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ON THE NUMBER OF EXPANSIONS OF COUNTABLE MODELS OF FIRST ORDER THEORIES

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ABSTRACT. Let **A** be a countable model of a countable first-order language L, and T be a first-order theory of a countable expansion $L' \supseteq L$. Let $\mathcal{S}$ denote the set of all expansions of **A** to L' that are models of T. It is proved that $\mathcal{S}$ can be embedded into a metric Stone space as a G_δ subset, and therefore $k = |\mathcal{S}|$ satisfies CH, i.e. either $k \leq \aleph_0$ or $k = 2^{\aleph_0}$. Several examples that illustrates this theorem are presented, too.

Works of Kueker [5], Reyes [9], Barwise [1], Makkai [7] and others, show that certain sets of model-theoretic objects related to a countable model **A**, as Aut**A** for example, behave as analytic subsets of the Cantor discontinuum. This property can be proved in several ways, and we shall present here two methods. The first one is based on the coding of model-theoretic objects by real numbers (or characteristic functions of certain subsets of real numbers). The second one is based on the properties of Lindenbaum algebras, and it has more model-theoretic nature.

1. Coding by reals

We shall present this method by example, i.e. we shall illustrate it in the case of the Kueker's theorem:

Theorem 1.1. *Let* **A** *be a countable model of a countable language. Then CH holds for* Aut**A**, *i.e.*

$$|\mathrm{Aut}\mathbf{A}| \leq \aleph_0 \quad or \quad |\mathrm{Aut}\mathbf{A}| = 2^{\aleph_0}.$$

The proof of this theorem that we shall present, is based on the following well known facts:

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1.1° For Borel subsets of real numbers R, CH holds (M. Suslin), i.e. if $X \subseteq R$ then $|X| \leq \aleph_0$ or $|X| = 2^{\aleph_0}$.

1.2° Cantor's triadic set K is a closed subset of R and it has the cardinality of continuum.

1.3° Suppose X is a countable set. Then 2^X is homeomorphic to the Cantor space K. Here $2 = \{0, 1\}$ has the discrete topology, and 2^X has Tychonoff product topology.

Now we proceed to the proof of Theorem 1.1. For the simplicity of notation, we shall assume that $\mathbf{A}$ is grupoid, i.e. $\mathbf{A} = (A, \cdot)$, where $\cdot$ is a binary relation on domain A. Let $\mathcal{F}$ be the set of all mappings (characteristic functions) $k : A^2 \to 2$ such that:

(1) $\forall a, a', b, b' \in A \ (k(a, a') = 1 \wedge k(b, b') = 1 \Rightarrow k(a \cdot b, a' \cdot b') = 1)$,
(2) $\forall a, a', b \in A \ (k(a, a') = 1 \wedge b \neq a \Rightarrow k(b, a') = 0)$,
(3) $\forall b \in A \ \exists a \in A \ k(a, b) = 1$.
(4) $\forall a, b, b' \in A \ (k(a, b) = 1 \wedge k(a, b') = 1 \Rightarrow b = b')$

If $f \in \mathrm{Aut}\mathbf{A}$, then let $k_f \colon A^2 \to 2$ be defined by $k_f(a, b) = 1$ iff $b = f(a)$, $a, b \in A$. Then it is not difficult to see that k_f satisfies the properties (1), (2), (3), (4), and that to different automorphisms f and f' correspond different k_f and $k_{f'}$ respectively (for example, if $f(a) = b \neq f'(a)$ then $k_f(a, b) = 1$, while $k_{f'}(a, b) = 0$). On the other hand, if $f \in \mathcal{F}$ and $f \colon A \to A$ is defined by $b = f(a)$ iff $k(a, b) = 1$, $a, b \in A$, then f is a well-defined function and $f \in \mathrm{Aut}\mathbf{A}$. Therefore, if the map ϕ is defined by $\phi \colon f \mapsto k_f$, $f \in \mathrm{Aut}\mathbf{A}$, then $\phi : \mathrm{Aut}\mathbf{A} \underset{\text{1-1}}{\overset{\text{onto}}{\longrightarrow}} \mathcal{F}$, thus

(5) $|\mathrm{Aut}\mathbf{A}| = |\mathcal{F}|$.

Further, let $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ be sets of functions $k \colon A^2 \to 2$ that satisfy conditions (1), (2), (3) and (4) respectively. Then:

$$\mathcal{F}_1 = \bigcap_{a,a',b,b' \in A} (\{k \in 2^{A^2} \mid k(a, a') = 0\} \cup \{k \in 2^{A^2} \mid k(b, b') = 0\} \cup \{k \in 2^{A^2} \mid k(a \cdot b, a' \cdot b') = 1\}),$$

$$\mathcal{F}_2 = \bigcap_{a,a' \in A} (\{k \in 2^{A^2} \mid k(a, a') = 0\} \cup \bigcap_{b \in A, b \neq a} \{k \in 2^{A^2} \mid k(b, a') = 0\}),$$

$$\mathcal{F}_3 = \bigcap_{b \in A} \bigcup_{a \in A} \{k \in 2^{A^2} \mid k(a, b) = 1\}$$

$$\mathcal{F}_4 = \bigcup_{a,b} (\{k \in 2^{A^2} \mid k(a, b) = 0\} \cup \bigcap_{b \in A, b' \neq b} \{k \in 2^{A^2} \mid k(a, b') = 0\}$$

Let 2^{A^2} be the product topology, where 2 has the discrete topology. Then the set $\{k \in 2^{A^2} \mid k(a, b) = \alpha\}, a, b \in A, \alpha \in 2$, is a clopen set. Thus, $\mathcal{F}_1$,

$\mathcal{F}_2$ and $\mathcal{F}_4$ are closed, while $\mathcal{F}_3$ is a countable intersection of open sets. As $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3 \cap \mathcal{F}_4$, it follows that $\mathcal{F}$ is a G_δ subset of the Cantor space 2^{A^2}, so by Fact 1.1° we have $|\mathcal{F}| \leq \aleph_0$ or $|\mathcal{F}| = 2^{\aleph_0}$. Hence, by (4) the theorem 1.1. is proved. □

2. Countable expansions in first-order logic

Now we shall present a proof of a general theorem, based on the properties of Lindenbaum algebras, that the set of all countable expansions of a countable model **A** of a first-order theory T satisfies CH. Let us introduce and review first some notation and terminology. Let L be a first-order language and **A** be a model of L. Then For_L denotes the set of all formulas of L, while Sent_L denotes the set of all sentences of L. Further, $L_A = L \cup \{\underline{a} \mid a \in A\}$, where $\underline{a}$ is the name of a, and $(A, a)_{a\in A}$ is the simple expansion of **A** to a model of L_A. By Th**A** we shall denote the set $\{\varphi \in \mathrm{Sent}_L \mid \mathbf{A} \models \varphi\}$. The Lindenbaum algebra of T over L is $\mathcal{L}(T, L) = \{[\varphi] \mid \varphi \in \mathrm{Sent}_L\}$, where $[\varphi] = \{\psi \in \mathrm{Sent}_L \mid T \vdash \varphi \leftrightarrow \psi\}$. The boolean operations $\{\cdot, +\}$ and constants $\{0, 1\}$ in $\mathcal{L}(T, L)$ are defined in the usual way: $[\varphi] \cdot [\psi] = [\varphi \wedge \psi]$, $[\varphi] + [\psi] = [\varphi \vee \psi]$, $[\varphi]' = [\neg\varphi]$, and $1 = [\theta]$, $0 = [\neg\theta]$, where θ is a tautology. In the following, we shall identify T with $\{[\varphi] \mid \varphi \in T\}$. If $T = \emptyset$ we shall write simply $\mathcal{L}(L)$ instead of $\mathcal{L}(T, L)$.

If **B** is an arbitrary Boolean algebra then $\mathbf{B}^*$ is the Stone space of **B**, i.e. the set of all ultrafilters of **B** with clopen sets $a^* = \{p \in \mathbf{B}^* \mid a \in p\}$, $a \in \mathbf{B}$, as a topological basis. Thus the dual space $\mathcal{L}(T, L)^*$ of $\mathcal{L}(T, L)$ is the set of all complete consistent theories of L that extend theory T. We remind the reader that the Cantor space 2^N is the Stone space of the free Boolean algebra Ω_ω with countable many free generators. The dual of an ideal $I \subseteq \mathbf{B}$ is $I^* = \{p \in \mathbf{B}^* \mid p \cap I \neq 0\}$. Observe that $I^* = \cup_{a\in I} a^*$ is an open set. For the rest of notation and terminology, we shall follow [3].

Lemma 2.1. *Let* **B** *be a countable Boolean algebra. Then* $\mathbf{B}^*$ *can be embedded into* 2^N *as a closed subset.*

Proof. Since Ω_ω is a free Boolean algebra, there is a homomorphism

$$h : \Omega_\omega \xrightarrow{onto} \mathbf{B}.$$

If $I = \ker(h)$, then I is an ideal of Ω_ω, thus $\mathbf{B} \cong \Omega_\omega / I$ and $\mathbf{B}^* \cong \Omega^*_\omega - I^*$. Further, I^* is open, hence $\Omega^*_\omega - I^*$ is closed in 2^N. □

Remark that for above I and $\mathcal{F} = \{p \in \Omega_\omega \mid \bigwedge_{a\in p} ha > 0\}$ we have $\mathcal{F} = \Omega^*_\omega - I^*$.

Lemma 2.2. *Let* $S \subseteq 2^N$ *be closed, and* $H \subseteq S$ *be* G_δ *in* S. *Then* H *is* G_δ *in* 2^N.

Proof. There are open subsets V_i, $i \in N$, of S so that $H = \cap_{i\in N} V_i$. Hence there are open $U_i \subseteq 2^N$ such that $V_i = S \cap U_i, i \in N$. Thus $H = S \cap (\cap_i U_i)$, and as S is a countable intersection of open subsets of 2^N, it follows that H is G_δ in 2^N. □

Let L be in the following a first-order language, $L' \supseteq L$ an expansion of L, T a theory of L' and $\mathbf{A}$ an arbitrary model of L. By $\boldsymbol{S}(\mathbf{A},T)$ we shall denote the set of all expansions $\mathbf{B}$ of the model $\mathbf{A}$ to L' such that $\mathbf{B} \models T$. Finally, let $k(\mathbf{A},T) = |\boldsymbol{S}(\mathbf{A},T)|$.

Theorem 2.3. *Let* $\mathbf{A}$ *be a countable model of a countable first-order language* L, $L' \supseteq L$ *be a countable expansion, and* T *a consistent theory of* L. *Then the number* $k(\mathbf{A},T)$ *satisfies CH, i.e. either* $k(\mathbf{A},T) \leq \aleph_0$ *or* $k(\mathbf{A},T) = 2^{\aleph_0}$.

Proof. Let $\mathbf{A}'$ be an expansion of $\mathbf{A}$ to L'_A such that $\mathbf{A}' \models T$. Let

$$P = P_{A'} = \{\varphi \in \mathrm{Sent}_{L'_A} \mid (\mathbf{A}', a)_{a\in A} \models \varphi\}$$

Then:

i. $T \cup \mathrm{Th}(\mathbf{A}, a)_{a\in A} \subseteq P$.
ii. If $\exists x \varphi x \in P$ then there is $a \in A$ such that $\varphi \underline{a} \in P$.
iii. P is a complete consistent theory.

Since $T \subseteq P$, by iii. we may assume that P is an ultrafilter of the Lindenbaum algebra $\mathcal{L}(T, L'_A)$.

Claim The correspondence $\Phi\colon \mathbf{A}' \mapsto P'_{\mathbf{A}}$ between expansions $\mathbf{A}'$ of $\mathbf{A}$ to L' such that $\mathbf{A}' \models T$, and ultrafilters of $\mathcal{L}(T, L'_A)$ satisfying conditions i.- iii. is one-to-one and onto.

Proof of Claim Suppose $\mathbf{A}'_1$ and $\mathbf{A}'_2$ are different expansions of $\mathbf{A}$, and let P_1 i P_2 be the corresponding sets satisfying conditions i.- iii. Since $\mathbf{A}'_1 \neq \mathbf{A}'_2$ there is, for example, an n-ary relation symbol R of L' such that for some $a_1, a_2, \ldots, a_n \in \mathbf{A}$, $R_{\underline{a}_1} \ldots \underline{a}_n \in P_1$ while $\neg R\underline{a}_1 \ldots \underline{a}_n \in P_2$. Thus $P_1 \neq P_2$, so Φ is 1-1.

Now, let P be any set of sentences satisfying conditions i.-iii. Since P is consistent, there is a model $\mathbf{B}'$ of P, and without loss of generality we may assume $\mathbf{A} \prec \mathbf{B}$, where $\mathbf{B}$ is a reduct of $\mathbf{B}'$ to L. Further, define $\mathbf{A}'$, an expansion of $\mathbf{A}$ by

$$R^{\mathbf{A}'} a_1 a_2 \ldots a_n \quad \text{iff} \quad R\underline{a}_1\underline{a}_2 \ldots \underline{a}_n \in P \qquad\qquad R \in \mathrm{Rel}_{L'},$$

$$F^{\mathbf{A}'} a_1 a_2 \ldots a_n = b \quad \text{iff} \quad (F\underline{a}_1\underline{a}_2 \ldots \underline{a}_n = \underline{b}) \in P \qquad F \in \mathrm{Fnc}_{L'},$$

where $\mathrm{Rel}_{L'}$ is the set of all relation symbols of L', and $\mathrm{Fnc}_{L'}$ is the set of all function symbols of L'. The structure $\mathbf{A}'$ is well-defined. For example, if $F \in \mathrm{Fnc}'_L$ and $a_1, a_2 \dots a_n \in A$ then $\exists x(F\underline{a}_1\underline{a}_2 \dots \underline{a}_n = x) \in P$, so by the property ii. there is $b \in A$ such that $(F\underline{a}_1\underline{a}_2 \dots \underline{a}_n = b) \in P$.

Now we shall prove $\mathbf{A}' \prec \mathbf{B}'$. Really, suppose $\mathbf{B}' \models \exists x\varphi(x, \underline{a}_1, \underline{a}_2, \dots \underline{a}_n)$, where $a_1, a_2 \dots a_n \in A$ and $\exists x\varphi(x, \underline{a}_1, \underline{a}_2, \dots \underline{a}_n) \in \mathrm{For}_{L_{A'}}$. Since P is complete, and $\mathbf{B}'$ is a model of P, we have $\exists x\varphi(x, \underline{a}_1, \underline{a}_2, \dots \underline{a}_n) \in P$, so by the property ii. there is $b \in \mathbf{A}$ such that $\varphi(b, \underline{a}_1, \underline{a}_2, \dots \underline{a}_n) \in P$ i.e. $\mathbf{B}' \models \varphi(b, \underline{a}_1, \underline{a}_2, \dots \underline{a}_n)$. Therefore, by Tarski-Vaught's Theorem, it follows $\mathbf{A}' \prec \mathbf{B}'$. Hence $P = P_{A'}$ i.e. Φ is onto, and this proves the claim.

Now, by Lemma 2.1. we may assume that the Stone space $\mathcal{L}(T, L'_A)^*$ of the Lindenbaum algebra $\mathcal{L}(T, L'_A)$ is closed subset of 2^N. Let $\mathcal{Y}$ be the set of all ultrafilters of $\mathcal{L}(T, L'_A)$ satisfying properties i.-iii. Then $\mathcal{Y}$ is the intersection of the following sets:

1. $\mathcal{U} = \{p \in \mathcal{L}(T, L'_A)^* \mid \mathrm{Th}(\mathbf{A}, a)_{a\in A} \subseteq p\} = \bigcap\{[\varphi]^* \mid \varphi \in \mathrm{Th}(\mathbf{A}, a)_{a\in A}\}$
2. $\mathcal{V} = \bigcap_{\exists x\varphi x \in \mathrm{For}_{L'_A}} ([\neg\exists x\varphi x]^* \cup \bigcup_{a\in A} [\varphi\underline{a}]^*)$.

The set $\mathcal{U}$ is obviously closed and $\mathcal{V}$ is G_δ in $\mathcal{L}(T, L'_A)^*$ as a countable intersection of open sets. Observe, for example, that $[\neg\exists x\varphi x]^a st$ is open. Therefore, by Lemma 2.2. $\mathcal{Y}$ is G_δ in 2^N, and so $|\mathcal{Y}|$ satisfies CH. By Claim $|\mathcal{S}(\mathbf{A}, T)| = |\mathcal{Y}|$, so the theorem follows. □

In the case of finite expansions, the above theorem is a simple consequence of Perfect Subset Theorem in [7]. However, the presented proof of Theorem 2.3. relies on rather basic model theory, and besides it gives an estimate of the complexity of the set $\mathcal{S}(\mathbf{A}, T)$ in the analytic hierarchy (G_δ).

3. Examples

In this part we shall list some examples that are consequences of Theorem 2.3. In the following, $\mathbf{A}$ is a countable model of a countable language L.

Example 3.1. We revise the Kueker's example from Section 1: if a $\mathbf{A}$ is a countable model of a countable language L, then $|\mathrm{Aut}\mathbf{A}|$ satisfies CH. Really, let $L(F) = L \cup \{F\}$ where F is an unary function symbol, and T be a theory of $L(F)$ which states that F is an automorphism in respect to symbols of L. If F is a new (i.e. $F \notin L$) unary function symbol, then axioms of T are:

1° $F(G(x_1, x_2, \dots, x_n)) = G(F(x_1), F(x_2), \dots, F(x_n))$, $G \in L$ is an n-ary function symbol.

2° $R(x_1, x_2, \dots, x_n) \Leftrightarrow R(F(x_1), F(x_2), \dots, F(x_n))$, $R \in L$ is an n-ary relation symbol.

3° Axioms which says that F is one-to-one and onto function.

Then obviously there is one-to-one correspondence between expansions of $\mathbf{A}$ to $L \cup \{F\}$ that are models of T and automorphism of $\mathbf{A}$. Therefore $|\mathrm{Aut}\ \mathbf{A}|$ satisfies CH.

Example 3.2. (Burris and Kwatinetz, see [4, p.35]) Let $\mathbf{A}$ be a countable algebra of a countable language. The set of all subalgebras Sub$\mathbf{A}$, the set of all endomorphisms End$\mathbf{A}$ and the set of all congruences Con$\mathbf{A}$ satisfy CH. To prove the first assertion, let U be a new unary relation symbol. Then subalgebras of $\mathbf{A}$ can be described as interpretations $U^{\mathbf{A}}$ in the expansion $(\mathbf{A}, U^{\mathbf{A}})$ which satisfy the conditions:

$c^{\mathbf{A}} \in U^{\mathbf{A}}$, c is a constant symbol of L,
For all $x_1, x_2, \ldots, x_n \in U^{\mathbf{A}}) \, F(x_1, x_2, \ldots, x_n) \in U^{\mathbf{A}}$, $F \in L$ is a function symbol,

i.e. the axioms

$U(c)$, c is a constant symbol of L,
$\forall x_1, x_2, \ldots, x_n (U(x_1) \wedge U(x_2) \wedge \ldots U(x_n) \Rightarrow U(F(x_1, x_2, \ldots, x_n)),$

where F is a function symbol of L.

Other cases are described in a similar way.

Example 3.3. As in the previous example one can find that the set of submodels of $\mathbf{A}$ (or countable sequences of submodels) that satisfy certain first order properties, also satisfies CH. For example, with the same notation as in the previous example, the set of all elementary submodels $\mathcal{E}(\mathbf{A})$ of $\mathbf{A}$ are described with following sentences:

1° Axioms for Sub$\mathbf{A}$,
2° $(\forall x_1, x_2, \ldots, x_n \in U)(\exists y \varphi \Rightarrow \exists y \in U \varphi)$, or more formally
$(\forall x_1, x_2, \ldots, x_n)((U(x_1) \wedge U(x_2) \wedge \ldots \wedge U(x_n)) \Rightarrow (\exists y \varphi \Rightarrow \exists y (U(y) \wedge \varphi)))$

By Tarski-Vaught theorem then easily follows that $U^{\mathbf{A}} \prec \mathbf{A}$ iff $(\mathbf{A}, U^{\mathbf{A}})$ satisfies the listed axioms.

Example 3.4. The set of all prime ideals of a countable commutative ring also satisfies CH. In other words, the Zariski space of a countable commutative ring satisfies CH. To see this observe that "I is a ring ideal" is a first-order property. It is described by universal closures of the following formulas in the language of rings $L = \{+, \cdot, 0\}$ with added unary predicate I which represent an ideal:

$$I(0),\ I(x) \wedge I(y) \Rightarrow I(x+y),\ I(x) \Rightarrow I(x \cdot y),\ I(xy) \Rightarrow I(x) \vee I(y).$$

In a similar way one can show that CH holds for the set of all maximal ideals of a countable ring.

Example 3.5. Let $\mathbf{P} = (P, \leq_P)$ be a countable, partially ordered set, $L = \{\leq\}$ and $L' = \{\leq, \preceq\}$. Taking for T the set of axioms of the linear ordering for $\preceq$ extending $\leq$, we find that the number of linear extensions of $\mathbf{P}$ satisfies CH. It is easy to design for each $0 < k \leq \aleph_0$ or $k = 2^{\aleph_0}$ a partially ordered set P which has exactly k linear extensions.

Some other families of subsets of $\mathbf{P}$ for which CH holds includes the set of all (maximal) chains, the set of all (maximal) antichains, and the set of all dense subsets of $\mathbf{P}$.

Example 3.6. If $\mathbf{A} = (A, G)$ is a planar graph, then by simple compactness argument one can show that $\mathbf{A}$ can be 4-colored, i.e. to elements of A can be assigned four colors so that the adjacent vertices are in different colors (assuming that the Four-coloring theorem for finite planar graphs is true). If $\mathbf{A}$ is infinite countable, let $a, b, c, d \in A$ be four distinct elements. Then every map $f: A \to \{a, b, c, d\}$ defines a coloring of $\mathbf{A}$. It is not difficult to write down first-order axioms which describes colorings of the above type. Thus, all 4-colorings of a countable planar graph satisfy CH.

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