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ON ONE CONGRUENCE RELATION ON A GLOBAL SEMIGROUP

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ABSTRACT. A group G end its global semigroup $Glb\,G$ is considered. Every congruence relation in the group G induces a congruence relation in the semigroup $Glb\,G$. These congruence relations and their classes are studied.

1. The induced equivalence relation.

1. Let ϱ be a binary relation defined on a set S and $Pow\,S$ the power set of S. The relation ϱ induces a series of relations on $Pow\,S$, defined by quantifiers. One of these relations is defined by

$$A\sigma B \iff (\forall a \in A, \exists b \in B) \quad a\varrho b, \tag{1}$$

where $A, B \in Pow\,S$.

Theorem 1. *If ϱ is a reflexive or transitive relation, then σ is also reflexive and transitive respectively.*

Proof. If ϱ is reflexive, then for every $a \in A$ there exists $a \in A$ such that $a\varrho a$ holds true. Therefore, $A\sigma A$ is valid for every $A \in Pow\,S$, so that σ is reflexive.

Let ϱ be transitive. If $A\sigma B$ and $B\sigma C$ hold true, then for every $a \in A$ and $b \in B$ there exist $b_1 \in B$ and $c \in C$ such that $a\varrho b_1$ and $b\varrho c$ hold true. Therefore, for $b_1 \in B$ there also exists $c_1 \in C$ such that $b_1\varrho c_1$. Now from $a\varrho b_1$ and $b_1\varrho c_1$ we get $a\varrho c_1$, i.e. for every $a \in A$ there exists $c_1 \in C$ such that $a\varrho c_1$ holds true. From this it follows that $A\sigma C$ holds true, so that σ is transitive. □

Using σ we can define a new induced relation δ on $Pow\,S$ by

$$\delta = \sigma \cap \sigma^{-1}. \tag{2}$$

In other words

$$A\delta B \iff [(\forall a_1 \in A, \exists b_1 \in B)\ a_1 \varrho b_1\ ] \wedge [(\forall b_2 \in B, \exists a_2 \in A)\ b_2 \varrho a_2\ ],$$

where $A, B \in Pow\,S$.

Theorem 2. *If ϱ is a reflexive and transitive relation on S , then δ is an equivalence relation on $Pow\,S$.*

Proof. From the definition of δ we can directly see that δ is symmetric.

We shell use the following characteristics of relations ([3]) : If relations α and β are reflexive (transitive), then so are the following relations : $\alpha \cap \beta$ and α^{-1} . Thus, since ϱ is reflexive and transitive so is the relation δ . Therefore, δ is an equivalence relation. □

In what follows we shell assume that ϱ is an equivalence relation.

We shell say that δ is an *equivalence relation induced by* ϱ and denote it by $\hat{\varrho}$.

Let us introduce the notation

$$\varrho_X = \{\varrho_x \mid x \in X \in Pow\,S\}$$

(Elements of ϱ_X are equivalence classes.)

Theorem 3. *The following logical equivalence*

$$A\hat{\varrho}B \iff \varrho_A = \varrho_B \tag{3}$$

holds true, where $A, B \in Pow\,S$.

Proof. Let $A\hat{\varrho}B$ be valid. If $\varrho_x \in \varrho_A$, then there exists $a \in A$ such that $\varrho_a = \varrho_x \in \varrho_A$. For this $a \in A$ there exists $b \in B$ such that $a\varrho b$ holds true, i.e. $\varrho_a = \varrho_b \in \varrho_B$. Therefore, $\varrho_x \in \varrho_B$, so that $\varrho_A \subseteq \varrho_B$ is valid. In exactly the same way we conclude that $\varrho_B \subseteq \varrho_A$ holds true. Therefore, $\varrho_A = \varrho_B$.

Conversely, let $\varrho_A = \varrho_B$ be valid. If $a \in A$, then $\varrho_a \in \varrho_A = \varrho_B$, so that there exists $b \in B$ such that $\varrho_a = \varrho_b \Leftrightarrow a\varrho b$ holds true. Similarly, from $b \in B$ it follows that there exists $a \in A$ such that $b\varrho a$. This means that $A\hat{\varrho}B$ is valid.

Therefore, (3) holds true. □

On account of Theorem 3 we conclude that we can describe equivalence classes for $\hat{\varrho}$ by

$$\hat{\varrho}_A = \{X \in Pow\,S \mid A\hat{\varrho}X\} = \{X \in Pow\,S \mid \varrho_A = \varrho_X\}.$$

Let us introduce the notation

$$[\hat{\varrho}_A] = \bigcup_{a\in A} \varrho_a . \tag{4}$$

Tho following logical equivalences

$$A\hat{\varrho}B \iff \hat{\varrho}_A = \hat{\varrho}_B \iff \varrho_A = \varrho_B \iff [\hat{\varrho}_A] = [\hat{\varrho}_B]$$

are now evident.

Let us now describe in detail the equivalence classes for $\hat{\varrho}$.

Theorem 4. *$X \in \hat{\varrho}_A$ if and only if X contains at least one element from every equivalence class of the family ϱ_A , but does not contain any element from any class out of ϱ_A . In other words*

$$X \in \hat{\varrho}_A \iff X \subseteq [\hat{\varrho}_A] \quad \wedge \quad [(\forall a \in A) \quad X \cap \varrho_a \neq \emptyset] .$$

Proof. From Theorem 3 we directly get the validity of that assertion. □

It is obvious that $[\hat{\varrho}_A]$ is the maximal element of $\hat{\varrho}_A$ (in the sense that there is no element of $\hat{\varrho}_A$ which contains $[\hat{\varrho}_A]$).

If $A \subseteq \varrho_b$, then $[\hat{\varrho}_A] = \varrho_b$, so that $X \in \hat{\varrho}_{\varrho_b} \Leftrightarrow X \subseteq \varrho_b$. This means that

$$\hat{\varrho}_{\varrho_b} = Pow \varrho_b . \tag{5}$$

2. Let $f : X \to Y$, then the relation kernel of f , $\kappa = ker f$, is an equivalence relation on X , defined by

$$(\forall a, b \in X) \qquad a\kappa b \iff f(a) = f(b) .$$

The equivalence classes of κ are defined by $\kappa_a = f^{-1}(f(a))$.

The kernel κ induces the equivalence relation $\hat{\kappa}$ on $Pow X$.

Since

$$\begin{aligned}[\hat{\kappa}_A] &= \bigcup_{a\in A} \kappa_a = \bigcup_{a\in A} f^{-1}(f(a)) = f^{-1}(\bigcup_{a\in A} f(a)) \\ &= f^{-1}\{f(a) \mid a \in A\} = f^{-1}(f(A))\end{aligned}$$

we have $[\hat{\kappa}_A] = f^{-1}(f(A))$, so that

$$A\hat{\kappa}B \iff f^{-1}(f(A)) = f^{-1}(f(B))$$

holds true.

2. The congruence relation on $Glb\,G$.

1. Let us now consider a group G . The set $Pow\,G$ with the global operations

$$AB = \{ab \mid a \in A, b \in B\} ,$$

$$A^{-1} = \{a^{-1} \mid a \in A\}$$

is the global monoid (i.e. semigroup with identity) with involution $^{-1}$ ([1], [2]).

This monoid we shall denote by $Glb\,G$. Suppose ϱ is a relation on G and σ and $\hat{\varrho}$ are the relations on $Pow\,G$, defined by (1) and (2) respectively.

Theorem 5. *If ϱ is a relation on $(G,.)$, compatible with the operations in G , then the relation σ is compatible with the global operations in $Glb\,G$.*

Proof. Let $A\sigma B$ and $X\sigma Y$ be valid $(A,B,X,Y \in Glb\,G)$. If $u \in AX$, then there exist $a \in A$ and $x \in X$ such that $u = ax$ holds true. In the other hand, from $A\sigma B$ $(X\sigma Y)$ it follows that for every $a \in A$ $(x \in X)$ there exists $b \in B$ $(y \in Y)$ such that $a\varrho b$ $(x\varrho y)$ is valid. Since ϱ is compatible with the binary operation, $ax\varrho by$ holds true. Thus, for any $u = ax \in AX$ there exists $v = by \in BY$ such that $ax\varrho by$ is valid. This means that $AX\sigma BY$ holds true, so that σ is compatible with the global binary operation.

Let $A\sigma B$ $(A, B \in Glb\,G)$ be valid, then for every $a \in A$ there exists $b \in B$ such that $a\varrho b$ holds true. Since ϱ is compatible with the unary operation $^{-1}$ in G , from $a\varrho b$ it follows $a^{-1}\varrho b^{-1}$. This means that for every $a^{-1} \in A^{-1}$ there exists $b^{-1} \in B^{-1}$ such that $a^{-1}\varrho b^{-1}$, i.e. $A^{-1}\sigma B^{-1}$ holds true. □ .

Theorem 6. *If ϱ is a congruence relation on G , then $\hat{\varrho}$ is also a congruence relation on $Glb\,G$.*

Proof. Using Theorem 5 and the well known result: If α and β are relations on G , compatible with operations in G , then so are the following relations: β^{-1} and $\alpha \cap \beta$, we can conclude that $\hat{\varrho}$ is a congruence relation on $Glb\,G$. □

2. If H is a subgroup of G , then the relation μ , defined by

$$(\forall x, y \in G) \quad x\mu y \iff x^{-1}y \in H \iff xH = yH ,$$

is an equivalence relation on G . The equivalence classes for μ are defined by $\mu_x = xH$.

Let us consider the equivalence relation $\hat{\mu}$ on $Glb\,G$ induced by μ. Since

$$[\hat{\mu}_A] = \bigcup_{a\in A} \mu_a = \bigcup_{a\in A} aH = AH ,$$

on account of (4),

$$A\hat{\mu}B \iff AH = BH$$

holds true.

Let us put $E = \{e\}$, where e is the identity of G. Then E is the identity of $Glb\,G$ ([4]). On account of $\mu_E = eH = H$ and (5) we have $\hat{\mu}_H = Pow\,H$ and $[\hat{\mu}_H] = H$.

Theorem 7. *Every normal subgroup H of G induces the congruence relation $\hat{\mu}$ on $Glb\,G$, defined by*

$$(\forall A, B \in Glb\,G) \quad A\hat{\mu}B \iff AH = BH , \tag{6}$$

where $\hat{\mu}_H = Glb\,H$.

Proof. If H is a normal subgroup of G, then μ is an equvalence relation on G, so that, account of Theorem 6, the induced equivalence relation $\hat{\mu}$ is a congruence relation on $Glb\,G$.

Since $\hat{\mu}_H = Pow\,H$ and H is a group, the equality $\hat{\mu}_H = Glb\,H$ must hold. □

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