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PARTIAL COMPLETIONS OF BOOLEAN ALGEBRAS

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ABSTRACT. Similarly to notion of completeness and k-completeness we define a notion of partial completness with respect to a subalgebra C. We also give a categorial characterization of partially complete Boolean algebras, and a construction of partial completion.

Since the completion of a relatively small Boolean algebra could be very large, it is reasonable to consider the possibility of making a completion of just a subalgebra of given Boolean algebra. This makes even more sense for various types of lattices, where the completion of the whole lattice is not anymore in the same variety.

1. Partially complete Boolean algebras

To make a motivation for our definition, we will list some known facts on complete Boolean algebras.

Proposition 1.1. *(i) Boolean algebra B is complete iff for every partition $(b_i)_{i\in I}$, the mapping $\varphi : B \mapsto \prod_{i\in I}(b_i)$ defined by $\varphi(b) = (b \cdot b_i)_{i\in I}$ is an isomorphism. (b_i), as usually, denotes the principal ideal generated bu b_i.*

(ii) Let κ be a cardinal. Boolean algebra B is κ-complete iff every disjoint family D, of cardinality less than κ, could be extended to a partition $(b_i)_{i\in I}$ so that the mapping $\varphi : B \mapsto \prod_{i\in I}(b_i)$ defined by $\varphi(b) = (b \cdot b_i)_{i\in I}$ is an isomorphism.

Proof. (i) φ is trivialy a monomorphism, and it is onto since B is complete. On the other hand, let $(b_i)_{i\in I}$, be a disjoint family in B. Let now $(b_i)_{i\in I_1}$ be its extension to a maximal disjoint family which is a partition of one in B. Consider a sequence $(e_i)_{i\in I_1} \in \prod_{i\in I_1}(b_i)$ such that $e_i = b_i$ for $i \in I$, and 0 otherwise. The element corresponding to this sequence in the above isomorphism is $\sum_{i\in I} b_i$. Since every disjoint family in B has supremum, B is complete.

(ii) If B is κ-complete, then for every disjoint family of cardinality κ we just add the complement of its summ, to make the desired partition, and construct the isomorphism in the same way as in (i). For the other side, proof is just the same as in (i). $\square$

Having this proposition in mind we define a notion of partial completeness of a Boolean algebra over its subalgebra.

Definition. Let B be a Boolean algebra, and C its subalgebra. B is partially complete with respect to C (shorter "C-complete") if for every partition $(c_i)_{i\in I}$ of C, the mapping $\varphi : B \mapsto \prod_{i\in I}(c_i)$, defined by $\varphi(b) = (b\cdot c_i)_{i\in I}$ is an isomorphism.

Proposition 1.2. *Let B be a Boolean algebra, and C its subalgebra so that B is C-complete.*

i) Let $b \in B$, $(b_i)_{i\in I} \subset B$. For every partition $(c_i)_{i\in I}$, in C, $b = \sum_{i\in I} b\cdot c_i$, and there exists $\sum_{i\in I} b_i \cdot c_i$.

ii) C is a regular subalgebra of B.

iii) for every $D \subset C$, there exists $supD$ in B.

Property i) is equivalent to B being C-complete.

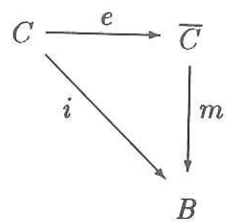
Proof. i) Follows from the fact that the induced mapping $\varphi : B \cong \prod_{i\in I}(c_i)$ is an onto mapping.

ii) It is enough to prove the preservation for disjoint sums. So let $c = \sum_{i\in I} c_i$ in C. Then, $\{c_i : i \in I\} \cup \{c'\}$ is a partition in C, and the element in $\prod_{i\in I}(c_i) \times (c')$ having coordinates c_i, $i \in I$, and 0 on the coordinate c' corresponds to c.

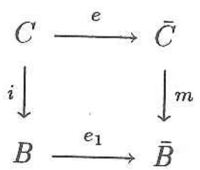
iii) In the case of disjoint family it is just special case of i) for $b = 1$. In the general case we can disjointize it on every step. $\square$

Definition. Let $C < B, S$. A homomorphism $\varphi : B \to S$ is C-complete if it preservs existing C-sums i.e. for every $\{c_i : i \in I\} \subset C$, $\varphi(\sum_{i\in I} c_i) = \sum_{i\in I} \varphi(c_i)$.

Proposition 1.3. *If B is C-complete then for any completion $\bar{C}$ of C there exists a C-complete embedding $m : \bar{C} \to B$, so that the following diagram commutes:*



Proof. Let $\bar{B}$ be a completion of B, and let $e_1 : B \to \bar{B}$ be the inclusion embedding. Since $f = e_1 \circ i$ is a complete embedding of C into complete Boolean algebra $\bar{B}$, by the Sikorski extension theorem, there exists a complete embedding $m : \bar{C} \to \bar{B}$, so that $f = m \circ e$.

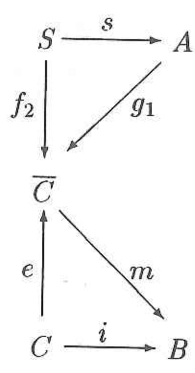


Let us prove that $Im(m) \subset B$. Since every $d \in \bar{C}$ is of the form $a = \sum e(D)$, for some $D \subset C$, we have:
$m(a) = m(\sum e(D)) = \sum m(e(D)) = \sum f(D) = \sum D \in B$
Hence m is the embedding with the desired properties. □

We will prove the anologue of Sikorski's extension theorem.

Proposition 1.4. *Let B be a Boolean algebra, and C its subalgebra. If B is C-complete, then for every Boolean algebra A, and its subalgebra S and any homomorphism $f : S \to B$, such that $Im(f) \subset C$, there exists a homomorphism $g : A \to B$ satisfying $g \circ s = f$.*

Proof. Let us consider the following diagram:



Let f_1 denote f with the restricted codomen C, and $f_2 = e \circ f_1$. m is the mapping from the preceeding proposition. Since $\bar{C}$ is a complete Boolean algebra, by the Sikorski extension criterion, there exists homomorphism g_1, so that the upper diagram commutes. $g = m \circ g_1$. Let us prove that $g \circ s = f$. Really,

$g \circ s = m \circ g_1 \circ s = m \circ f_2 = m \circ e \circ f_1 = i \circ f_1 = f$

□

We define a completion of a Boolean algebra B over a subalgebra C analogously to definition of a completion.

Definition. Let $C < B < S$. B is C-dense in S, if for every element $a \in S$ there exists a partition $(c_i)_{i \in I}$ of 1 in C and a family $(b_i)_{i \in I} \subset B$, so that $a = \sum i \in I b_i \cdot c_i$.

Definition. Let $C < B$. Boolean algebra S is a C-completion of B if:
(i) S is a C-complete Boolean algebra.
(iii) B is C-dense in S.

2. Construction of partial completions

Constructing a partial completion of a Boolean algebra we will use sheafs over a subalgebra following [2]. Sheaf of Boolean algebras is a generalization of the notion of subdirect product of an indexed family of sets so that the index set and the members of the family have topological structure. Then Boolean algebra is represented as a set of continuous choice functions. We will just mention here the definition and the main representation theorem.

Definition. Let S and X be topological spaces, π a mapping $\pi : S \to X$ and $\mathcal{B} = (B_p)_{p \in X}$ a family of Boolean algebras indexed by a set X. $\mathcal{S} = (S, \pi, X, \mathcal{B})$ is a sheaf of Boolean algebras if it satisfies the following conditions:
(i) $(B_p)_{p \in X}$ is a partition of S.
(ii) π is a projection i.e. $\pi[B_p] = \{p\}$ and π is continuous, open and a local homeomorphism.
(iii) Let $u \subset X$ be an open subset, and $f_1, \ldots, f_n \in \prod_{p \in u} B_p$ continuous functions from u to S, and $t(x_1, \ldots, x_n)$ and $t_1(x_1, \ldots, x_n)$ Boolean terms. Then, the set

$$\{p \in u : t(f_1(p), \ldots, f_n(p)) = t_1(f_1(p), \ldots, f_n(p))\}$$

is open.

Boolean algebras B_p are called stalks of the sheaf $\mathcal{S}$. The set of sections over u is usually denoted by $\Gamma_u(\mathcal{S})$, and the set of global sections by $\Gamma(\mathcal{S})$. For two sections f, g over u, $\|f = g\|_u = \{p \in u : f(p) = g(p)\}$. It is easy to see that algebra of global sections is a subdirect product of the stalks.

Definition. Let B be a Boolean algebra, $C < B$ and $X = UltC$. Let for $p \in X$, $< p >^{fi}$ be filter in B generated by p, $B_p = B/\langle p\rangle^{fi}$. Let also, for $b \in B$, $f_b : X \to S$ be the mapping defined by $f_b(p) = b/\langle p\rangle^{fi}$ and $\pi_p : B \to B_p$ be the canonical homomorphism. Finally, let $S = \bigcup_{p\in X} B_p$ be the topological space having $\mathcal{D} = \{f_b[u] : b \in B, u \subset X, \text{open}\}$ as a base of topology, and $\pi : S \to X$ projection. $\mathcal{S} = (S, \pi, X, (B_p)_{p\in X})$ is called the sheaf of B over subalgebra C.

Theorem 2.1. *Let the notation be as in the preceeding definition. $\mathcal{S}$ is a sheaf of Boolean algebras. Boolean algebra B is isomorphic to Boolean algebra of global sections of $\mathcal{S}$.*

Following ideas from [1], we define an algebra of dense open sections over a subalgebra.

Definition. We say a function $f \in \prod B/P$ is a dense open section of B if the set of all points at which f is continuous is a dense open set of X. $\Gamma_D(B)$ is the set of dense open sections of B. For the congruence relation $\cong$ on $\prod B/P$ defined by: $f \sim g$ if they agree on a dense open subset of X, $\Gamma_D B/\sim$ will be denoted by $\mathcal{R}(B)$, the algebra of dense-open sections of B.

We summarize a few properties of $\mathcal{R}(B)$. s denotes the isomorphism from B onto $ClopUltB$ from the Stone duality.

Proposition 2.2. *(i) A function $f \in \prod B/P$ is continuous at point $P \in X$ iff there is some $c \in P$ and some $b \in B$ so that f agrees with f_b on $s(c)$.*

(ii) The mapping $\varphi : \Gamma(B) \mapsto \mathcal{R}(B)$ defined by $\varphi(f) = f/\sim$, is an embedding, into a dense subalgebra.

Proof. (i) It is easy to get from the definition the known fact that $f_b(u)$, $b \in B$, u open in X, constitute the base for the topology of S. Therefore, $\{f_b(x) | x \in P\}$ is a neighboorhood basis for the point a/P. Hence, if f agrees with f_a on $s(x)$ then f is continuous at each point of $s(x)$. On the other hand, if f is continuous at P, then $f(P)$ equals b/P, for some $b \in B$, hence $f(P) = f_b(P)$. Since $f_b[X]$ is an open neighboorhood of b/P, by continuity of f, there exists a neighboorhood u of P which is mapped into $f_b[X]$, meaning that f and f_b agree on u. c is then, any member of B such that $s[c] \subset u$.

(ii) It is obviously a homomorphism. Let us check that it is $1-1$. So suppose $\varphi(f) = 0/\sim$. This would mean that $\|f \neq 0\|$ is of first category, which is impossible since it is a non-empty open set. To prove that $Im(\varphi)$ is dense in $\mathcal{R}(B)$, suppose that f is a nonzero dense open section. By part (i) of this proposition, there exists $c \in C$ and $b \in B$, so that $f = f_b$ on $S(c)$. Then for the global section g defined as f_b on $S(c)$, and zero otherwise, we have $g \leq f$. $\square$

In the sequal C-section will denote a memeber f of ΓB such that there exists $c \in C$ such that $f(P) = 1$ for $c \in P$, and zero otherwise. The set of C sections will be denoted by ΓC. By Theorem 2.1, C is isomorphic to ΓC, and we will identify them.

Proposition 2.3. *$\mathcal{R}(B)$ is C-complete.*

Proof. Consider $\{f_c : c \in D\}$, a family of C-sections. Let f be the dense open section defined by: $f(P) = 1$ iff $c \in P$ for some $c \in D$, 0 otherwise. It is really a dense open section, since , for $U = \bigcup\{S(c); c \in D\}$, it is continuous in every point of $U \cup intU^c$. □

Proposition 2.4. *$\Gamma B/\sim$ is C-dense in $\mathcal{R}(B)$.*

Proof. Let $f \in \mathcal{R}(B)$. Let it be continuous on a dense open set U. Let further $\{c_i : i \in I\}$ be a maximal disjoint family of clopen sets in U. Wlog, we can suppose that for each $i \in I$ there exists $b_i \in B$ so that for every $P \in c_i$, $f(P) = b_i/P$. If this is not the case, for every $P \in c_i$, we can find a neighbourhood so that for some $b \in B$, $f = f_b$ on that neighbourhood, and then find the finite subcover of every c_i, and finally take the refinement of $\{c_i : i \in I\}$. It is obviously $f/\sim = \sum_{i \in I} f_{b_i \cdot c_i}/\sim$. □

The folowing theorem directly follows from the preceeding propositions.

Theorem 2.5. *Let $C < B$. $\mathcal{R}(B)$ is a C-completion of B.*

References

[1] J. Harding, *Completions of orthomodular lattices II*, Order **10** (1993), 283-294.
[2] S. Koppelberg, *Handbook of Boolean algebras v.1 (J.D. Monk, R. Bonnet, eds.)*, North Holand, New York, 1989.

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