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THE INDUCED RELATIONS ON A POWER SET

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ABSTRACT. Let ϱ be a relation on the set S. By using the quantifiers and the relations ϱ i ϱ^{-1} a series of relations on the power set $Pow\,S$ are defined. The characteristics of these relations are studied and their classification is made.

1. The elementary induced relations

Let ϱ be a binary relation defined on a set S and $Pow\,S$ the power set of S. The relation ϱ on S induces a series of relations on $Pow\,S$.

Induced relations determined only by quantifiers we shall call the *elementary induced relations.* There are four such relations and they are defined by:

$$\text{(1)}\qquad \begin{cases} A\varrho_{11}B & \iff & (\exists a\in A, \exists b\in B) \quad a\varrho b\,, \\ A\varrho_{12}B & \iff & (\forall a\in A, \exists b\in B) \quad a\varrho b\,, \\ A\varrho_{21}B & \iff & (\exists a\in A, \forall b\in B) \quad a\varrho b\,, \\ A\varrho_{22}B & \iff & (\forall a\in A, \forall b\in B) \quad a\varrho b\,. \end{cases}$$

If we introduce the notations

$$A\varrho B \iff (\forall a\in A, \forall b\in B) \quad a\varrho b$$

and

$$a\varrho B \iff \{a\}\varrho B\,,$$

then we can write

$$A\varrho_{21}B \iff (\exists a\in A) \quad a\varrho B\,,$$

$$A\varrho_{22}B \iff A\varrho B\,.$$

It is obvious that

$$\text{(2)} \qquad \varrho_{12}, \varrho_{21} \subseteq \varrho_{11} \quad \textit{and} \quad \varrho_{22} \subseteq \varrho_{12}, \varrho_{21}$$

hold true.

With the help of the operation $^{-1}$ we obtain the new four induced relations. For them, on account of (2),

$$\text{(2')} \qquad \varrho_{12}^{-1}, \varrho_{21}^{-1} \subseteq \varrho_{11}^{-1} \quad \textit{and} \quad \varrho_{22}^{-1} \subseteq \varrho_{12}^{-1}, \varrho_{21}^{-1}$$

hold true.

Theorem 1. *The inclusions*

$$\text{(3)} \qquad \varrho_{21}^{-1} \subseteq \varrho_{12} \quad \textit{and} \quad \varrho_{21} \subseteq \varrho_{12}^{-1}$$

hold true.

Proof. If $A\varrho_{21}^{-1}B \Leftrightarrow B\varrho_{21}A$ is valid, then there exists at least one element of B which is ϱ-related to every element of A, so that every element of A is ϱ-related to at least one element of B, i.e. $A\varrho_{12}B$ holds true. Conversely, let $A\varrho_{12}B$ is valid, i.e. every element of A is ϱ-related to at least one element of B, but this does not guarantee that any two elements of A are ϱ-related to the same element of B. Therefore, the inclusion $\varrho_{21}^{-1} \subseteq \varrho_{12}$ holds true.

From the proved inclusion it directly follows that the second inclusion is also valid. □

With the help of the elementary induced relations and operations $\cap$ and $^{-1}$ with relations we can obtain new induced relations. On account of (2), (2') and (3) there are only ten new induced relations more.

Let us put

$$\alpha_{ij} = \varrho_{ij} \cap \varrho_{ij}^{-1} \qquad (i,j = 1,2)\,,$$

$$\beta_1 = \varrho_{11} \cap \varrho_{12}^{-1}\,, \quad \beta_2 = \varrho_{22} \cap \varrho_{21}^{-1}\,, \quad \gamma = \varrho_{12} \cap \varrho_{21}\,.$$

The relations α are obviously symmetric.

2. $\tilde{}$-induced relations

Let us consider the elementary relations induced by ϱ^{-1} and put

$$\tilde{\varrho}_{ij} = (\varrho^{-1})_{ij} \quad (i,j = 1,2)\,.$$

According to (1), using $a\varrho^{-1}b \Leftrightarrow b\varrho a$, we have

$$(1')\qquad \begin{cases} A\tilde{\varrho}_{11}B & \Longleftrightarrow \quad (\exists a \in A, \exists b \in B) \quad b\varrho a\,, \\ A\tilde{\varrho}_{12}B & \Longleftrightarrow \quad (\forall a \in A, \exists b \in B) \quad b\varrho a\,, \\ A\tilde{\varrho}_{21}B & \Longleftrightarrow \quad (\exists a \in A, \forall b \in B) \quad b\varrho a\,, \\ A\tilde{\varrho}_{22}B & \Longleftrightarrow \quad (\forall a \in A, \forall b \in B) \quad b\varrho a\,. \end{cases}$$

For these induced relations and their inverse relations the inclusions analogous to the inclusions (2), (2') and (3) hold true.

The relations $\tilde{\alpha}_{ij}$, $\tilde{\beta}_i$ and $\tilde{\gamma}$ $(i,j = 1,2)$ are defined like the corresponding relations from the point 1.

The relations induced by ϱ^{-1} we shall call $\tilde{}$- *induced relations.*

Theorem 2. *The equalities*

$$(4)\qquad \varrho_{11}^{-1} = (\varrho^{-1})_{11} \quad \text{and} \quad \varrho_{22}^{-1} = (\varrho^{-1})_{22}$$

hold true.

Proof. Since

$$\begin{aligned} A\varrho_{11}^{-1}B \quad & \Longleftrightarrow \quad B\varrho_{11}A \quad \Longleftrightarrow \quad (\exists b \in B, \exists a \in A) \quad b\varrho a \\ & \Longleftrightarrow \quad (\exists a \in A, \exists b \in B) \quad b\varrho a \quad \Longleftrightarrow \quad A\tilde{\varrho}_{11}B\,, \end{aligned}$$

we conclude that $\varrho_{11}^{-1} = \tilde{\varrho}_{11}$ is valid.

In a similar manner we can prove the second equality. □

3. The classification of the induced relations

Let σ be one of the induced relations, then the relations $\tilde{\sigma}$ and $\widetilde{\sigma^{-1}}$ we receive by using the conventions:

$$\widetilde{\mu \cap \nu} = \tilde{\mu} \cap \tilde{\nu}, \quad \widetilde{\mu^{-1}} = (\tilde{\mu})^{-1} \qquad \text{and} \qquad \tilde{\tilde{\mu}} = \mu\,.$$

For example, $\tilde{\beta}_1 = \tilde{\varrho}_{11} \cap (\tilde{\varrho}_{12})^{-1}$.

For every induced relation σ there exist three corresponding induced relations more: σ^{-1} , $\tilde{\sigma}$ and $(\tilde{\sigma})^{-1}$. We shall say that these four relations are *conjugate* to each other. If σ is an elementary induced relation, then we shall call all such relations the *elementary induced relations in the broader sense.*

Using the elementary induced relations in the broader sense and the operations $\cap$ and $^{-1}$ with relations, we receive the large number of new induced relations. There are 98 such relations. On account of inclusions (2), (2') and (3) and the corresponding inclusions for $\tilde{}$-induced relations, as well as the equalities (4), these new induced relations may be formed by at the most four elementary induced relations in the broader sense. Because of this fact all the induced relations may be classified into four classes:

Class A. The elementary induced relations in the broader sense; there are 12 such relations.

Class B. The induced relations formed by using only two relations of the Class A ; there are 34 such relations.

Class C. The induced relations formed by using only three relations of the Class A ; there are 36 such relations.

Class D. The induced relations formed by using only four relations of the Class A ; there are 16 such relations.

All the induced relations may be grouped into groups of four conjugate relations each. There are groups which contain the same relations (for example, $\tilde{\varrho}_{11} = \varrho_{11}^{-1}$).

In the Class A there are 2 groups of four and 2 groups of two relations each. Let the representatives of these groups be just the elementary induced relations. If we denote the groups by Ai $(i = 1,2,3,4)$, then those representatives are:

$$A1 : \varrho_{11} , \quad A2 : \varrho_{12} , \quad A3 : \varrho_{21} , \quad A4 : \varrho_{22} .$$

The groups $A1$ and $A4$ contain only two elements each.

In the Class B there are 5 groups of four, 6 groups of two and 2 groups of only one relations each. Let the representatives of the groups of the Class B be:

$$B1 : \varrho_{11} \cap \varrho_{11}^{-1} , \quad B2 : \varrho_{11} \cap \varrho_{12}^{-1} , \quad B3 : \varrho_{12} \cap \varrho_{21} , \quad B4 : \varrho_{12} \cap \varrho_{12}^{-1} ,$$

$$B5 : \varrho_{12} \cap \tilde{\varrho}_{12} , \quad B6 : \varrho_{12} \cap \tilde{\varrho}_{21} , \quad B7 : \varrho_{12} \cap (\tilde{\varrho}_{12})^{-1} ,$$

$$B8 : \varrho_{12} \cap (\tilde{\varrho}_{21})^{-1} , \quad B9 : \varrho_{21} \cap \varrho_{21}^{-1} , \quad B10 : \varrho_{21} \cap \varrho_{22}^{-1} ,$$

$$B11 : \varrho_{21} \cap \tilde{\varrho}_{21} , \quad B12 : \varrho_{21} \cap (\tilde{\varrho}_{21})^{-1} , \quad B13 : \varrho_{22} \cap \varrho_{22}^{-1} .$$

The groups $B4$, $B5$, $B7$, $B9$, $B11$ and $B12$ contain two and the groups $B1$ and $B13$ only one relations each.

In the Class C there are 8 groups of four and 2 groups of two relations each. Let the representatives of the groups of the Class C be:

$$C1 : \varrho_{11} \cap \varrho_{12}^{-1} \cap \tilde{\varrho}_{12} , \quad C2 : \varrho_{12} \cap \varrho_{21} \cap \tilde{\varrho}_{12} , \quad C3 : \varrho_{12} \cap \varrho_{21} \cap \tilde{\varrho}_{21} ,$$

$$C4: \varrho_{12} \cap \varrho_{21} \cap (\tilde{\varrho}_{12})^{-1}\,, \quad C5: \varrho_{12} \cap \varrho_{21} \cap (\tilde{\varrho}_{21})^{-1}\,, \quad C6: \varrho_{12} \cap \varrho_{12}^{-1} \cap \tilde{\varrho}_{12}\,,$$
$$C7: \varrho_{12} \cap \varrho_{12}^{-1} \cap \tilde{\varrho}_{21}\,, \quad C8: \varrho_{12} \cap \tilde{\varrho}_{21} \cap (\tilde{\varrho}_{21})^{-1}\,, \quad C9: \varrho_{21} \cap \varrho_{21}^{-1} \cap \tilde{\varrho}_{21}\,,$$
$$C10: \varrho_{21} \cap \varrho_{22}^{-1} \cap (\tilde{\varrho}_{21})^{-1}\,.$$

The groups $C1$ and $C10$ contain only two relations each.

In the Class D there are 2 groups of four, 3 groups of two and 2 groups of one relation each. Let the representatives of the groups of the Class D be:

$$D1: \varrho_{12} \cap \varrho_{21} \cap \tilde{\varrho}_{12} \cap \tilde{\varrho}_{21}\,, \quad D2: \varrho_{12} \cap \varrho_{21} \cap (\tilde{\varrho}_{12})^{-1} \cap (\tilde{\varrho}_{21})^{-1}\,,$$
$$D3: \varrho_{12} \cap \varrho_{21} \cap \tilde{\varrho}_{12} \cap (\tilde{\varrho}_{12})^{-1}\,, \quad D4: \varrho_{12} \cap \varrho_{21} \cap \tilde{\varrho}_{21} \cap (\tilde{\varrho}_{21})^{-1}\,,$$
$$D5: \varrho_{12} \cap \varrho_{12}^{-1} \cap \tilde{\varrho}_{12} \cap (\tilde{\varrho}_{12})^{-1}\,, \quad D6: \varrho_{12} \cap \varrho_{12}^{-1} \cap \tilde{\varrho}_{21} \cap (\tilde{\varrho}_{21})^{-1}\,,$$
$$D7: \varrho_{21} \cap \varrho_{21}^{-1} \cap \tilde{\varrho}_{21} \cap (\tilde{\varrho}_{21})^{-1}\,.$$

The groups $D1$, $D2$ and $D6$ contain two relations and the groups $D5$ and $D7$ only one relation each.

4. Some characteristics of the elementary induced relations

The following characteristics of relations: reflexivity, antireflexivity, symmetry, antisymmetry and transitivity are said to be the fundamental characteristics of relations.

Lemma. *If relations σ and τ have one of the fundamental characteristics, then the relations σ^{-1} and $\sigma \cap \tau$ have the same characteristic* ([5]).

If ϱ has one of the fundamental characteristics, then we shall find all the induced relations which have the same characteristic. We shall not be interested for such characteristics of induced relations which the relation ϱ does not have, i.e. which are not hereditary characteristics. For example, α are symmetric relations, but ϱ need not be.

Theorem 3. *If ϱ is a reflexive relation then ϱ_{11} and ϱ_{12} are also reflexive but ϱ_{21} and ϱ_{22} are not.*

Proof. Let ϱ be reflexive, i.e.

$$(\forall a \in S) \quad a\varrho a \tag{5}$$

and let $A \in Pow\, S$.

Since from (5) it follows $(\forall a \in A)\ a\varrho a$, we conclude that

$$(\exists a \in A) \quad a\varrho a \quad \Longleftrightarrow \quad A\varrho_{11}A\,,$$

so that ϱ_{11} is reflexive.

From (5) we obtain $(\forall a \in A, \exists a \in A)\ a\varrho a$, i.e. $A\varrho_{12}A$ so that ϱ_{12} is reflexive.

However, it need not exist $a \in A$ such that $a\varrho A$ holds true. From this it follows that ϱ_{21} and ϱ_{22} are not reflexive. □

Theorem 4. *If ϱ is an antireflexive relation then ϱ_{21} and ϱ_{22} are also antireflexive, but ϱ_{11} and ϱ_{12} are not.*

Proof. Let ϱ be antireflexive, i.e.

$$(\forall a \in S) \quad (a,a) \notin \varrho$$

and let $A \in Pow\ S$.

It may exist $a_1, a_2 \in A$ such that $a_1 \varrho a_2$ holds true, so that it may be $A \varrho_{11} A$. Similarly, for every $a_1 \in A$ it may exist $a_2 \in A$ such that $a_1 \varrho a_2$ holds true, so that it may be $A \varrho_{12} A$. Therefore, ϱ_{11} and ϱ_{12} are not antireflexive.

Since $(\forall a \in A) \quad (a,a) \notin \varrho$, it may not exist $a_1 \in A$ such that, for every $a_2 \in A$, $a_1 \varrho a_2$ holds true. Hence, $(A,A) \notin \varrho_{21}$ and $(A,A) \notin \varrho_{22}$. □

Theorem 5. *If ϱ is a symmetric relation then ϱ_{11} and ϱ_{22} are also symmetric but ϱ_{12} and ϱ_{21} are not and the equalities $\varrho_{12} = \tilde{\varrho}_{12}$ and $\varrho_{21} = \tilde{\varrho}_{21}$ hold true.*

Proof. Let ϱ be symmetric, i.e.

$$a \varrho b \Rightarrow b \varrho a \tag{6}$$

and let $A, B \in Pow\, S$.

From (1) it is obvious that ϱ_{11} and ϱ_{22} are symmetric and that ϱ_{12} and ϱ_{21} are not symmetric.

Since

$$\begin{aligned} A \varrho_{12} B \quad &\Longleftrightarrow \quad (\forall a \in A, \exists b \in B) \quad a \varrho b \\ &\Longrightarrow \quad (\forall a \in A, \exists b \in B) \quad b \varrho a \quad \Longleftrightarrow \quad A \tilde{\varrho}_{12} B , \end{aligned}$$

the equality $\varrho_{12} = \tilde{\varrho}_{12}$ holds true.

In exactly the same way we can prove the second equality. □

Theorem 6. *If ϱ is a transitive relation then ϱ_{11} is not transitive, but the other three elementary induced relations are transitive.*

Proof. Let ϱ be a transitive, i.e.

$$a \varrho b \wedge b \varrho c \Longrightarrow a \varrho c. \tag{7}$$

If $A \varrho_{11} B \wedge B \varrho_{11} C$ holds true, then

$$[(\exists a \in A, \exists b_1 \in B) \quad a \varrho b_1] \quad \wedge \quad [(\exists b_2 \in B, \exists c \in C) \quad b_2 \varrho c] .$$

Since, generally speaking, $b_1 \neq b_2$, then $A\varrho_{11}C$ need not hold, so that ϱ_{11} is not transitive.

If $A\varrho_{12}B \wedge B\varrho_{12}C$ holds true, then

$$[(\forall a \in A, \exists b_1 \in B) \quad a\varrho b_1\,] \quad \wedge \quad [(\forall b \in B, \exists c \in C) \quad b\varrho c\,]\,.$$

Since, for every $b \in B$, there exists $c \in C$ such that $b\varrho c$, then for $b_1 \in B$ must exist $c_1 \in C$ such that $b_1\varrho c_1$. Now from $a\varrho b_1$ and $b_1\varrho c_1$ it follows $a\varrho c_1$. Thus, for every $a \in A$ there exists $c_1 \in C$ such that $a\varrho c_1$, so that $A\varrho_{12}C$ holds true.

From

$$A\varrho_{21}B \ \wedge \ B\varrho_{21}C \quad \Longleftrightarrow \quad [(\exists a \in A) \quad a\varrho B\,] \wedge [(\exists b \in B) \quad b\varrho C\,]$$

it obviously follows that $(\exists a \in A) \quad a\varrho C$, i.e. $A\varrho_{21}C$.

The logical equivalences

$$\begin{aligned} A\varrho_{22}B \ \wedge \ B\varrho_{22}C \quad &\Longleftrightarrow \quad A\varrho B \ \wedge \ B\,\varrho C \\ &\Longrightarrow \quad A\varrho C \quad \Longleftrightarrow \quad A\varrho_{22}C \end{aligned}$$

evidently hold true. □

If ϱ is an antisymmetric relation, then it is easy to prove that the elementary induced relations may not be antisymmetric.

Theorem 7. *If relation ϱ and the representative of a group of conjugate relations have one of the fundamental characteristics, then the same characteristic have all the relations of that group.*

Proof. The assertion of the Theorem follows directly from the Lemma. □

5. The hereditary characteristics of the induced relations

According to Theorem 7 we can conclude that:

I. If ϱ is a reflexive relation, then the relations of the following groups: $A1-2$, $B1-2$, $B4-5$, $B7$, $C1$, $C6$ and $D5$ are also reflexive.

II. If ϱ is an antireflexive relation, then the relations from the following groups: $A3-4$, $B9-13$, $C9-10$ and $D7$ are also antireflexive.

III. If ϱ is a symmetric relation, then $\varrho^{-1} = \varrho$ holds true, so that the ˜-induced relations coincide with the ordinary induced relations. According to (2), (2'), (3) and Theorem 4, we can conclude that:

(i) The Classes C and D do not exist.

(ii) In the Class B the groups:

(a) $B1$ coincides with the group $A1$;

(b) $B2$ and $B5$ coincide with the group $A2$;

(c) $B8$ and $B11$ coincide with the group $A3$;

(d) $B10$ and $B13$ coincide with the group $A4$.

(iii) The following groups mutually coincide: $B3$ and $B6$, $B4$ and $B7$, $B9$ and $B12$.

Therefore, in this case we have only the following induced relations:

$A1: \varrho_{11}$,
$A2: \varrho_{12}, \varrho_{12}^{-1}$,
$A3: \varrho_{21}, \varrho_{21}^{-1}$,
$A4: \varrho_{22}$,
$B3: \varrho_{12} \cap \varrho_{21}, \varrho_{12}^{-1} \cap \varrho_{21}^{-1}$,
$B4: \varrho_{12} \cap \varrho_{12}^{-1}$,
$B9: \varrho_{21} \cap \varrho_{21}^{-1}$.

The relations from $A1$ and $A4$ are symmetric relations if ϱ is symmetric, but the relations from the groups $B4$ and $B9$ are always symmetric. The other relations are not symmetric.

IV. If ϱ is a transitive relation, then the relations of the following groups: $A2-4$, $B3-13$, $C2-10$ and $D1-7$ are also transitive.

V. If ϱ is a quasi-order, i.e. a reflexive and transitive relation, then the relations from the following groups: $A2$, $B4-5$, $B7$, $C6$ and $D5$ are also quasi-orders. In particular, the relations from the group $B4$ are equivalence relations.

VI. If ϱ is a strict order, i.e. an antireflexive and transitive relation, then the relations of the following groups: $A3-4$, $B9-13$, $C9-10$ and $D7$ are also strict orders.

VII. If ϱ is a tolerance, i.e a reflexive and symmetric relation, then ϱ_{11} is tolerance. Moreover, if ϱ is only reflexive, then $\varrho_{12} \cap \varrho_{12}^{-1}$ is also a tolerance.

VIII. If ϱ is an equivalence relation, then only

$$\delta = \varrho_{12} \cap \varrho_{12}^{-1}$$

is an equivalence relation. Therefore, the unique equivalence relation on $Pow\,S$ induced by an equivalence relation on S is just the relation δ.

It is easy to show that if ϱ is the equality relation on S, then δ is the equality relation on $Pow\,S$.

Remark. We may obtain induced relations also by using the operations $\cup$ and $^{-1}$ or by using all the three operations $\cap$, $\cup$ and $^{-1}$, but we shall not do it in this paper.

6. Some applications

Let $\mathbb{K}^n$ be a Euclidean n-dimensional linear space and $Int\,\mathbb{R}$ be the set of all the closed real intervals.

In [1] the relations on $Pow\ \mathbb{K}^n$ induced by the orthogonality of vectors in $\mathbb{K}$ are considered. In [2] and [3] some relations on $Int\ \mathbb{R}$ induced by the relation $\leq$ (less or equal) on $\mathbb{R}$ are studied.

In [4] the congruence relations on global semigroup of a group G induced by a congruence relation on G is considered.

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