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ON DIMENSIONS OF CLASS SPACES

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ABSTRACT. In our previous papers, we have introduced the notion of a class space, i.e. topologies on proper classes, and we defined and studied there the main topological concepts on such spaces. In this paper we shall discuss the notion of dimensions of class spaces. Analogues of Ind and ind for class spaces are defined, and their properties are studied.

1. Introduction

In our previous papers [3], [4], we have introduced the notion of a class space, i.e. topologies on proper classes, and explained the reasons for studying so defined spaces. In this paper we shall discuss and study the notion of dimensions of class spaces.

First we shall review some notation. We shall use the notation and definitions introduced in [3], [4]. For example, by capital letters $X, Y, Z, \ldots$ we denote classes, and by $x, y, z, \ldots$ sets. Greek letters may stand both for classes and for sets. For our metatheory we shall take NBG class theory if not otherwise stated. Further, we shall assume the usual constructions and definitions from set theory and class theory. For example, we remind the reader that a class X is transitive if from $x \in y \in X$ it follows $x \in X$. Throughout the paper K will denote a transitive class. Now we review the axioms for class spaces as we shall often refer to them.

Let K be a class and τ and σ be classes of subsets of K. We call triple $\mathcal{K} = (K, \tau, \sigma)$ a topological class if the following axioms are satisfied:

0. $\emptyset \in \tau, \quad \emptyset \in \sigma$
1. $x, y \in \tau \Rightarrow x \cap y \in \tau$
2. For any i, and $\langle x_j |\ j \in i \rangle$, $(\forall j \in i\ x_j \in \tau) \Rightarrow \cup_j x_j \in \tau$
3. For any $a \in K$ there is $x \in \tau$ such that $a \in x$.
4. $\forall x \in \tau \forall y \in \sigma\ x - y \in \tau$.

1'. $x, y \in \sigma \Rightarrow x \cup y \in \sigma$.
2'. For any i, and $\langle x_j |\ j \in i\rangle$, $(\forall j \in i\ x_j \in \sigma) \Rightarrow \bigcap_j x_j \in \sigma$
3'. For any subset x of K there is $y \in \sigma$ such that $x \subseteq y$.
4'. $\forall x \in \tau \forall y \in \sigma\ y - x \in \sigma$.

Elements of τ are open subsets while elements of σ are closed subsets of $\mathcal{K}$. The following proposition from [3] states that σ is uniquely determined by τ, and vice versa.

Proposition 1.1. *Let $\mathcal{K} = (K, \tau, \sigma)$ and $\mathcal{K}' = (K, \tau, \sigma')$ be class spaces. Then $\sigma = \sigma'$.*

Various topological notions for class spaces, such as the continuity, the compactness, product of class spaces etc were introduced in our previous papers, and various results concerning these notions were proved. The most important result obtained is that the finite product of compact class spaces is also a compact class space. In this paper we shall discuss and develop the notion of dimensions of topological class spaces.

The functions Ind (Brower-Čech, or large inductive dimension), ind (Menger-Urysohn, or small inductive dimension), dim (see [1]) are the most important dimension functions for topological spaces. By use of these functions we can classify topological spaces according to their dimensions. Let us remind that the notion of the *compartment* plays the main role in the definitions of Ind and ind, while in the definition of dim this role have the notion of covering and the order of covering.

If X is a standard topological space, then B is a compartment between disjoint, closed subspaces P and Q if $X \backslash B = O_1 \cup O_2$, where O_1, O_2 are disjoint open subsets of X which contain P and Q respectively. Every compartment B in X defines a partition of X of the form $X = O_1 \cup B \cup O_2$. Now, suppose $\mathcal{K} = (K, \sigma, \tau)$ is a class space. If K is a proper class, then obviously there is no a partition of K into sets in the above form, neither there is a covering of K by a set-family of sets. Therefore, there is no straightforward way for defining of dimension functions. Our aim is to propose possible definitions of dimension functions on class spaces.

In the following, $\operatorname{cl} x$, $\operatorname{int} x$, $\operatorname{fr} x$, $\operatorname{acc} x$ denote respectively the closure, the interior, the boundary, and the set of accumulation points of a set $x \subseteq X$ in a class space $\mathcal{X}$. If $x \subseteq a \subseteq X$, then these terms in respect of the subspace a are denoted by $\operatorname{cl}_a x$, etc. If not otherwise stated, N denotes the set of non-negative integers.

2. Dimension functions

Let $\mathcal{K} = (K, \tau, \sigma)$ be a topological class space. A set $B \in \sigma$ is a compartment between sets $P, Q \in \sigma$ if there is $U \in \tau$ such that $P \cup Q \cup B \subset U$ and

$U\backslash B = O_1 \cup O_2$, where $O_1, O_2 \in \tau$, $P \subset O_1$, $Q \subset O_2$, and $O_1 \cap O_2 = \emptyset$. Let us notice that the notion of the compartment is well defined. Namely, by the Axiom 3, for every $x \in P \cup Q \cup B$ there is $U_x \in \tau$ such that $x \in U_x$ and $U = \cup_{x \in P \cup Q \in B} U_x \supset P \cup Q \cup B$. As we have $U \in \tau$, $B \in \sigma$, by Axiom 4 it follows $U \backslash B \in \tau$, and also $O_1 \cup O_2 \in \tau$. For the compartment B we shall say that it is a *thin compartment* if it has the empty interior, i.e. $\operatorname{int} B = \emptyset$.

Theorem 2.1. *For every compartment B between sets $P, Q \in \tau$ there is a thin compartment $B' \subset B$.*

Proof. As B is a compartment between P and Q, there is $U \in \tau$ such that $P \cup Q \cup B \subset U$, $U \backslash B = O_1 \cup O_2$, $P \subset O_1$, $Q \subset O_2$, $O_1, O_2 \in \tau$, and $O_1 \cap O_2 = \emptyset$. Let us choose $O_1^* = U \backslash \operatorname{cl} O_2$. As $\operatorname{cl} O_2 \in \tau$ and $U \in \tau$, it follows $O_1^* \in \tau$. Also $O_1^* \supset O_1$ and $O_1^* \cap O_2 = \emptyset$. Now we show that $\operatorname{int}(\operatorname{cl} O_2 \cap B) = \emptyset$. Suppose, in contrary, that $\operatorname{int}(\operatorname{cl} O_2 \cap B) \neq \emptyset$. Then there is $x \in U_x \subset \operatorname{cl} O_2 \cap B$, so $U_x \subset B$ and $U_x \cap O_2 \neq \emptyset$, i.e. $B \cap O_2 \neq \emptyset$, a contradiction. Then $B' = \operatorname{cl} O_2 \cap B$ is a thin compartment between P and Q. □

Theorem 2.2. *Let $\mathcal{K} = (K, \tau, \sigma)$ be a topological class space, $P, Q \in \sigma$ and B a compartment between P and Q. If $X_0 \in \sigma$ is such that $P \cap X_0 \neq \emptyset$ and $Q \cap X_0 \neq \emptyset$, then $B_0 = B \cap X_0$ is a compartment between sets $P_0 = X_0 \cap P$ and $Q_0 = X_0 \cap Q$ in the space X_0 with the topology induced by $\mathcal{K}$.*

Proof. As B is a compartment between P and Q in $\mathcal{K}$, and $X_0 \in \sigma$, we have $B_0 \in \sigma$, and also B_0 is a closed subset of X_0. Further, there is $U \in \tau$ such that $P \cup Q \cup B \subset U$, $U \backslash B = O_1 \cup O_2$, $P \subset O_1$, $Q \subset O_2$, and $O_1 \cap O_2 = \emptyset$. Let $O_1^* = (U \cup U^*) \cap O_1 \cap X_0$ and $O_2^* = (U \cup U^*) \cap O_2 \cap X_0$, where $U^* \in \tau$ such that $X_0 \subset U^*$ exist by the axioms for class spaces. Further, O_1^* and O_2^* are open in X_0 and $X_0 \backslash B_0 = O_1^* \cup O_2^*$, thus B_0 is a compartment between P_0 and Q_0. □

Definition 2.3. Let $\mathcal{K} = (K, \tau, \sigma)$ be a topological class space where K is a transitive class. The function $\operatorname{Ind}_\sigma \colon \sigma \to N \cup \{-1\} \cup \{\infty\}$ is defined in the following way for $F \in \sigma$:

$\operatorname{Ind}_\sigma(F) = -1$ if and only if $F = \emptyset$.

Suppose that we have defined values $\operatorname{Ind}_\sigma(F) \leq n-1$. Then $\operatorname{Ind}_\sigma(F) \leq n$ if for any disjoint and closed sets P, Q in F there is a compartment $B \in \sigma$ between P and Q in K such that $\operatorname{Ind}_\sigma(B) \leq n-1$.

If $\operatorname{Ind}_\sigma(F) \leq n$ and there is a pair of disjoint and closed subsets of F such that for no compartment B between them in $\mathcal{K}$, $\operatorname{Ind}_\sigma(B) \leq n-2$ the we shall say that $\operatorname{Ind}_\sigma(F) = n$. If for no $n \geq -1$, $\operatorname{Ind}_\sigma(F) \leq n$, then we put $\operatorname{Ind}_\sigma(F) = \infty$.

By use of Ind_σ we define $\mathrm{Ind}_C\colon K \to N \cup \{-1, \infty\}$, and $\mathrm{Ind}_C(\mathcal{K})$. For $X \in K$ we put $\mathrm{Ind}_C(X) = -1$ if and only if $X = \emptyset$. Suppose that we have defined values $\mathrm{Ind}_C(X) \leq n-1$, $X \in K$. Then for $X \in K$, $\mathrm{Ind}_C(X) \leq n$ if for any disjoint and closed sets P, Q in space X there is a compartment $B \in \sigma$ between P and Q in K such that $\mathrm{Ind}_\sigma(B) \leq n-1$. Specially, $\mathrm{Ind}_C(X) \leq n$ if for any disjoint sets $P, Q \in \sigma$ there is a compartment $B \in \sigma$ such that $\mathrm{Ind}_\sigma(B) \leq n-1$. Similarly we define $\mathrm{Ind}_C(\mathcal{K}) \leq n$. Namely, $\mathrm{Ind}_C(\mathcal{K}) \leq n$ iff for all disjoint $P, Q \in \sigma$ there is a compartment $B \in \sigma$ for P and Q such that $\mathrm{Ind}_C(B) \leq n-1$.

If $\mathrm{Ind}_C(X) \leq n$ and if in the space X there is a pair of closed, disjoint sets such that for every compartment B between these sets in $\mathcal{K}$, $\mathrm{Ind}_\sigma(B) \geq n-1$, then we say that $\mathrm{Ind}_C(X) = n$. If for no $n \geq -1$, $\mathrm{Ind}_C(X) \leq n$, then we put $\mathrm{Ind}_C(X) = \infty$. Similarly we define $\mathrm{Ind}_C(\mathcal{K}) \leq n$.

Note 2.4 As K is a transitive class then $X \in K$ implies $X \subset K$, so $\mathcal{K}$ inherits a topological structure on X. Thus $\mathrm{Ind}_C(X)$ is well defined. But in general, elements of σ are not the elements of K, so $\mathrm{Ind}_C(F)$ is not necessarily defined for all $F \in \sigma$.

Note 2.5 $\mathrm{Ind}_C(\mathcal{K})$ is well-defined, and we see that $\mathrm{Ind}_C(\mathcal{K})$ is a numerical characteristic of $\mathcal{K}$ in respect to dimensions of elements of K. In this way we avoid the problem of defining of Ind_C on higher order classes (and type theory), at least for transitive topological class spaces.

Theorem 2.6. *Let $\mathcal{K} = (K, \tau, \sigma)$ be a class space. If $F, H \in \sigma \cap K$ and $F \subset H$, then $\mathrm{Ind}_C(F) \leq \mathrm{Ind}_C(H)$. Also, for all $F \in \sigma \cap K$, $\mathrm{Ind}_C(F) \leq \mathrm{Ind}_C(\mathcal{K})$.*

Proof. We prove $\mathrm{Ind}_C(H) \leq n \Rightarrow \mathrm{Ind}_C(F) \leq n$ by induction on n. If n=-1, then $H = \emptyset$, and so $F = \emptyset$, thus the inequality holds for $n = -1$. Suppose the inductive hypothesis for $n-1$, and let $\mathrm{Ind}_C(H) \leq n$. Suppose P and Q are disjoint, closed subsets of F. These sets are disjoint and closed subsets of H as well, so by the inductive hypothesis there is a compartment $B \in \sigma$ for these sets in K such that $\mathrm{Ind}_\sigma(B) \leq n-1$. Then B is obviously a compartment for P and Q in F, thus $\mathrm{Ind}_C(F) \leq n$.

As disjoint and closed subsets of $F \in \sigma \cap K$ are members of σ, it follows $\mathrm{Ind}_C(F) \leq \mathrm{Ind}_C(\mathcal{K})$. □

Corollary 2.7. *If $\sigma \subset K$ then the following assertions are equivalent.*

(a) *For all $F \in \sigma$, $\mathrm{Ind}_C(F) \leq n$.*

(b) $\mathrm{Ind}_C(\mathcal{K}) \leq n$.

Proof. The implication (b)⇒(a) follows from the above theorem. Now suppose (a). Then there are disjoint $P, Q \in \sigma$ such that $F = P \cup Q$. Then $F \in \sigma$. By the hypothesis $\mathrm{Ind}_C(F) \leq n$, so there is a compartment $B \in \sigma$ such that $\mathrm{Ind}_\sigma(B) \leq n-1$. Therefore $\mathrm{Ind}_C(\mathcal{K}) \leq n$. □

We note also the following statement.

Proposition 2.8. *If $\mathcal{K}$ is a topological class space and* $\mathrm{Ind}_C(\mathcal{K}) < \infty$, *then $\mathcal{K}$ is a normal class space.*

Theorem 2.9. *Let* Ind *be the large inductive dimension of (standard) topological spaces. Then for any class space $\mathcal{K}$ such that $\sigma \subset K$, and $F \in K$,* $\mathrm{Ind}(F) \leq \mathrm{Ind}_C(F)$.

Proof. We shall prove the statement of the theorem by induction on dimension. For $F = \emptyset$ the inequality obviously is true. Suppose the that the inequality holds holds for all natural numbers up to $n-1$. Suppose $\mathrm{Ind}_C(F) \leq n$ and let P, Q be disjoint closed, subsets of F. As $F \in \sigma$ then $P, Q \in \sigma$, and as $\mathrm{Ind}_C(F) \leq n$ there is a compartment $B \in \sigma$ in $\mathcal{K}$ such that $\mathrm{Ind}_\sigma(B) \leq n-1$. Then by Definition 2.3 it follows $\mathrm{Ind}_C|\sigma \cap K = \mathrm{Ind}_\sigma|\sigma \cap K$, thus $\mathrm{Ind}_C(B) \leq n-1$. Let $B_0 = B \cap F$. As $B_0 \in \sigma$, and σ is a subclass of K , by the inductive hypothesis it follows $\mathrm{Ind}(B) \leq \mathrm{Ind}_C(B)$. The dimension function Ind is monotonous on closed subsets, so $\mathrm{Ind}(B_0) \leq n-1$, as by Theorem 2.2 for the compartment B_0 between closed sets P and Q in F we have $\mathrm{Ind}(F) \leq n$. □

Definition 2.10. Let K be a transitive class and $\mathcal{K} = (K, \tau, \sigma)$ be a class space. The function $\mathrm{ind}_\sigma\colon \sigma \to N \cup \{-1, \infty\}$ is defined in the following way: $\mathrm{ind}_\sigma(F) = -1$ if and only if $F = \emptyset$. Suppose that we have defined values $\mathrm{ind}_\sigma(F) \leq n-1$, $F \in \sigma$. Now, we put $\mathrm{ind}_\sigma(F) \leq n$ if for any point $p \in F$ and any closed $Q \subset F$ such that $p \notin Q$ there is a compartment $B \in \sigma$ in $\mathcal{K}$ between p and Q such that $\mathrm{ind}_\sigma(B) \leq n-1$. If $\mathrm{ind}_\sigma(F) \leq n$ and if there is $p \in F$ and closed $Q \subset F$ such that $p \notin Q$ so that for all compartment B in $\mathcal{K}$ between p and Q we have $\mathrm{ind}_\sigma(B) \geq n-1$, then we put $\mathrm{ind}_\sigma(F) = n$. If for no integer $n \geq -1$, $\mathrm{ind}_\sigma(F) \leq n$, then we put $\mathrm{ind}_\sigma(F) = \infty$.

By use of ind_σ we define new dimension function $\mathrm{ind}_C\colon K \to N \cup \{-1, \infty\}$ and the value $\mathrm{ind}_C(K)$ as follows. If $X = \emptyset$ then we put $\mathrm{ind}_C(X) = -1$. Suppose that we have defined values $\mathrm{ind}_C(X) \leq n-1$, $X \in K$. Then we put $\mathrm{ind}_C(X) \leq n$ if for any point p and any closed subset Q of space X such that $p \notin Q$ there is a compartment $B \in \sigma$ for p and Q in $\mathcal{K}$ such that $\mathrm{ind}_\sigma(B) \leq n-1$. Similarly we define $\mathrm{ind}_C(\mathcal{K}) \leq n$. Namely, $\mathrm{ind}_C(\mathcal{K}) \leq n$ iff for any point $p \in K$ and closed $Q \in \sigma$ such that $p \notin Q$ there is a compartment $B \in \sigma$ for p and Q such that $\mathrm{ind}_C(B) \leq n-1$.

In particular, $\mathrm{ind}_C(\mathcal{K}) \leq n$ if for every point $p \in K$ and $Q \in \sigma$, $p \notin Q$, there is a compartment $B \in \sigma$ in $\mathcal{K}$ with $\mathrm{ind}_\sigma(B) \leq n-1$.

If $\mathrm{ind}_C(X) \leq n$ and if there is $p \in X$ and closed $Q \subset X$ such that $p \notin Q$ so that for all compartment B in $\mathcal{K}$ between p and Q we have $\mathrm{ind}_C(X) \geq n-1$, then we put $\mathrm{ind}_C(X) = n$. If for no integer $n \geq -1$, $\mathrm{ind}_C(X) \leq n$, then we put $\mathrm{ind}_C(X) = \infty$.

Theorem 2.11. *Let $\mathcal{K}$ be a class space and $X, Y \in K$. Then $X \subset Y$ implies $\mathrm{ind}_C(X) \leq \mathrm{ind}_C(Y)$. Also for all $X \in K$, $\mathrm{ind}_C(X) \leq \mathrm{ind}_C(K)$.*

Proof. For $n = -1$, $Y = \emptyset$ implies $X = \emptyset$ so in this case the inequality holds. Suppose the inductive hypothesis, that the inequality holds for $\mathrm{ind}_C(Y) \leq n-1$. Suppose $\mathrm{ind}_C(Y) \leq n$, and p be a point and Q a closed subset of space X. Then there is Q^* in $\mathcal{K}$ such that $Q = Q^* \cap X \subset Q^* \cap Y$ and $p \notin Q^* \cap Y$. Since $\mathrm{ind}_C(Y) \leq n$ there is a compartment $B \in \sigma$ for p and Q^* such that $\mathrm{ind}_\sigma(B) \leq n-1$. Then B is a compartment between p and Q thus $\mathrm{ind}_C(X) \leq n$. In a similar way we prove $\mathrm{ind}_C(X) \leq \mathrm{ind}_C(K)$ for all $X \in K$. □

Corollary 2.12. *The following statements are equivalent:*

(a) *For all $X \in K$, $\mathrm{ind}_C(X) \leq n$.*
(b) $\mathrm{ind}_C(K) \leq n$.

Proof. The part (b)⇒(a) follows from the above theorem. Suppose (a). Let us choose $p \in K$ and $Q \in \sigma$ such that $p \notin Q$. Let us put $X = \{p\} \cup Q$. As $\mathrm{ind}_C(X) \leq n$ and Q is closed in X there is a compartment $B \in \sigma$ such that $\mathrm{ind}_C(B) \leq n-1$, and this means that $\mathrm{ind}_C(K) \leq n$. □

An immediate consequence of the definition of ind_C is the following assertion.

Proposition 2.13. *If $\mathrm{ind}_C(\mathcal{K}) < \infty$ then (K) is a regular topological class space.*

Proposition 2.14. *Let $\mathcal{K}$ be a class space. Then for every $X \in K$ we have $\mathrm{ind}(X) \leq \mathrm{ind}_C(X)$, where $\mathrm{ind}(X)$ is the small inductive dimension of X.*

Proof. The proof is by induction. This inequality is obviously true for $X = \emptyset$. Suppose the inductive hypothesis, that the inequality holds for all dimensions $\leq n-1$. Suppose $\mathrm{ind}_C(X) \leq n$ and let p be a point and Q a closed subset of X such that $p \notin Q$. As $\mathrm{ind}_C(X) \leq n$, there is a compartment B in $\mathcal{K}$ such that $\mathrm{ind}_\sigma(B) \leq n-1$. From Definition 1.10 it follows that $\mathrm{ind}_C|\sigma = \mathrm{ind}_\sigma$, so $\mathrm{ind}_C(B) \leq n-1$. Further, $B_0 = B \cap X$ is a compartment in X between the point p and the subset Q, and by the inductive hypothesis and Theorem 1.11 we have $\mathrm{ind}_C(B_0) \leq \mathrm{ind}_C(B) \leq n-1$ and $\mathrm{ind}(B_0) \leq \mathrm{ind}_C(B_0)$, hence $\mathrm{ind}(B_0) \leq n-1$. Therefore $\mathrm{ind}(X) \leq n$. □

Theorem 2.15. *Let $\mathcal{K}$ be a topological class space. Then for every $X \in K \cap \sigma$ we have* $\mathrm{ind}_C(X) \leq \mathrm{Ind}_C(X)$. *Specially, if $\mathcal{K}$ is a T_1 class space then* $\mathrm{ind}_C(K) \leq \mathrm{Ind}_C(K)$.

Proof. The proof is by induction. This inequality is obviously true for $X = \emptyset$. Suppose the inductive hypothesis, that the inequality holds for all dimensions $\leq n-1$. Suppose $\mathrm{ind}_C(X) \leq n$ and let p be a point and Q a closed subset of X such that $p \notin Q$. As $\mathcal{K}$ is T_1 class space then $\{p\}$ is a closed subset of X. As $\mathrm{Ind}_C(X) \leq n$ there is a compartment $B \in \sigma$ such that $\mathrm{Ind}_\sigma(B) \leq n-1$. Further, on $K \cap \sigma$ we have $\mathrm{Ind}_\Sigma = \mathrm{Ind}_C$ therefore $\mathrm{Ind}_C(B) \leq n-1$. By the inductive hypothesis $\mathrm{ind}_C(B) \leq \mathrm{Ind}_C(B) \leq n-1$, so $\mathrm{ind}_C(X) \leq n$ □

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