Harte's theorem.

1. Introduction.

Let S be a semigroup (ring) with identity. The element $a \in S$ is (*(von Neumann) regular* if $a \in aSa$. That is, there is a solution of the equation $axa = a$. These solutions are usually called *inner* or 1-*inverses* of a, and will be denoted by a^-. If in addition, $xax = x$, then we call x a *reflexive* inverse of a, and denote it by a^+. The set of all regular elements in S will be denoted by $\widehat{S}$ and it obviously includes the invertible group S^{-1} and the idempotents $S^\bullet = \{a \in S : a^2 = a\}$. An element a is *unit regular* or *decomposably regular* provided there is $b \in S^{-1}$ such that $aba = a$ ([1], [2]). It is easy to prove that

$$S^{-1}S^\bullet = S^\bullet S^{-1} = \{a \in A : a \in aS^{-1}a\}. \tag{1.0.1}$$

When A is a Banach algebra with the identity 1, Harte [14, Theorem 1.1] (see also [15], [26]) has shown that the decomposably invertible elements are the intersection of the regular elements with the closure of the invertibles (for a subset M of A let δM and $cl\, M$ denote, respectively, the boundary and the closure of M) :

$$A^{-1}A^\bullet = \widehat{A} \cap cl\,(A^{-1}). \tag{1.0.2}$$

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Let us remark that the left side in the equality (1.0.2) is purely algebraic, while the right side in (1.0.2) depends on metric properties of A. Hence, the remarkable characteristics of Harte's theorem is that it proves the equality of two different quantities. In this paper we would like to present some results and problems connected with Harte's theorem.

2. Harte's type theorems

In this section A denotes a Banach algebra with identity 1.

Theorem 2.1. *Let A be a Banach algebra with identity 1, and S be a multiplicative semigroup of A, such that $A^{-1} \subset S \subset \widehat{A}$. Then*

$$SA^{\bullet} = \widehat{A} \cap cl(S) \Longleftrightarrow SA^{\bullet} \subset \widehat{A}. \tag{2.1.1}$$

Proof. It is enough to prove $\Longleftarrow$. If $a \in \widehat{A} \cap cl(S)$ then there are $a^+ \in A$ and $b \in S$ such that $1 + (b-a)a^+ = c \in A^{-1}$. Hence $a + (b-a)a^+a = ca$, i.e., $a = (c^{-1}b)a^+a$, and $a \in SA^{\bullet}$. To prove '$\subset$', suppose that $a \in SA^{\bullet}$. Hence, $a \in \widehat{A}$, and there are $c \in S$ and $p \in A^{\bullet}$ such that $a = cp$. Set $p_n = p - 1/n$, $n = 2, 3, \ldots$, and $a_n = cp_n$. It is clear that $p_n \in A^{-1}$, $a_n \in S$ and $a_n \to a$. Hence $a \in cl(S)$. $\square$

Corollary 2.2. *Let A be a Banach algebra with identity 1, and S be a multiplicative semigroup of A, such that $A^{-1} \subset S \subset \widehat{A}$. Then*

$$A^{\bullet}S = \widehat{A} \cap cl(S) \Longleftrightarrow A^{\bullet}S \subset \widehat{A}. \tag{2.2.1}$$

Proof. By the proof of Theorem 2.1; let us only remark that now if $a \in \widehat{A} \cap cl(S)$ then there are $a^+ \in A$ and $b \in S$ such that $1 + a^+(b-a) = c \in A^{-1}$. Hence $a + aa^+(b-a) = ac$, i.e., $a = aa^+(bc^{-1})$, and $a \in A^{\bullet}S$. $\square$

Let A_l^{-1} (A_r^{-1}) denotes the semigrop of all left (right) invertible elements of A. Now we have

Corollary 2.3. *Let A be a Banach algebra with identity 1. Then*

$$A_l^{-1}A^{\bullet} = \widehat{A} \cap cl(A_l^{-1}), \tag{2.3.1}$$

$$A^{\bullet}A_r^{-1} = \widehat{A} \cap cl(A_r^{-1}), \tag{2.3.2}$$

$$A_l^{-1}A^{\bullet} \cap A^{\bullet}A_r^{-1} = \widehat{A} \cap cl(A_l^{-1}) \cap cl(A_r^{-1}). \tag{2.3.3}$$

Proof. By Theorem 2.1 and Corollary 2.2; let us only remark that $A^{-1} \subset A_l^{-1} \subset \widehat{A}$, $A_l^{-1}A^{\bullet} \subset \widehat{A}$, $A^{-1} \subset A_r^{-1} \subset \widehat{A}$, and $A^{\bullet}A_r^{-1} \subset \widehat{A}$. $\square$

Remark 2.4. Let us remark that

$$A_l^{-1}A^\bullet \subset \{a \in A : a \in aA_l^{-1}a\} \subset A^\bullet A_l^{-1} \subset \widehat{A}, \tag{2.4.1}$$

and

$$A^\bullet A_r^{-1} \subset \{a \in A : a \in aA_r^{-1}a\} \subset A_r^{-1}A^\bullet \subset \widehat{A}. \tag{2.4.2}$$

Only for a special semigroup $S \subset \widehat{A}$, say S is a subgroup of A^{-1}, one can has

$$SA^\bullet = \{a \in A : a \in aSa\} = A^\bullet S \subset \widehat{A}. \tag{2.4.3}$$

With this observation, we now come to Harte's theorem [14, Theorem 1.1].

Corollary (Harte's theorem) 2.5. *Let A be a Banach algebra with identity 1. Then*

$$A^{-1}A^\bullet = \widehat{A} \cap cl(A^{-1}). \tag{2.5.1}$$

Proof. By Theorem 2.1. □

Corollary 2.6. *Let A be a Banach algebra with identity 1, $a \in \widehat{A}$ and S be an open multiplicative semigroup of A, such that $A^{-1} \subset S \subset \widehat{A}$ and $SA^\bullet \subset \widehat{A}$. Then the following conditions are equivalent:*

(i) $a \in \delta S$,

(ii) $a = sp$, $s \in S$, $p \in A^\bullet$ *and* $sp \notin S$.

Proof. By Theorem 2.1. □

Corollary 2.7. *Let A be a Banach algebra with identity 1, $a \in \widehat{A}$ and S be an open multiplicative semigroup of A, such that $A^{-1} \subset S \subset \widehat{A}$ and $A^\bullet S \subset \widehat{A}$. Then the following conditions are equivalent:*

(i) $a \in \delta S$,

(ii) $a = ps$, $s \in S$, $p \in A^\bullet$ *and* $ps \notin S$.

Proof. By Corollary 2.2. □

Corollary 2.8. *Let A be a Banach algebra with identity 1, $a \in \widehat{A}$ and S be an open multiplicative semigroup of A, such that $A^{-1} \subset S \subset \widehat{A}$, $A^\bullet S \subset \widehat{A}$ and $A^\bullet S \subset \widehat{A}$. Then the following conditions are equivalent:*

(i) $a \in \delta S$,

(ii) $a = s_1p_1 = p_2s_2$, $s_i \in S$, $p_i \in A^\bullet$ $(i = 1, 2)$, $s_1p_1 \notin S$ *and* $p_2s_2 \notin S$.

Proof. Clear. □

Corollary 2.9. *Let A be a Banach algebra with identity 1 and $a \in \widehat{A}$. Then the following conditions are equivalent:*

(i) $a \in \delta A_l^{-1}$,

(ii) $a = sp$, $s_i \in A_l^{-1}$, $p \in A^\bullet$ *and* $p \neq 1$.

Proof. Clear. □

Corollary 2.10. *Let A be a Banach algebra with identity 1 and $a \in \widehat{A}$. Then the following conditions are equivalent:*

(i) $a \in \delta A_r^{-1}$,

(ii) $a = ps$, $s_i \in A_r^{-1}$, $p \in A^\bullet$ *and* $p \neq 1$.

Proof. Clear. □

Corollary 2.11. *Let A be a Banach algebra with identity 1 and $a \in \widehat{A}$. Then the following conditions are equivalent:*

(i) $a \in \delta A^{-1}$,

(ii) $a = s_1 p_1 = p_2 s_2$, $s_i \in A^{-1}$, $p_i \in A^\bullet$ *and* $p_i \neq 1\,(i = 1, 2)$.

Proof. Clear. □

Recall that the generalised exponential, $\mathrm{Exp}(A)$, [15, Theorem 7.11.4] form the connected component of 1 in A^{-1};

$$\mathrm{Exp}(A) = \{e^{c_1} e^{c_2} \dots e^{c_k} : c_i \in A,\ i = 1, \dots k\}.$$

It is well known that $\mathrm{Exp}(A)$ is an open subset of A and a closed normal subgroup of A^{-1}. Also, (see [19, (5.5)]

$$\mathrm{Exp}(A)A^\bullet = \{a \in A : a \in a\mathrm{Exp}(A)a\} = A^\bullet \mathrm{Exp}(A) \subset \widehat{A}.$$

For the proof of the next result see [19, Theorem 6]

Theorem (Harte-Raubenheimer) 2.12. *Let A be a Banach algebra with identity 1. Then*

$$Exp(A)A^\bullet = \widehat{A} \cap cl\, Exp(A). \tag{2.12.1}$$

Recall that $\mathrm{Exp}(A)$ is the unique open subset of A^{-1} which is a connected subgroup of A^{-1} [21, Theorem 4.4.2]. In addition to Theorem 2.12 we have

Theorem 2.13. *Let A be a Banach algebra with identity 1, and S be an open subset of A^{-1} and subgroup of A^{-1}. Then*

$$SA^\bullet = \widehat{A} \cap cl\,(S) \Longleftrightarrow A^\bullet \subset cl\, S. \tag{2.13.1}$$

Proof. It is enough to prove the $\Longleftarrow$. From $A^\bullet \subset cl\, S$ we have $SA^\bullet \subset cl\, S$. Now (2.13.1) follows from the proof of Theorem 2.1. □

Corollary 2.14. *Let A be a Banach algebra with identity 1, $a \in \widehat{A}$, S be an open subset of A^{-1} and subgroup of A^{-1}, and $A^{\bullet} \subset cl\, S$. Then the following conditions are equivalent:*

(i) $a \in \delta S$,

(ii) $a = s_1 p_1 = p_2 s_2$, $s_i \in S$, $p_i \in A^{\bullet}$ *and* $p_i \neq 1\, (i = 1, 2)$.

Proof. By Theorem 2.13. □

Recall that an element a in A is *hermitian* if $\|\exp(ita)\| = 1$ for all real t [28]. Let us denote the set of all hermitian idempotents in A by $A^{\bullet}_h$. In connection with the Moore-Penrose generalized inverse, Rakočević [22] (see also [6], [17], [18], [23], [25]) has studied the set of elements a in A for which there exists an x in A satisfying the following conditions:

(2.14.1) $$axa = a,$$

(2.14.2) $$xax = x,$$

(2.14.3) $$ax \quad \text{is hermitian},$$

(2.14.4) $$xa \quad \text{is hermitian}.$$

By [22, Lemma 2.1] there is at most one x such that equations (2.14.11), (2.14.12), (2.14.13) and (2.14.14) hold. The unique x is denoted by $a^{\dagger}$ and colled the Moore-Penrose inverse of a. Let $A^{\dagger}$ denote the set of all elements in A which have Moore-Penrose inverses. Clearly $A^{\dagger} \subset \widehat{A}$, and if A is a C^*-algebra then $A^{\dagger} = \widehat{A}$ [18, Theorem 6].

For the proof of the next two results see [22, Theorem 2.5, Corollary 2.6]

Theorem (Rakočević) 2.15. *Let A be a Banach algebra with identity 1. Then*

(2.15.1) $$A^{-1} A^{\bullet}_h \cap A^{\bullet}_h A^{-1} = A^{\dagger} \cap cl\,(A^{-1}).$$

Corollary 2.16. *Let A be a Banach algebra with identity 1 and $a \in A^{\dagger}$. Then the following conditions are equivalent:*

(i) $a \in \delta A^{-1}$,

(ii) $a = s_1 p_1 = p_2 s_2$, $s_i \in A^{-1}$, $p_i \in A^{\bullet}_h$ *and* $p_i \neq 1\, (i = 1, 2)$.

3. Semigroups in $B(X)$.

Now we shall describe others semigroups which obey condition (2.4.3). Let X be an infinite-dimensional complex Banach space and denote the set of bounded (compact) linear operators on X by $B(X)\,(K(X))$. The fact that

$K(X)$ is a closed two-sided ideal in $B(X)$ enables us to define the *Calkin* algebra over X as the quotient algebra $C(X) = B(X)/K(X)$. $C(X)$ is itself a Banach algebra in the quotient algebra norm

$$\|T + K(X)\| = \inf_{K \in K(X)} \|T + K\|. \tag{3.0.1}$$

We shall use π to denote the natural homomorphism of $B(X)$ onto $C(X)$; $\pi(T) = T + K(X)$, $T \in B(X)$. Throughout this paper $N(T)$ and $R(T)$ will denote, respectively, the null space and the range space of T. Set $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim X/R(T)$. An operator $T \in B(X)$ is *Fredholm* if $R(T)$ is closed, and both $\alpha(T)$ and $\beta(T)$ are finite. If $T \in B(X)$ and $R(T)$ is closed, it is said that T is *semi-Fredholm* operator if either $\alpha(T) < \infty$ or $\beta(T) < \infty$. Set

$$\Phi_+(X) = \{T \in B(X) : R(T) \quad \text{is closed and} \quad \alpha(T) < \infty\}, \tag{3.0.2}$$

and

$$\Phi_-(X) = \{T \in B(X) : R(T) \quad \text{is closed and} \quad \beta(T) < \infty\}. \tag{3.0.3}$$

It is clear that $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$. Let us mention that $\Phi(X)$, $\Phi_+(X)$ and $\Phi_-(X)$ are multiplicative open semigroup in $B(X)$ ([7], [15]) and by Atkinson's theorem ([7, Theorem 3.2.8], [15, Theorem 6.4.3]) we have

$$\Phi(X) = \pi^{-1}(C(X)^{-1}). \tag{3.0.4}$$

The index of an operator $T \in B(X)$ is defined by $i(T) = \alpha(T) - \beta(T)$, if at least one of $\alpha(T)$ and $\beta(T)$ is finite. It is well known that $B(X)^{-1} + K(X) \subset \Phi(X)$, and that $T \in B(X)^{-1} + K(X)$ if and only if $T \in \Phi(X)$ and $i(T) = 0$. Set

$$\Phi_0(X) = \{T \in \Phi(X) : i(T) = 0\}, \tag{3.0.5}$$

$$\Phi_l(X) = \pi^{-1}(C(X)_l^{-1}), \tag{3.0.6}$$

$$\Phi_r(X) = \pi^{-1}(C(X)_r^{-1}). \tag{3.0.7}$$

It is well-known that $\Phi_0(X)$, $\Phi_l(X)$ and $\Phi_r(X)$ are open semigroups in $B(X)$ ([7], [15]). Further, $T \in \Phi_l(X)$ if and only if $T \in \Phi_+(X)$ and there exists a bounded projection of X onto $R(T)$; $T \in \Phi_r(X))$ if and only if $T \in \Phi_-(X)$ and there exists a bounded projection of X onto $N(T)$ ([6], [7], [15]). Recall that an operator T is regular, i. e., $T \in \widehat{B(X)}$, if and only if $N(T)$ and $R(T)$ are closed, complemented subspaces of X ([6], [15], [26]). Let us mention that Gonzalez [11, Theorem] has proved [14, Theorem 1.1] for operators. His proof was based on a theorem of Caradus [6, Chapter 5, Theorem 13] involving two kinds of "gap" between the subspaces (see a good comment [14, pp. 329], and for further related results see e.g. [3], [4], [5], [10], [27]).

Theorem (Gonzalez) 3.1. *Let X be a Banach space, $T \in \widehat{B(X)}$ and P is a projection in $B(X)$ with $N(P) = N(T)$. Then the following conditions are equivalent.*

(i) *There is a sequence $\{U_n\}$ in $B(X)^{-1}$ with $\{\|U_n^{-1}\|\}$ bounded, such that $\|T - U_nP\| \to 0$.*
(ii) *There is $U \in B(X)^{-1}$ such that $T = UP$.*
(iii) $T \in \delta B(X)^{-1}$.
(iv) *$N(T)$ is isomorphic to a complement of $R(T)$.*

Theorem 3.2. *If X is a Banach space, then*

$$\Phi_l(X)B(X)^\bullet = \widehat{B(X)} \cap cl\,(\Phi(X)_l), \tag{3.2.1}$$

$$B(X)^\bullet\Phi_r(X) = \widehat{B(X)} \cap cl\,(\Phi_r(X)), \tag{3.2.2}$$

$$\Phi_l(X)B(X)^\bullet \cap B(X)^\bullet\Phi_r(X) = \widehat{B(X)} \cap cl\,(\Phi(X)_l) \cap cl\,(\Phi_r(X)). \tag{3.2.3}$$

Proof. By [6, p. 132, Theorem 2] we have that $\Phi_l(X)B(X)^\bullet \subset \widehat{B(X)}$ and $B(X)^\bullet\, \Phi_r(X) \subset \widehat{B(X)}$. Hence the proof followes by Corollary 2.3. □

Corollary 3.3. *Let X be a Banach space and $A \in \widehat{B(X)}$. Then the following conditions are equivalent:*

$$T \in \delta\Phi_l(X), \tag{3.3.1}$$

$$T = PB, \quad P \in B(X)^\bullet \setminus \Phi_l(X) \quad \text{and} \quad B \in \Phi_l(X), \tag{3.3.2}$$

Proof. By (3.2.1) and the fact that $\Phi_l(X)$ is an open subset of $B(X)$. □

Corollary 3.4. *Let X be a Banach space and $A \in \widehat{B(X)}$. Then the following conditions are equivalent:*

$$T \in \delta\Phi_r(X), \tag{3.4.1}$$

$$T = CQ, \quad Q \in B(X)^\bullet \setminus \Phi_r(X) \quad \text{and} \quad C \in \Phi_r(X), \tag{3.4.2}$$

Proof. By (3.2.2) and the fact that $\Phi_r(X)$ is an open subset of $B(X)$. □

Let us mention that it has been proved in [24, (3.5)] that

$$\{A \in B(X) : A \in A\Phi(X)A\} = B(X)^\bullet\Phi(X) = \Phi(X)B(X)^\bullet. \tag{3.4.3}$$

The following three results are from [24].

Theorem (Rakočević) 3.5. *If X is a Banach space then*

$$B(X)^{\bullet}\Phi(X) = \widehat{B(X)} \cap cl\,\Phi(X). \tag{3.5.1}$$

Corollary 3.6. *Let X be a Banach space and $A \in \widehat{B(X)}$. Then the following conditions are equivalent:*

$$A \in \delta\Phi(X), \tag{3.6.1}$$

$$A = PB, \quad P \in B(X)^{\bullet} \setminus \Phi(X) \quad and \quad B \in \Phi(X), \tag{3.6.2}$$

$$A = CQ, \quad Q \in B(X)^{\bullet} \setminus \Phi(X) \quad and \quad C \in \Phi(X), \tag{3.6.3}$$

For any Hilbert space X, let $\dim_H X$ denote the Hilbert dimension of X, that is the cardinality of an orthonormal basis of X. We set $\mathrm{nul}_H(T) = \dim_H N(T)$ and $\mathrm{def}_H(T) = \dim_H R(T)^{\perp}$ for $T \in B(X)$. If X is a separable Hilbert space, then with connection according to Theorem 3.5 we have

Theorem 3.7. *Let X be a separable Hilbert space. Then*

$$\begin{aligned} &\widehat{B(X)} \cap cl\,\Phi(X) \\ &\quad = \Phi(X) \cup \{T \in B(X) : nul_H(T) = def_H(T) \quad \text{and} \quad R(T) \quad \text{closed}\}. \end{aligned} \tag{3.7.1}$$

Theorem 3.8. *If X is a Banach space then*

$$B(X)^{\bullet}\Phi_0(X) = \Phi_0(X)B(X)^{\bullet} = \widehat{B(X)} \cap cl\,\Phi_0(X). \tag{3.8.1}$$

Proof. By [6, p. 132, Theorem 2] we have that $\Phi_0(X)B(X)^{\bullet} \subset \widehat{B(X)}$ and $B(X)^{\bullet}\Phi_0(X) \subset \widehat{B(X)}$. Hence we can apply Theorem 2.1 and Corollary 2.2. □

Corollary 3.9. *Let X be a Banach space and $A \in \widehat{B(X)}$. Then the following conditions are equivalent:*

$$A \in \delta\Phi_0(X), \tag{3.9.1}$$

$$A = PB, \quad P \in B(X)^{\bullet} \setminus \Phi_0(X) \quad and \quad B \in \Phi_0(X), \tag{3.9.2}$$

$$A = CQ, \quad Q \in B(X)^{\bullet} \setminus \Phi_0(X) \quad and \quad C \in \Phi_0(X), \tag{3.9.3}$$

Proof. By Theorem 3.8, Corollary 2.7 and Corollary 2.9. □

Remark 3.10. Let X be a Banach space. By Theorem 3.8 we have

$$B(X)^{\bullet}\Phi_0(X) = \Phi_0(X)B(X)^{\bullet}. \tag{3.10.1}$$

From the proof of [24, Theorem 3, (3.3)] we can conclude that

$$\{A \in B(X) : A \in A\Phi_0(X)A\} \subset B(X)^{\bullet}\Phi_0(X). \tag{3.10.2}$$

Now we have the following question (problem): If X is a Banach space, must we have

$$B(X)^{\bullet}\Phi_0(X) = \Phi_0(X)B(X)^{\bullet} = \{A \in B(X) : A \in A\Phi_0(X)A\}? \tag{3.9.3}$$

Recall that by Atkinson's theorem a bounded linear operator on a Banach space is Fredholm if and only if it has an invertible coset in the Calkin algebra. Motivated by this Harte ([12], [13], [15], [16], [19]) has associated (and has investigated) "Fredholm" elements of a Banach algebra A with an arbitrary homomorphism $T : A \mapsto B$; (A and B are complex Banach algebras with identity $1 \neq 0$, T is bounded with $T(1) = 1$). An element $a \in A$ is *Fredholm* (more precise *T-Fredholm*) iff $T(a) \in B^{-1}$. The set of all T-Fredholm elements of A is denoted by $\Phi_T(A)$. Recall that the homomorphism $T : A \mapsto B$ is *finitely regular* if

$$T^{-1}(0) \subset \widehat{A},$$

and an ideal I of A is *inessential* if the set of accumulation points of the spectrum of $x \in I$ is a subset of $\{0\}$ for each $x \in I$.

Recently Djordjević ([8], [9]) has investigated regular and T-Fredholm elements and, among other things, he has proved

Theorem (Djordjević) 3.11. *Suppose that the inessential ideals I_i, $i = 1, 2$, of A have the same sets of idempotents, I_2 is a closed subset of A, and let $P_i : A \mapsto A/I_i$ be the natural homomorphisms of A onto A/I_i, $i = 1, 2$. Now, if P_1 is a finitely regular, then*

$$A^{\bullet}\Phi_{P_1}(A) = \widehat{A} \cap cl\,(\Phi_{P_2}(A)). \tag{3.8.1}$$

Djordjević has got Theorem 3.5 as a corollary of Theorem 3.11. (The proof is based on the facts that the ideal of finite-rank operators in $B(X)$, $F(X)$, and $K(X)$ have the same sets of idempotents and $F(X) \subset \widehat{B(X)}$, and then applying Theorem 3.11 with $B(X)$ in place of A, $F(X)$ in place of I_1 and $K(X)$ in place of I_2.).

4. Partial order and regular boundary elements

Recall that in semigroup S the relation

$$e \leq f \Longleftrightarrow e = ef = fe, \quad e, f \in S^\bullet, \tag{4.0.1}$$

is well-known standard partial ordering relation on the set of idempotents, if any. Hartwig [20] has introduced the following, so colled *plus-relation.*

Definition (Hartwig) 4.1. Let S be a semigroup. For $a, b \in S$ set $a \leq b$, if

(4.1.1) (i) a is regular, and
(ii) there is some $a^+ \in S$, such that $a^+a = a^+b,\ aa^+ = ba^+$.

It is well known [20, Theorem 1] that the plus-relation of (4.1.1) defines a partial-order on S. This partial order is colled *plus-partial order*, shortly +-order, and for idempotents the standard order (4.0.1) coincides with the +-order.

Remark 4.2. Let $(G, \leq)$ be a partially ordered set. By a *closed interval* in G we shall mean any subset of the form $\{x \in G : a \leq x \leq b\}, \{x \in G : x \geq a\}$, or $\{x \in G : x \leq a\}$, where a and b are arbitrary elements of G. There are many known ways of using the order properties of G to define a topology on G. Recall that a base for the open set in the well-known *interval topology* of G consists of all subsets of the form $\cap\{C_i : i = 1, 2, \dots, n\}$, where each C_i is the complement of a closed interval. We let $\mathcal{I}$ denote the interval topology on G. It is natural to set the following question (problem):

If S is a semigroup, $(S, \leq)$ is a partial ordered set with the plus-partial order and $cl_{\mathcal{I}}(S^{-1})$ the closure of S^{-1} in interval topology on S, must we have

$$S^{-1}S^\bullet = \widehat{S} \cap cl_{\mathcal{I}}(S^{-1})? \tag{4.2.1}$$

Clearly, instead of interval topology, we can consider other topologies defined by plus-partial order (or other partial order) on S, and set the similar question to (4.2.1).

If we specialize to the case where $S = R$ is a ring with unity, then we have

Theorem 4.3. *Let R be a ring with unity, and $L(R^{-1}) = \{y \in R : y \leq x \ \text{for some} \ x \in R^{-1}\}$ be the set of predecessors of R^{-1}, where $\leq$ is the plus-partial order. Then we have*

$$R^{-1}R^\bullet = \widehat{R} \cap L(R^{-1}) = L(R^{-1}). \tag{4.3.1}$$

Proof. By [20, Proposition 3, (i), (v)] and (1.0.1). □

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