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THE ALGORITHMS AND DATA STRUCTURES FOR FORMING SYMBOLIC MODELS OF THE ROBOTIC SYSTEMS

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ABSTRACT. One of the basic problems in forming mathematical models of the robotic systems in their symbolic form is the formation of calculation graph for the analytical expessions of robotic quantities. The first step in formation of the calculation graph is the splitting of the expression into the products of the other expressions and the remaining expression. In this paper the necessary and sufficient conditions for splitting of the analytical expressions into the set of products of two expressions are found, and the algorithm for solving this problem is described.

1. Introduction

Significant advancements in the development of mathematical models of robotic systems have been made by numeric-symbolic [1] and symbolic [1-3] methods. An algorithm has been constructed [2,3] to form the mathematical model of a simple kinematic chain in symbolic form, and the program environment SYM [4] has been implemented for modelling of the robotic systems. This algorithm has been modified for complex kinematic chains, using the programming package *Mathematica* [5].

In [6], a network model of database has been proposed for generating the mathematical models of robotic systems in symbolic form. The navigation through the database allows easy formation of the analytical expressions and obtaining numerical values for the corresponding robotic quantities.

The analytical expressions, obtained with the aid of the algorithm from [6] can be simplified by applying trigonometric identities. These expressions

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are of the following form:

$$Y = \sum_{i=1}^{N} S_i \tag{1}$$

where each of the addends is of the form:

$$S_i = k_i \cdot \prod_{j=1}^{L} x_j^{e_{ij}} \tag{2}$$

and where:

- Y - is the robotic quantity to be calculated;
- k_i - is a constant coefficient related to the i-th addend;
- x_j -is one of the basic variables of the robotic system model represented by its name $(q, \dot{q}, \ddot{q}, \sin q, \cos q)$. For each addend the same sequence of variables x_j, $j = 1, ..., L$ is used.
- e_{ij} - is the exponent of the j-th variable of the i-th addend. The algorithm for forming the mathematical model ensures that each of the exponents is a nonnegative integer number.

The main task is to form the calculation graph for the chosen analytical expressions of the type (1), with the minimum of mathematical operations. To obtain the maximal reduction in the number of mathematical operations this paper proposes splitting of the chosen expression in the form:

$$Y = \sum_{l=1}^{M} (Y_{l1} \cdot Y_{l2}) + Y_{M+1} \tag{3}$$

where Y_{l1}, Y_{l2}, $l = 1, ..., M$ and Y_{M+1} are also the expressions of the type (1).

The expressions Y_{l1}, Y_{l2}, $l = 1, ..., M$ have two addends at least and are determined in a way which maximize the reduction of the number of mathematical operations. Y_{M+1} represents the remainder of the expression Y which can not be split into products any more.

This paper gives the necessary and sufficiant conditions for reducing the expression for Y into that of type (3). The concept of structural matrices which is used in solving this problem, is briefly described in the next paragraph.

2. Structural matrices

The concept of structural matrices was introduced in [1] to represent analytical expressions of the robotic quantities.

Structural matrix S of the expression Y is represented with the vector of coefficients $K_S = \left[k_1^S, ..., k_N^S\right]^\top$, the vector of variables $X_S = \left[x_1^S, ..., x_L^S\right]^\top$ and the matrix of exsponents:

$$E_S = \begin{bmatrix} e_{11}^S & e_{12}^S & \cdots & e_{1L}^S \\ e_{21}^S & e_{22}^S & \cdots & e_{2L}^S \\ \cdots & \cdots & \cdots & \cdots \\ e_{N1}^S & e_{N2}^S & \cdots & e_{NL}^S \end{bmatrix}$$

given in the previous paragraph.

In [1], the algebra of structural matrices is introduced, and here is described only the multiplication of the structural matrices because this is important for sloving the assigned problem.

Let us observe the multiplication of two structural matrices which have the same vestors of variables.

$$A = (K_A, X_A, E_A)$$
$$B = (K_B, X_B, E_B)$$
$$X_A = X_B$$

If the exponent matrices E_A i E_B are given with:

$$E_A = \begin{bmatrix} e_{11}^A & e_{12}^A & \cdots & e_{1L}^A \\ e_{21}^A & e_{22}^A & \cdots & e_{2L}^A \\ \cdots & \cdots & \cdots & \cdots \\ e_{I1}^A & e_{I2}^A & \cdots & e_{IL}^A \end{bmatrix} \quad E_B = \begin{bmatrix} e_{11}^B & e_{12}^B & \cdots & e_{1L}^B \\ e_{21}^B & e_{22}^B & \cdots & e_{2L}^B \\ \cdots & \cdots & \cdots & \cdots \\ e_{J1}^B & e_{J2}^B & \cdots & e_{JL}^B \end{bmatrix}$$

then their product is the exponent matrix E_C where:

$$E_C = \begin{bmatrix} e_{11}^C & e_{12}^C & \cdots & e_{1L}^C \\ e_{21}^C & e_{22}^C & \cdots & e_{2L}^C \\ \cdots & \cdots & \cdots & \cdots \\ e_{M1}^C & e_{M2}^C & \cdots & e_{ML}^C \end{bmatrix}$$

and where:

$$M = I \cdot J$$
$$e_{ml}^C = e_{il}^A + e_{jl}^B$$
$$m = (i-1) \cdot J + j$$
$$l = 1, ..., L; i = 1, ..., I; j = 1, ..., J.$$

Also, if the vectors of coefficients K_A and K_B are given by:

$$K_A = \begin{bmatrix} k_1^A \\ k_2^A \\ \dots \\ k_I^A \end{bmatrix}; K_B = \begin{bmatrix} k_1^B \\ k_2^B \\ \dots \\ k_J^B \end{bmatrix}$$

then, their product is the vector of coefficients K_C where:

$$K_C = \begin{bmatrix} k_1^C \\ k_2^C \\ \dots \\ k_M^C \end{bmatrix}$$

and where:

$$M = I \cdot J$$
$$k_m^C = k_i^A \cdot k_j^B$$
$$m = (i-1) \cdot J + j$$
$$i = 1, ..., I; j = 1, ..., J.$$

Thus the structural matrix C is obtained as a product of the structural matrices A and B.

$$C = (K_C, X_C, E_C)$$
$$X_C = X_A = X_B$$

In solving our problem we have an opossite situation. The structural matrix exists and two structural matrices whose product will be the given matrix, are to be found.

3. Splitting of the structural matrix in the form of products

The problem can be broken in two separate problems. The first is to split the exponent matrix and the second is to split the vector of coefficients.

Let us observe the equation

$$E_A \cdot E_B = E_C \tag{4}$$

where E_A, E_B i E_C are the exsponent matrices defined in the previous paragraph. The matrix E_C represent the exsponents of the expression of the type (1) which we want to write in the form of product of the expressions of the same type. The matrices E_A and E_B are the unknowns in this equation. If the solution of the system exists, then the matrices E_A and E_B which satisfy the equation will represent the exponent matrix of the expressions which form the product.

If we write equation (4) as a system we obtain an $I \cdot J \cdot L$ linear equations with $(I + J) \cdot L$ unknowns. The system is of the form:

(5)
$$\begin{gathered} e^A_{il} + e^B_{jl} = e^C_{ml} \\ m = (i-1) \cdot J + j \\ l = 1, ..., L; i = 1, ..., I; j = 1, ..., J. \end{gathered}$$

Let us denote vectors $\left[e^A_{i1}, ..., e^A_{iL}\right]^\top$, $\left[e^B_{j1}, ..., e^B_{jL}\right]^\top$ and $\left[e^C_{m1}, ..., e^C_{mL}\right]^\top$ (rows of the matrices E_A, E_B and E_C) with e^A_i, e^B_j and e^C_m respectively. Now, the system can be written in a vector form:

(6)
$$\begin{gathered} e^A_i + e^B_j = e^C_m \\ m = (i-1) \cdot J + j \\ i = 1, ..., I; j = 1, ..., J. \end{gathered}$$

where the addition of the vectors is defined in the usual way.

Let us choose $i1$ and $i2$ so that $i1, i2 \in \{1, ..., I\}$ and $i1 \neq i2$, and $j1$ and $j2$ so that $j1, j2 \in \{1, ..., J\}$ and $j1 \neq j2$. Then, we pick the four following equations from the system (6):

(7)
$$\begin{gathered} e^A_{i1} + e^B_{j1} = e^C_{m1} \\ m1 = (i1 - 1) \cdot J + j1 \end{gathered}$$

(8)
$$\begin{gathered} e^A_{i1} + e^B_{j2} = e^C_{m2} \\ m2 = (i1 - 1) \cdot J + j2 \end{gathered}$$

(9)
$$\begin{gathered} e^A_{i2} + e^B_{j1} = e^C_{m3} \\ m3 = (i2 - 1) \cdot J + j1 \end{gathered}$$

(10)
$$\begin{gathered} e^A_{i2} + e^B_{j2} = e^C_{m4} \\ m4 = (i2 - 1) \cdot J + j2 \end{gathered}$$

By summing equations (7) and (10) we obtain

(11)
$$e^A_{i1} + e^B_{j1} + e^A_{i2} + e^B_{j2} = e^C_{m1} + e^C_{m4}$$

Also, by summing equations (8) and (9) we obtain

$$e_{i1}^{A} + e_{j2}^{B} + e_{i2}^{A} + e_{j1}^{B} = e_{m2}^{C} + e_{m3}^{C} \tag{12}$$

It follows that following condition must be satisfied

$$e_{m1}^{C} + e_{m4}^{C} = e_{m2}^{C} + e_{m3}^{C} \tag{13}$$

or in another form

$$e_{m1}^{C} - e_{m3}^{C} = e_{m2}^{C} - e_{m4}^{C} \tag{14}$$

If the condition (13) is not fulfilled it follows that the system (6) has no solution.

If this procedure is repeated for each combinatin of $i1$, $i2$ and $j1$, $j2$ on the basis of transitivity law for equality, the $I \cdot J$ conditions are obtained which have to be satisfied to provide the solvability of the system (6).

If these conditions are satisfied it can be shown that the system has an infinite set of solutions in the set R^L. For solving the concrete problem where all the exponents which are found in analytical expressions of the robotic quantities have nonnegative integer values, we need to find our solutions of the system (6) in the set $(N \bigcup \{0\})^L$. The solution which belongs to the set $(N \bigcup \{0\})^L$ we will call the allowed solution. First, we can prove that there is at least one solution in the set Z^L.

Lemma 1. *If the conditions (13) are satisfied then exists at least one solution of the system (6) in the set* Z^L.

The lemma is proved by derivation of the solution.

Suppose that conditions (13) hold. Then, for e_1^A we take the 0 vector. From J equations in which e_1^A participates we have that:

$$e_j^B = e_j^C \qquad j = 1, ..., J. \tag{15}$$

Now for each $e_i^A, i \neq 1$ remains J equations from which follows that:

$$e_i^A = e_{m_1}^C - e_1^C = ... = e_{m_J}^C - e_J^C \qquad m_j = (i-1) \cdot J + j \qquad i = 2, ..., I. \tag{16}$$

Let us check the correctness of equation (16). We choose $i1 \in \{2, .., I\}$. For $i2$ we take 1, and for $j1, j2$ we take any combination from the set $\{1, ..., J\}$. Equation (16) follows from the condition (14) for this combination and transitivity law for equality.

Thus the solutions which are in the set Z^L are found for all e_i^A, $i = 1, ..., I$ and e_j^B, $j = 1, ..., J$ because all e_m^C, $m = 1, ..., M$ are from the set Z^L.

Now we can prove that exists an allowed solution.

Theorem 1. *If the conditions (13) are satisfied then exists at least one allowed solution of the system (6).*

Suppose that conditions (13) hold. For e_1^A we obtain each of the components so that $e_{1l}^A = min(e_{1l}^C, ..., e_{Jl}^C)$ for $l = 1, ..., L$. The obtained vector is allowed because all the e_m^C are allowed. From J equations in which e_1^A participates we have that:

$$e_j^B = e_j^C - e_1^A \qquad (17)$$
$$j = 1, ..., J.$$

These solutions are also allowed because $e_{jl}^C \geq e_{1l}^A$ for $j = 1, ..., J$ and $l = 1, ..., L$. Now for each $e_i^A, i \neq 1$ remains J equations from which follows that:

$$e_i^A = e_{m_1}^C - e_1^C + e_1^A = ... = e_{m_J}^C - e_J^C + e_1^A \qquad (18)$$
$$m_j = (i-1) \cdot J + j$$
$$i = 2, ..., I.$$

The correctness of equation (18) is proved in an analogous way as equation (16) from Lemma 1. Now we must prove that all the e_i^A for $i = 2, ..., I$ are alowed.

Suppose that there exists $i1$ and $l1$ for which $e_{i1l1}^A < 0$. Then, choose the $j1$ so that $e_{j1l1}^C = e_{1l1}^A = min(e_{1l1}^C, ..., e_{Jl1}^C)$. Now follows that $e_{j1l1}^B = e_{j1l1}^C - e_{1l1}^A = 0$. The vector e_{j1}^B paticipates in another $I-1$ equations. From these equations we choose the one in which e_{i1}^A also participates. It follows that $e_{j1l1}^B = e_{m1l1}^C - e_{i1l1}^A = 0$ where $m1 = (i1-1) \cdot J + j1$. From this equation follows that $e_{m1l1}^C = e_{i1l1}^A$ and the contradiction is obtained with the assumption that $e_{i1l1}^A < 0$ because $e_{ml}^C \geq 0$, for $m = 1, ..., M$ and $l = 1, ..., L$.

In an analogous way we can solve the problem of splitting the vector of coefficients. Let us observe the equation

$$K_A \cdot K_B = K_C \qquad (19)$$

where K_A, K_B and K_C are the vectors of coefficients defined in the previous paragraph.

If we write equation (19) as a system we obtain $I \cdot J$ equations with $(I+J)$ unknowns. The system is of the form:

$$k_i^A \cdot k_j^B = k_m^C \qquad (20)$$
$$m = (i-1) \cdot J + j$$
$$i = 1, ..., I; j = 1, ..., J.$$

By a procedure analogous to the one for splitting the exponent matrices we obtain the following conditions:

$$k_{m1}^{C} \cdot k_{m4}^{C} = k_{m2}^{C} \cdot k_{m3}^{C} \tag{21}$$

or in another form

$$\frac{k_{m1}^{C}}{k_{m3}^{C}} = \frac{k_{m2}^{C}}{k_{m4}^{C}} \tag{22}$$

which are analogous to the conditions (13) and (14).

The following theorem can be proved:

Theorem 2. *If the conditions (21) are satisfied, then exists at least one solution of the system (20) in a set of real numbers.*

Suppose that conditions (21) hold. For k_1^A we take the value 1.00. From J equations in which k_1^A participates we have that:

$$k_j^B = k_j^C \qquad j = 1, ..., J. \tag{23}$$

Now, for each $k_i^A, i \neq 1$ remains J equations from which follows that:

$$k_i^A = \frac{k_{m_1}^C}{k_1^C} = ... = \frac{k_{m_J}^C}{k_J^C} \qquad m_j = (i-1) \cdot J + j \qquad i = 2, ..., I. \tag{24}$$

The correctness of equation (24) is proved in an analogous way as equation (16) from Lemme 1.

Now, the complete problem can be formulated. Let us observe the equation

$$A \cdot B = C \tag{25}$$

where A, B and C are the structural matrices. The goal is to obtain the matrices A and B so that equation (25) holds.

Theorem 3. *The structural matrices A and B which satisfy the equation (25) exist if and only if conditions (13) and (21) are fulfilled.*

If the conditions (13) and (21) hold, the existence of the matrices A and B is proved by applying Theoreme 1 and Theoreme 2, and assemblying the corresponding exponent matrix and vector of coefficients into a structural matrix. The theorem in the opposite direction obviously follows from the procedure for obtaining the conditions (13) and (21). In the case when the conditions (13) are not satisfied the contradiction is obtained in the system (6), and if the conditions (21) are not satisfied the contradiction is obtained in the system (20).

4. The algorithm

When the conditions (13) and (21) are derived two rows from the matrix E_A and E_B, i.e. two elements of the vectors K_A and K_B, are chosen. Also, by choosing four rows from the matrix E_C, four elements of the vector K_C are chosen. This means that these four addends from the expression described by the exponent matrix E_C and the vector of coefficients K_C, are obtained by multiplying two expressions each containing two addends which are described by the chosen parts of the matrices E_A and E_B and vectors K_A and K_B. Thus every condition of the type (13) and (21) represent a "2×2" multiplication.

In the same way equations (16) and (18) describe a "$2 \times J$" multiplication because the two rows of the matrix E_A and all J rows of matrix E_B are chosen as well as two elements of the vector K_A and all J elements of the vector K_B. Also are chosen the $2 \cdot J$ rows of the matrix E_C and the same number of elements of the vector K_C.

The algorithm task is to reduce the starting expression Y to the type (3) by splitting the structural matrix S into the products.

The first step is to obtain all possible "$2 \times n$" multiplications, where n is determined as large as possible for each multiplication. The description of this part of the algorithm, written in pseudocode, is given bellow.

```
for (i1 = 1 ; i1 < N ; i1++)
 {
  for (i2 = i1 + 1 ; i2 ≤ N ; i2++)
   {
    if (NotMem(i1,i2))
     {
      MemPair(i1,i2,mu);
      s = e^S_{i1} − e^S_{i2};
      q = k^S_{i1} / k^S_{i2};
      for (i3 = 1 ; i3 < N ; i3++)
       {
        for (i4 = i3 + 1 ; i4 ≤ N ; i4++ )
         {
          if (s == e^S_{i3} − e^S_{i4} && q == k^S_{i3} / k^S_{i4})
            MemPair(i3,i4,mu);
          else if (s == e^S_{i4} − e^S_{i3} && q == k^S_{i4} / k^S_{i3})
            MemPair(i4,i3,mu);
         }
       }
```

```
        }
        MemMul(mu);
        FreeMul(mu);
    }
}
```

Function **NotMem** checks if the given pair is memorised in the knowledge database. If it is the function returns **0** because of the transitivity law for equality there is no sense to check the conditions for that pair.

Procedure **MemPair** memorises the given pair in variable **mu** which will contain the complete multiplication.

Procedure **MemMul** memorises complete multiplication **mu** in the knowledge database.

Procedure **FreeMul** frees the variable **mu** for a new multiplication.

After applying this part of the algorithm we have memorised in the knowledge database all possible "$2 \times n$" multiplications over the structural matrix S. Now, the manipulating over the knowledge database is used to provide such a selection of multiplications which enable the reducing the starting expression Y in the type (3) with a minimum of mathematical operations. Here will be explained only how to produce the "$m \times n$" multiplications using $m-1$ "$2 \times n$" multiplications.

If $m-1$ "$2 \times n$" multiplications (or their parts) are found in the knowledge database having the same first elements of each pair and all different second elements of the each pair, it is easy to prove that a "$m \times n$" multiplying is constructed. From the transitivity law for equality follows that all the conditions for the existing solution of the systems (6) and (20) are satisfied when $I = m$ and $J = n$.

6. Conclusion

The algorithms for forming symbolic mathematical models of complex robotic systems have been developed. With the aim of reducing the number of mathematical operations, the investigations are directed towards the development of the algorithms for symplification of analytical expressions and the formation of a corresponding calculation graph.

A theorem is proved with the necessary and sufficient conditions for splitting the analytical expressions into the products of two expressions. On the basis of this theorem an algorithm for forming the set of candidates for splitting the expressions into products is given. This algorithm is implemented and tested on an example of the robotic mechanism with six rotational degrees of freedom. Of the set of obtained candidates we can choose the ones which reduce the number of mathematical operations. On the concrete example significant reduction of analytical expressions has been achieved.

Further investigations will be concerned with the development of algorithms for grouping the expression candidates memorised in the knowledge database which reduce the number of mathematical operations.

References

[1] M. Vukobratović and N. Kirćanski, *Real-Time Dynamics of Manipulation Robots*, Springer-Verlag, 1985.

[2] A. Timčenko, *A Program Package for Generating Symbolic Models of Manipulation Robots*, M. Sc. Thesis, Beograd, 1989. (In Serbian)

[3] A. Timčenko, N. Kirćanski and M. Vukobratović, *A Two-Step Algorithm for Generating Efficient Manipulator Models in Symbolic Form*, Proceedings of the 1991 IEEE International Conference on Robotics and Automation, Sacramento, California - April 1991, pp. 1887–1892.

[4] A. Timčenko, N. Kirćanski, D. Urošević and M. Vukobratović, *SYM - Program Environment for Manipulator Modeling, Control and Simulation*, Proceedings of the 1991 IEEE International Conference on Robotics and Automation, Sacramento, California - April 1991, pp. 1122–1127.

[5] M. Racković and D. Surla, *Generating the Analytic Expressions for Driving Moments of the Complex Kinematic Chain*, XXXVIII Yugoslav Conference of ETRAN, Niš 1994., pp. 105–106. (In Serbian)

[6] D. Surla and M. Racković, *The Implementation of Symbol Operations into Symbolic Modelling of Robotic Systems*, IX Conference on Aplied Mathematics PRIM'94, Budva 1994, pp. 371–377.

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