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SOME CONGRUENCES ON AN AG^{**}-GROUPOID

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ABSTRACT. Some congruences on AG^{*}-groupoids have been considered in [3]. In this paper we shall describe some congruences on AG^{**}-groupoids.

1. Preliminaries

If on groupoid S the following is true

$$(1) \qquad (\forall a,b,c \in S)\ (ab)c = (cb)a,$$

then S is an *AG-groupoid* (Abel-Grassmann's groupoid), [4]. In an AG-groupoid, clearly, holds *medial law*

$$(2) \qquad (ab)(cd) = (ac)(bd),$$

for every $a,b,c,d \in S$.

Let on AG-groupoid S the following is true

$$(3) \qquad a(bc) = b(ac)$$

for every $a,b,c \in S$. This class of AG-groupoids we shall call an AG^{**}-groupoids. If S has left identity then S is an AG^{**}-groupoid, [8]. Let S be an AG^{**}-groupoid and $a,b,c,d \in S$, then

$$(4) \qquad (ab)(cd) = c((ab)d) = c((db)a) = (db)(ca).$$

An AG-groupoid S is called an *inverse AG-groupoid* if for every $a \in S$ there exists $a' \in S$ such that $(aa')a = a, (a'a)a' = a'$ and a' is an *inverse* for a, [9]. As usually we shall denote by $V(a)$ the set of all inverses of $a \in S$. If $a,b \in S,\ a' \in V(a),\ b' \in V(b)$ then $a'b' \in V(ab)$, [9].

For undefined notions and notations we refer to [1],[2], [6] and [10].

2. The congruence σ

Lemma 2.1. *Let S be an AG^{**}-groupoid and $E(S) \neq \emptyset$, then $E(S)$ is a semilattice.*

Proof. If $e, f \in E(S)$, then by (4) and (2) it follows that

$$ef = (ee)(ff) = (fe)(fe) = (ff)(ee) = fe. \quad \square$$

The basic definitions of congruences on an AG-groupoid are given in [9] and those definitions are analogous with those in semigroup theory.

Theorem 2.1. *Let S be an AG^{**}-groupoid and let $E(S) \neq \emptyset$, then the relation σ defined on S with*

$$\sigma = \{(x, y) \in S \times S \mid (\exists e \in E(S))ex = ey\}$$

is a congruence relation on S and $e\sigma f$, for every $e, f \in E(S)$. Furthermore, $\sigma = \{(x, y) \in S \times S \mid (\exists e \in E(S))xe = ye\}$.

Proof. Clearly, σ is a reflexive and symmetric relation. Let $x\sigma y$, $y\sigma z$, then $ex = ey$, $fy = fz$ for some $e, f \in E(S)$. Now by (1), (2), (3) and (4) we have

$$\begin{aligned}(ef)x &= ((ee)f)x = (xf)(ee) = (ef)(ex) = (ef)(ey) = (ee)(fy)\\ &= (ee)(fz) = (ef)(ez) = (zf)(ee) = (zf)e = (ef)z.\end{aligned}$$

Since $ef \in E(S)$ we conclude $x\sigma z$ and σ is a transitive relation. Hence, σ is an equivalence relation.

Let $x\sigma y$ and $z \in S$, then $ex = ey$ for some $e \in E(S)$. Now we have

$$\begin{aligned}e(xz) &= (ee)(xz) = (ex)(ez) = (ey)(ez) = (ee)(yz) = e(yz),\\ e(zx) &= (ee)(zx) = (ez)(ex) = (ez)(ey) = (ee)(zy) = e(zy).\end{aligned}$$

Hence, σ is a congruence relation.

Let $e, f \in E(S)$, then since $E(S)$ is a semilattice we have

$$efe = eef = eff.$$

Hence, $e\sigma f$ for every $e, f \in E(S)$.

Let β be a relation defined on S with

$$\beta = \{(x, y) \in S \times S \mid (\exists e \in E(S))xe = ye\}.$$

If $(x,y) \in \beta$, then there exists $e \in E(S)$ such that $xe = ye$. Now, by (1) we have

$$ex = (ee)x = (xe)e = (ye)e = (ee)y = ey,$$

so $(x,y) \in \sigma$. Conversely, if $(x,y) \in \sigma$,then there exists $f \in E(S)$ such that $fx = fy$. Now, we have

$$\begin{aligned} xf &= x(ff) = f(xf) = (ff)(xf) = (fx)(ff) = (fy)(ff) \\ &= (ff)(yf) = f(yf) = y(ff) = yf, \end{aligned}$$

so $(x,y) \in \beta$. Hence, $\beta \equiv \sigma$.

Now, if $(x,y) \in \sigma$ and $z \in S$, then for some $e \in E(S)$ hold $(xz)e = (yz)e$ and $(zx)e = (zy)e$. □

Corollary 2.1. *Let S be an AG-groupoid with left identity 1, then the relation σ is a smallest congruence on S with property that $e\sigma f$ for every $e, f \in S$.*

Proof. Let τ be an arbitrary congruence with above property, then for $a, b \in S$ from $a\sigma b \iff ea = eb$ we have $ea\tau = eb\tau$. Now, $e\tau a\tau = e\tau b\tau$ and since $1 \in E(S)$ it follows that $1\tau a\tau = 1\tau b\tau$, whence $a\tau = b\tau$. Hence, $\sigma \subseteq \tau$. □

3. The maximum idempotent-separating congruence μ

Lemma 3.1. *Let S be an inverse AG^{**}-groupoid, then $\mid V(a) \mid = 1$ for each $a \in S$.*

Proof. Let $a \in S$ and $x, y \in V(a)$, then

$$xa = x((ay)a) = (ay)(xa) = (ax)(ya) = y((ax)a) = ya,$$

so

$$x = (xa)x = (ya)x = (xa)y = (ya)y = y.$$

Hence, $\mid V(a) \mid = 1$. □

If S is an inverse AG^{**}-groupoid then unique inverse for $a \in S$ we denote with a^{-1}. Notice that aa^{-1} is not necessary idempotent.

Example 3.1. Let S be an AG-groupoid defined by the following Cayley table:

	1	2	3	4
1	2	2	4	4
2	2	2	2	2
3	1	2	3	4
4	1	2	1	2

.

Then S is an inverse AG^{**}-groupoid, $e, f \in E(S)$. Elements 1 and 4 are mutually inverse and $1 \cdot 4 = 4$, $4 \cdot 1 = 1$ are not idempotents.

Remark 3.1. We notice that if ρ is a congruence relation on AG^{**}-groupoid S, then S/ρ is an AG^{**}-groupoid. Also, if S is an inverse AG^{**}-groupoid, then S/ρ is an inverse AG^{**}-groupoid and if $(x, y) \in \rho$ then $(x^{-1}, y^{-1}) \in \rho$ and conversely.

Theorem 3.1. *Let S be an inverse AG^{**}-groupoid and $E(S) \neq \emptyset$, then the relation*

$$\mu = \{(a, b) \in S \times S \ a^{-1}a = b^{-1}b\}$$

is an idempotent-separating congruence on S. If on S holds $a^{-1}a \in E(S)$ for all $a \in S$ then μ is a maximum idempotent-separating congruence on S.

Proof. It is clear that μ is an equivalence.

If $a\mu b$, $c \in S$ and $e \in E(S)$ then

$$\begin{aligned}(ac)^{-1}(ac) &= (a^{-1}c^{-1})(ac) = (a^{-1}a)(c^{-1}c)\\ &= (b^{-1}b)(c^{-1}c)) = (b^{-1}c^{-1})(bc) = (bc)^{-1}(bc),\end{aligned}$$

so $ac\mu bc$. Similarly, $ca\mu cb$. Thus μ is a congruence.

Let $e, f \in E(S)$ and $e\mu f$ then $e = ee = ff = f$. Hence, μ is an idempotent-separating.

Let $a^{-1}a \in E(S)$ holds for all $a \in S$. If ρ is an idempotent-separating congruence on S, $a, b \in S$ and $a\rho b$ then from $a^{-1}\rho b^{-1}$ we have $a^{-1}a\rho b^{-1}b$. Since ρ is idempotent-separating it follows that $a^{-1}a = b^{-1}b$, whence it follows that $a\mu b$ and $\rho \subseteq \mu$. □

4. A congruence pair

Example 4.1. Let S be an AG-groupoid defined by the following Cayley table:

	1	2	3	4	5
1	2	1	1	1	1
2	1	2	2	2	2
3	1	2	4	5	3
4	1	2	3	4	5
5	1	2	5	3	4

Then S is an inverse AG^{**}-groupoid, $a = a^{-1}$ for every $a \in S$ and $aa^{-1} = a^{-1}a$.

In this section with S we shall denote the inverse AG^{**}-groupoid in which $aa^{-1} = a^{-1}a$ holds for every $a \in S$.

Lemma 4.1. *If $a \in S$ then $a^{-1}a \in E(S)$.*

Proof. Let $a \in S$ then

$$(a^{-1}a)(a^{-1}a) = a^{-1}((a^{-1}a)a) = a^{-1}((aa^{-1})a) = a^{-1}a \in E(S). \quad \square$$

Hence, $E(S) \neq \emptyset$.

Definition 4.1. Let K be a subset of S, then: K is *full* if $E(S) \subseteq K$; K is *self-conjugate* if $x^{-1}(Kx) \subseteq K$ for every $x \in S$; K is *inverse closed* if from $x \in K$ it follows, $x^{-1} \in K$; K is *normal* if it is full, self-conjugate and inverse closed.

Let ρ be a congruence on S. The restriction $\rho_{|_{E(S)}}$ is the *trace* of ρ to be denoted by $tr\rho$, and the set $ker\rho = \{a \in S \mid (\exists e \in E(S))\ a\rho e\}$ is the *kernel* of ρ.

Lemma 4.2. *Let ρ be a congruence relation on S, then $ker\rho$ is a normal subgroupoid of S.*

Proof. If $a, b \in ker\rho$, then $a\rho e$, $b\rho f$ for some $e, f \in E(S)$. Now $ab\rho ef$ and since $ef \in E(S)$ we have that $ab \in ker\rho$. Hence, $ker\rho$ is a subgroupoid of S.

Clearly, $ker\rho$ is full.

Let $a \in S$, then $a(ker\rho \cdot a^{-1}) = \{a(ba^{-1}) \mid b \in ker\rho\}$. From $b \in ker\rho$ we have that $b\rho e$ for some $e \in E(S)$ so $a(ba^{-1})\rho a(ea^{-1})$. Since $a(ea^{-1}) = e(aa^{-1}) \in E(S)$, then $a(ba^{-1}) \in ker\rho$. Hence, $a(ker\rho \cdot a^{-1}) \subseteq ker\rho$ and $ker\rho$ is a self-conjugate subgroupoid of S.

If $x \in ker\rho$, then $x\rho e$ for some $e \in E(S)$ and $x^{-1}\rho e^{-1} = e$. Hence, $x^{-1} \in ker\rho$ and $ker\rho$ is inverse closed.

By above we conclude that $ker\rho$ is a normal subgroupoid of S. $\square$

Definition 4.2. Let K be a normal subgroupoid of S and τ congruence on semilattice $E(S)$ such that

$$ea \in K,\ e\tau a^{-1}a \Longrightarrow a \in K \tag{5}$$

for every $a \in S$ and $e \in E(S)$. Then the pair (K, τ) is a *congruence pair* for S.

In such a case, we can define a relation $\rho_{(K,\tau)}$ on S by

$$a\rho_{(K,\tau)}b \Longleftrightarrow a^{-1}a\tau b^{-1}b,\ ab^{-1},\ ba^{-1} \in K. \tag{6}$$

Lemma 4.3. *For a congruence pair (K,τ) for S, we have*

$$e(ab) \in K,\ e\tau a^{-1}a \Longrightarrow ab \in K$$

for any $a,b \in S,\ e \in E(S)$,

Proof. Let $a,b \in S,\ e \in E(S),\ e(ab) \in K$ and $e\tau a^{-1}a$, then

$$\begin{aligned} e(ab) &= (ee)(ab) = (be)(ae) = (((bb^{-1})b)e)(ae) = ((eb)(bb^{-1}))(ae) \\ &= (b((eb)b^{-1}))(ae) = (b((b^{-1}b)e))(ae) = (e((b^{-1}b)e))(ab) \\ &= ((b^{-1}b)e)(ab) = (e(b^{-1}b))(ab), \\ (ab)^{-1}(ab) &= (a^{-1}b^{-1})(ab) = ((ab)b^{-1})a^{-1} = ((b^{-1}b)a)a^{-1} = (a^{-1}a)(b^{-1}b) \\ &\quad \tau e(b^{-1}b). \end{aligned}$$

By above and (5) we have $ab \in K$. □

Theorem 4.1. *If (K,τ) is a congruence pair for S, then $\rho_{(K,\tau)}$ is the unique congruence ρ on S for which $ker\rho = K$ and $tr\rho = \tau$. Conversely, if ρ is a congruence on S, then $(ker\rho, tr\rho)$ is a congruence pair for S and $\rho_{(ker\rho,tr\rho)} = \rho$.*

Proof. Let (K,τ) be a congruence pair for S, and let $\rho = \rho_{(K,\tau)}$. Then ρ is reflexive since K is full, and it is symmetric since τ is symmetric. Let $a\rho b$ and $b\rho c$, so that $a^{-1}a\tau b^{-1}b\tau c^{-1}c$ and $ba^{-1}, bc^{-1} \in K$. Since K is inverse closed we have $(ba^{-1})^{-1} = b^{-1}a \in K$. Since K is a substructure we have

$$(b^{-1}a)(bc^{-1}) = (b^{-1}b)(ac^{-1}) \in K.$$

From above and $b^{-1}b\tau a^{-1}a$, by Lemma 4.3, it follows that $ac^{-1} \in K$. Thus $a\rho c$ and ρ is transitive.

Next let $a\rho b$ and $c \in S$. Then

$$\begin{aligned} (ac)^{-1}(ac) &= (a^{-1}c^{-1})(ac) = (a^{-1}a)(c^{-1}c) \\ &\quad \tau(b^{-1}b)(c^{-1}c) = (b^{-1}c^{-1})(bc) = (bc)^{-1})(bc). \end{aligned}$$

Also,

$$\begin{aligned} (ac)(bc)^{-1} &= (ac)(b^{-1}c^{-1}) = (ab^{-1})(cc^{-1}) \in K \cdot E(S) \subseteq K, \\ (bc)(ac)^{-1} &= (bc)(a^{-1}c^{-1}) = (ba^{-1})(cc^{-1}) \in K \cdot E(S) \subseteq K. \end{aligned}$$

Hence, $ac\rho bc$. Similarly, $ca\rho cb$. Therefore ρ is a congruence on S.

If $a \in ker\rho$, then $a\rho e$ for some $e \in E(S)$. Now, $aa^{-1} = a^{-1}a\tau e$ and $ae,\ ea^{-1} \in K$ whence by (5) it follows that $a^{-1} \in K$. Since K is inverse closed we have $a \in K$. Conversely, if $a \in K$, then from $a = (aa^{-1})a \in K$ and $a^{-1}a\tau(a^{-1}a)(a^{-1}a)$ we have $a\rho a^{-1}a$ and $a \in K$. Consequently, $ker\rho = K$; and obviously $tr\rho = \tau$.

Now let λ be a congruence on S such that $ker\lambda = K$ and $tr\lambda = \tau$. Assume first that $a\lambda b$. Then $a^{-1}\lambda b^{-1}$ so that $a^{-1}a\lambda b^{-1}b$ and also $ab^{-1}\lambda bb^{-1} = b^{-1}b,\ ba^{-1}\lambda aa^{-1} = a^{-1}a$. This shows that $a^{-1}a\tau b^{-1}b$ and $ab^{-1},\ ba^{-1} \in ker\lambda = K$, which implies that $a\rho b$ and $\lambda \subseteq \rho$. Conversely, assume that $a\rho b$. Then $a^{-1}a\lambda b^{-1}b$ and $ab^{-1},\ ba^{-1} \in K = ker\lambda$. Now there exist $e, f \in E(S)$ such that $ab^{-1}\lambda e, ba^{-1}\lambda f$ whence $a^{-1}b\lambda e,\ b^{-1}a\lambda f$. From above and $a^{-1}a = aa^{-1},\ b^{-1}b = bb^{-1}$ we have

$$\begin{aligned} ab^{-1} &= ((aa^{-1})a)b^{-1} = (b^{-1}a)(aa^{-1})\lambda f(bb^{-1}),\\ ba^{-1} &= b((a^{-1}a)a^{-1}) = (a^{-1}a)(ba^{-1})\lambda(b^{-1}b)f \end{aligned}$$

and since $E(S)$ is a semilattice it follows that

$$e\lambda = ab^{-1}\lambda = f(b^{-1}b)\lambda = ba^{-1}\lambda = f\lambda.$$

Now

$$\begin{aligned} a &= (aa^{-1})a\lambda(bb^{-1})a = (ab^{-1})b\lambda eb,\\ b &= (bb^{-1})b\lambda(aa^{-1})b = (ba^{-1})a\lambda ea \end{aligned}$$

and by (4)

$$\begin{aligned} a\lambda eb &= e(ea) = (ee)(ea) = (ae)(ee) = (ae)e = ((eb)e)e\\ &= (ee)(eb) = e(eb) = ea\lambda b. \end{aligned}$$

Hence, $\rho \subseteq \lambda$. Consequently, $\rho = \lambda$ which proves uniqueness.

Conversely, let ρ be a congruence on S. By Lemma 4.2 we have that $ker\rho$ is a normal substructure of S. For $a \in S,\ e \in E(S)$ let $ea \in ker\rho,\ e\ tr\rho a^{-1}a$, holds then $ea\rho f$ for some $f \in E(S)$. Now $a = (aa^{-1})a\rho ea\rho f$ and $a \in ker\rho$. Hence, statement (5) holds and $(ker\rho, tr\rho)$ is a congruence pair for S. From above it follows that $ker\rho_{(ker\rho,tr\rho)} = ker\rho,\ tr\rho_{(ker\rho,tr\rho)} = tr\rho$. Now the uniqueness just proved implies that $\rho_{(ker\rho,tr\rho)} = \rho$. □

References

[1] S. Bogdanović, *Semigroups with a system of subsemigroups*, Inst. of Math., Novi Sad, 1985.

[2] S. Bogdanović and M. Ćirić, *Polugrupe*, Prosveta, Niš, 1993.

[3] M. Božinović and P. V. Protić, *Some congruences on a π-inverse LA*-semigroups*, Facta Universitatis (to appear).

[4] J. Deneš and A. D. Keedwell, *Latin squares and their applications*, Akadémia Kiadó, Budapest, 1974.

[5] J. M. Howie, *The maximum idempotent-separating congruence on an inverse semigroup*, Proc. Edinburg Math. Soc. **14** (1964), 71-79.

[6] J. M. Howie, *An introduction to semigroup theory*, Academic Press, 1976.

[7] M. A. Kazim and M. Naseeruddin, *On almost semigroups*, The Aligarh Bull. Math. **2** (1972), 1-7.

[8] Q. Mushtaq and Q. Iqbal, *Decomposition of a locally associative LA-semigroup*, Semigroup Forum **41** (1990), 155-164.

[9] Q. Mushtaq and M. Iqbal, *Partial ordering and congruences on LA-semigroups*, Indian J. pure appl. Math. **22(4)** (1991), 331-336.

[10] M. Petrich, *Congruences on inverse semigroups*, J. of Algebra **55** (1978), 231-256.

[11] M. Petrich, *Inverse Semigroups*, John Wiley and Sons, New York, 1984.

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