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A CLASSIFICATION OF LOOPS ON AT MOST SIX ELEMENTS

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ABSTRACT. Eight kinds of equivalence classes (five of which are new) within the family $L(n)$ of finite loops on n elements ($n \leq 6$) are considered. The classes arise by combining the operations of isotopy over $L(n)$ (with some of its specializations) and loop-parastrophy (parastrophy followed by a special isotopy, which returns the image to $L(n)$).

The used isotopies are triples of permutations of the ground-set (applied successively to rows, columns and elements of the associated Cayley table) which map $L(n)$ onto $L(n)$. Classical isotopy and isomorphic classes correspond to the triples of the form (p, q, r) and (p, p, p) respectively. Three new natural kinds of interclasses, denoted as C-, R- and E-classes, correspond to the triples of the form (q, p, p), (p, q, p) and (p, p, q) respectively. The combinations "isotopy over $L(n)$ + loop-parastrophy" and "isomorphism + loop-parastrophy" lead to the classical main classes and to a new kind of classes, denoted as Π-classes. Finally, a new kind of classes, called parastrophic closures, corresponds to the transitive closure of the loop-parastrophy operator.

Cardinalities, intersections and dualities for all the eight kinds of equivalence classes of loops are completely determined for $n \leq 6$. In additicn, the following theorem, related to classical isomorphic, isotopy and main classes, is proved by using the new Π-classes: All the isotopy classes within a main class have the same family of cardinalities of their included isomorphic classes.

1. Introduction

Isotopy classes, isomorphic and main classes belong to the "folklore" of the theory of latin squares and loops. These classes were studied, for example, in [8], [6], [4], [7].

In particular, the figures 9408, 109, 22 and 12 of Table 1. were for the first time correctly determined in the papers [8], respectively [6]. These figures were confirmed by computer in [4]. A systematic tabulation of latin squares on at most six elements and of some their properties was given in [7]. An extensive review of the related results was given in the book [5], Sections 4.2 and 4.3.

In this paper are additionally considered five new ([2]) kinds od equivalence classes of loops: $C-$, $R-$, $E-$, $\Pi-$classes and parastrophic closures. The relationships among all the eight kinds of classes are studied in detail for the case of loops on at most six elements.

Isotopy and isomorphic classes of latin squares correspond to the isotopies determined by three and one permutation of the ground-set. $C-$, $R-$ and $E-$classes correspond to the cases when exactly two among the three permutations determining a loop-preserving isotopy – coincide.

A very small modification (abandoning of fixing the unit) of the algorithm for generating isomorphic classes of loops generates ([1]) $C-$ and $R-$classes. On the other hand, $C-$ and $R-$classes can be further used ([2]) for a construction of isotopy classes.

It is known ([3]) that iterative applications of parastrophic operators *within the class of loops* (to a fixed initial loop) – produce loops belonging to at most six different isomorphic classes. Parastrophic closures are obtained when the arising loops themselves are considered, instead of their isomorphic classes. The upper bound for the cardinality of parastrophic closures with loops of order n is equal ([2]) to $6 \cdot \max \text{g.c.d.}(s_1, \dots, s_k)$, where the maximum is taken over all the partitions $n - 1 = s_1 + \dots + s_k$.

The relationships between $\Pi-$classes and isomorphic classes are completely analogous to the relationships between main classes and isotopy classes.

The inclusion chart of the considered kinds of loop classes has the following outlook:

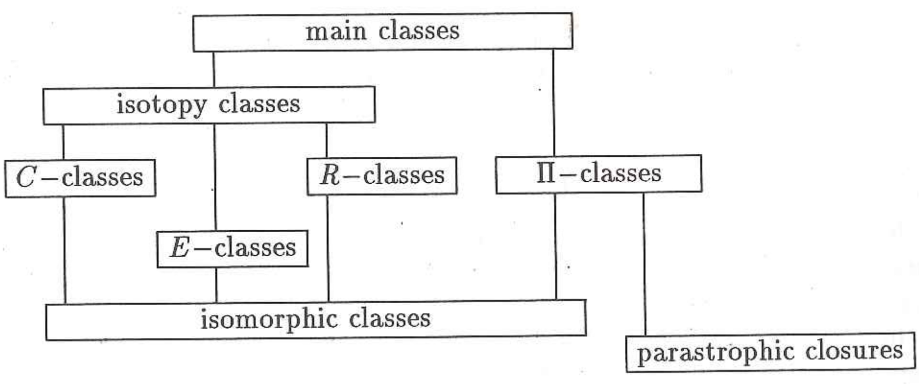

Figure 1.

In Table 1. are given some summary data for $n \leq 6$, which include cardinality of the family $L(n)$ of all loops of order n, as well as the number of all the above defined subclasses of $L(n)$. The figures for the three well-known kinds of classes can be also found in [5]:

n	≤ 3	4	5	6
cardinality of $L(n)$	1	4	56	9408
number of isomorphic classes in $L(n)$	1	2	6	109
number of E–classes in $L(n)$	1	2	5	103
number of C– (also number of R–) classes in $L(n)$	1	2	3	40
number of isotopy classes in $L(n)$	1	2	2	22
number of Π–classes in $L(n)$	1	2	4	40
number of main classes in $L(n)$	1	2	2	12
number of parastrophic closures in $L(n)$	1	4	14	832

Table 1.

It would be hard to extend such results to larger values of n, since $|L(7)| = 16.942.080$ ([5]).

The classes were enumerated and analysed with the aid of a PC computer, by using algorithms given in [1]. Most of the running time was spent for the generation of parastrophic closures. This is a consequence of the fact that parastrophic closures are not superclasses of isomorphic classes.

The questions concerning the relationships among the considered classes of loops of order n are obviously trivial for $n \leq 3$. The full description will be given for $n = 6$, while the corresponding data for $n \in \{5, 4\}$ will be briefly listed in the last section.

Isomorphic classes of loops as well as their cardinalities are listed in the Appendix. These classes are basic constituents of all the considered classes of loops except for the parastrophic closures.

2. Definitions and denotations

Let $S(n)$ denote the set $\{1, \dots, n\}$.

A *latin square* of order n is an $n \times n$ matrix A with elements in $S(n)$, which satisfies that there are no two coinciding elements in the same row or in the same column of A.

A *loop* (with unit 1) of order n is a latin square A of order n, which additionally satisfies $A[i,1] = A[1,i] = i$, for $1 \leq i \leq n$.

Let $L(n)$ denote the family of loops of order n.

We proceed with definitions of eight kinds of equivalence classes over $L(n)$.

Two loops X and Y of order n belong to the same *isotopy class* if there exists an *isotopy* , i.e., a triple $T = (p, q, r)$ of permutations of $S(n)$ satisfying $Y[p(i), q(j)] = r(X[i,j])$, for$1 \leq i, j \leq n$. In particular, if T is of the form

(p,p,p), (q,p,p), (p,q,p) or (p,p,q), then the loops X and Y are respectively said to belong to the same *isomorphic class*, *C−class*, *R−class* or *E−class*.

The *type of an isotopy class* is the family of cardinalities of the included isomorphic classes.

Let r_A and l_A respectively denote the permutations of $S(n)$ which produce the right and the left inverse elements of the loop A (thus $A[i,r_A(i)] = 1$ and $A[l_A(i),i] = 1$ for $i \in S(n)$).

Each loop A has six *loop-parastrophes* $A, \rho(A), \lambda(A), \tau(A), \lambda\tau(A), \rho\tau(A)$, associated to it, where τ is the transposition operator, while the operators ρ and λ have the following meaning (denotations ρ and λ are in accordance with the denotations used in [3]): $\rho(A)[r_A(i), A[i,j]] = j$, and $\lambda(A)[A[i,j], l_A(j)] = i$, for $1 \le i,j \le n$.

Two loops X and Y from $L(n)$ are said to belong to the same *main class* if there exists another loop $Z \in L(n)$, such that X and Z belong to the same isotopy class and Y is a loop-parastroph of Z. In particular, if the word "isotopy" in this definition is replaced by the word "isomorphic", then X and Y are said to belong to the same $\Pi-$*class*.

Two loops X and Y from $L(n)$ are said to belong to the same *parastrophic closure* if there exists a sequence $X = Z_1, Z_2, \ldots, Z_k = Y$ of loops from $L(n)$, such that Z_{i+1} is a loop-parastroph of Z_i, for $1 \le i \le k-1$. The parastrophic closure, associated to a loop A, will be denoted by $PC(A)$.

The *order* O of a permutation p is the smallest natural number such that p^O is the identical permutation.

The ordinal numbers of isotopy classes will be followed by the letter "I". The ordinal numbers of $C-$ and $R-$ classes will be usually followed by the letters "C" and "R", respectively. No additional letters will be used with the ordinal numbers of isomorphic classes.

3. $C-$, $R-$, $E-$ and isotopy classes

Given a permutation p of $S(n)$, the permutations q of $S(n)$, such that the isotopies (q,p,p), (p,q,p), and (p,p,q) map $L(n)$ to $L(n)$ – are characterized in [2]. Although the definitions of $C-$, $R-$ and $E-$classes are analogous, it turns out, when consideration is restricted to the loops in $L(n)$, that $E-$classes have a special role.

Namely, isotopies **a)** (q,p,p) **b)** (p,q,p) **c)** (p,p,q) map a loop X from $L(n)$ to another loop in $L(n)$ if and only if ([2]) for $1 \le i \le n$:

a) $q(i) = p(X[i,p^{-1}(1)])$
b) $q(i) = p(X[p^{-1}(1),i])$
c) $q(X[p^{-1}(1),i]) = q(X[i,p^{-1}(1)]) = p(i)$

The *commutator* of a loop $X \in L(n)$ is the set of those elements

$k \in S(n)$, which satisfy that $X[k,j] = X[j,k]$, for each $j \in S(n)$. The number of commutators is ([2]) an invariant of an $E-$class. An abridged search for $E-$classes can be gained by partitioning (representatives of) isomorphic classes w.r.t. this number.

Those $E-$classes, the loops of which have more than one commutator, are listed (by means of their isomorphic subclasses) in the separate fields of 1., 3. and 5. column of Table 2. (the remaining $E-$classes necessarily coincide with isomorphic classes). Each represented $E-$class has in the next column to the right associated an expression of the form $(f(A) \cdot c(A) + \dots)$, where:

- $c(A)$ is the cardinality of the isomorphic class determined by A
- $f(A)$ is the number of isotopies of the form **c**), which fix the loop A.

2 commutators		3 commutators		6 commutators	
3	(2·120)			1	$(12 \cdot 60)$
50	(2·120)			2	$(12 \cdot 60)$
92	(2·120)	4, 79	$(6 \cdot 20 + 6 \cdot 40)$	39	$(120 \cdot 6)$
94	(2·120)	8, 83	$(2 \cdot 60 + 2 \cdot 120)$	40, 42	$(8 \cdot 60 + 8 \cdot 30)$
103	(2·120)	47, 78	$(6 \cdot 20 + 6 \cdot 40)$	43, 55	$(4 \cdot 120 + 4 \cdot 60)$
104	(2·120)	54, 82	$(2 \cdot 60 + 2 \cdot 120)$	49	$(12 \cdot 60)$

Table 2.

The cardinality of the $E-$class determined by A is equal to

$$\frac{1}{f(A)} \cdot (n-1)! \cdot (\text{number of commutators of } A);$$

the numerator is equal to the number of isotopies of the form **c**).

The next two tables give the intersection and inclusion relationships among isotopy, $C-$, $R-$ and isomorphic classes over $L(6)$.

The denotations in the $x-$th row and the $y-$th column of Table 3. mean that the isomorphic class $10 \cdot x + y$ belongs to the intersection of the $C-$class C and the $R-$class R :

Each $C-$class has a non-empty intersection with each $R-$class within the same isotopy class ([2]). Consequently, each loop isotopy can be represented as a product of two special isotopies within $C-$classes and $R-$classes respectively. Isotopy classes of loops in $L(n)$ can be determined as the unions of those $R-$classes, which have non-empty intersections with the same $C-$class.

A further conclusion is that each $C-$class has at least one common isomorphic class with each $R-$class inside the same isotopy class. E.g., since the isotopy class $10I$ includes three $C-$ and three $R-$classes, it follows that

x	$y=0$	$y=1$	$y=2$	$y=3$	$y=4$	$y=5$	$y=6$	$y=7$	$y=8$	$y=9$
0		1C 1R	2C 2R	2C 2R	3C 3R	4C 4R	4C 5R	5C 4R	5C 5R	6C 3R
1	7C 6R	8C 7R	8C 7R	8C 8R	9C 9R	10C10R	11C 9R	12C 7R	13C11R	13C12R
2	14C13R	15C14R	16C15R	16C15R	14C16R	12C 8R	13C17R	16C15R	11C 9R	15C18R
3	17C16R	13C12R	17C19R	18C14R	14C19R	16C15R	18C18R	17C13R	19C10R	20C20R
4	21C21R	22C16R	23C22R	24C23R	25C24R	22C16R	26C25R	27C26R	28C27R	29C28R
5	24C23R	30C14R	22C13R	28C29R	31C27R	32C30R	26C24R	33C31R	34C 4R	31C29R
6	25C25R	33C32R	27C33R	28C34R	35C35R	36C 9R	22C19R	37C31R	33C36R	35C37R
7	37C36R	24C23R	29C28R	21C21R	32C30R	21C21R	20C20R	23C22R	27C38R	38C 3R
8	39C39R	29C28R	28C40R	40C 4R	24C23R	40C 4R	28C40R	28C34R	34C 4R	30C14R
9	30C18R	40C 5R	24C30R	26C25R	26C24R	24C30R	26C25R	22C16R	22C16R	34C 5R
10	26C25R	33C32R	37C32R	25C25R	32C23R	31C40R	21C22R	23C21R	31C34R	32C23R

Table 3.

the number of included isomorphic classes cannot be smaller than 9. This number is actually equal to 12; each one of the isomorphic classes 41, 45, 97 and 98 is included into the intersection of the classes $22C$ and $16R$.

Each row of Table 4. contains in order the ordinal number of an isotopy class, the included $C-$classes, the included $R-$classes and the set of included isomorphic classes:

$1I = 1C = 1R = \{1\}$
$2I = 2C = 2R = \{2, 3\}$
$3I = 3C + 6C + 38C = 3R = \{4, 9, 79\}$
$4I = 4C + 5C + 34C + 40C = 4R + 5R = \{5, 6, 7, 8, 58, 83, 85, 88, 91, 99\}$
$5I = 7C = 6R = \{10\}$
$6I = 8C + 12C = 7R + 8R = \{11, 12, 13, 17, 25\}$
$7I = 9C + 11C + 36C = 9R = \{14, 16, 28, 65\}$
$8I = 10C + 19C = 10R = \{15, 38\}$
$9I = 13C = 11R + 12R + 17R = \{18, 19, 26, 31\}$
$10I = 14C + 17C + 22C = 13R + 16R + 19R = \{20, 24, 30, 32, 34, 37, 41, 45, 52, 66, 97, 98\}$
$11I = 15C + 18C + 30C = 14R + 18R = \{21, 29, 33, 36, 51, 89, 90\}$
$12I = 16C = 15R = \{22, 23, 27, 35\}$
$13I = 20C = 20R = \{39, 76\}$
$14I = 21C + 23C = 21R + 22R = \{40, 42, 73, 75, 77, 106, 107\}$
$15I = 24C + 32C = 23R + 30R = \{43, 50, 55, 71, 74, 84, 92, 95, 104, 109\}$
$16I = 25C + 26C = 24R + 25R = \{44, 46, 56, 60, 93, 94, 96, 100, 103\}$
$17I = 27C = 26R + 33R + 38R = \{47, 62, 78\}$
$18I = 28C + 31C = 27R + 29R + 34R + 40R = \{48, 53, 54, 59, 63, 82, 86, 87, 105, 108\}$
$19I = 29C = 28R = \{49, 72, 81\}$
$20I = 33C + 37C = 31R + 32R + 36R = \{57, 61, 67, 68, 70, 101, 102\}$
$21I = 35C = 35R + 37R = \{64, 69\}$
$22I = 39C = 39R = \{80\}$

Table 4.

4. Π−classes and main classes

Π−classes play a central role among the classes in Figure 1. They can be used for establishing a relationship among the well-known kinds of classes (isotopy, isomorphic and main):

Theorem 1. *All the isotopy classes within the same main class have the same type.*

The proof is based on the intermmediate notion of Π−class. It easily follows from the following three lemmas:

Lemma 1. *Each Π−class and each isotopy class within the same main class have non-empty intersection.*

Proof. Suppose that a main class contains a Π−class Π and an isotopy class IT s.t $\Pi \cap IT = \emptyset$. If $L_1 \in \Pi$ and $L_2 \in IT$, then by definition of main class, there exists an isotopy i and a loop-parastrophy π satisfying $L_1 = \pi i L_2$. Thus the loop iL_2 belongs to the classes IT and Π, contradicting $\Pi \cap IT = \emptyset$. □

Lemma 2. *Isomorphic classes within a Π−class have the same cardinality.*

Proof. Consider two isomorphic classes IM_1 and IM_2 within the same Π−class. Let $L \in IM_1$ and π be a loop-parastrophy satisfying $\pi(L) \in IM_2$. The function π maps IM_1 to IM_2 since $\pi i L = i \pi L \in IM_2$ for arbitrary $iL \in IM_1$. The operator π is expressed by means of the operators λ, ρ and τ. Since all these operators are involutive [2], it follows that there exists the inverse function π^{-1}. This implies that the function π is a bijection between IM_1 and IM_2. □

Lemma 3. *The intersections of a Π−class with distinct isotopy classes from the same main class − have the same number of included isomorphic classes.*

Proof. Analogously to the proof of previous lemma, one primarily proves that the intersections of isotopy classes with the same Π−class have the same cardinality (the proof remains valid when the isomorphism i is replaced by the isotopy). The application of Lemma 2 to the equicardinal intersections completes the proof. □

Proof of Theorem 1. Lemmae 1, 2 and 3 give that the intersections of two isotopy classes with each Π−class within a main class − consist of the same number of equicardinal isomorphic subclasses. □

It turns out that each two isotopy classes, taken from any two distinct main classes over $L(6)$ have different types (such a conclusion need not be valid for larger ground-sets). Therefore, main classes over $L(6)$ can be reconstructed by use of the relationships between isotopy and isomorphic classes.

According to the following Table 5., the isotopy classes of loops on 6 elements can be collected into 12 wholes (denoted by I,II,... ,XII) w.r.t. the type. The families of cardinalities of the included isomorphic classes are given in the third column of the table (e.g., the family $\{60, 60, 120, 120\}$ is written as $2 \cdot 60 + 2 \cdot 120$).

I	$1I$	$1 \cdot 60$	VII	$22I$	$1 \cdot 40$
II	$2I$	$1 \cdot 60 + 1 \cdot 120$	VIII	$3I,\ 17I,\ 19I$	$1 \cdot 20 + 1 \cdot 40 + 1 \cdot 60$
III	$5I$	$1 \cdot 20$	IX	$4I,\ 15I,\ 18I$	$2 \cdot 60 + 8 \cdot 120$
IV	$6I$	$1 \cdot 60 + 4 \cdot 120$	X	$7I,\ 9I,\ 12I$	$2 \cdot 60 + 2 \cdot 120$
V	$10I$	$4 \cdot 30 + 8 \cdot 120$	XI	$8I,\ 13I,\ 21I$	$1 \cdot\ 6\ + 1 \cdot 30$
VI	$16I$	$9 \cdot 120$	XII	$11I,\ 14I,\ 20I$	$2 \cdot 30 + 2 \cdot 60 + 3 \cdot 120$

Table 5.

It follows from Theorem 1 and Table 5. that there are at least 12 main classes on 6 elements. The data from [5] confirm that each one of the 12 registered candidates is itself a main class. The same conclusion can be derived from Table 6; there are only 12 different collections of isotopy classes which have non-empty intersections with a $\Pi-$ class.

5. Duality

Loops L and $\tau(L)$ are said to be *dual* to each other. Two isotopy (isomorphic) classes are dual whenever they contain two mutually dual representatives. It easily follows from the definition that the dual of a $C-$class is an $R-$class within the same isotopy class, and conversely. On the other hand, $\Pi-$classes and main classes contain complete pairs of mutually dual isomorphic classes, since the duality operator is a special kind of a loop-parastroph operator.

Duality operator will be denoted by $\sim$; the denotation $\sim$ between two equicardinal sets of classes means that the underlying bipartite matching of mutually dual classes is not yet decided exactly.

The mutually dual pairs of isomorphic classes are (4, 47), (5, 53), (6, 59), (7, 48), (8, 54), (9, 62), (13,17), (14, 18), (15, 64), (16, 19), (21, 57), (24, 52), (26, 65), (28,31), (29, 67), (30,66), (33, 68), (34, 37), (36, 70), (38, 69), (51, 61), (56,60), (58, 63), (78, 79), (82, 83), (85, 86), (87, 88), (89, 101), (90, 102), (91, 105), (92, 104), (93, 96), (94, 103), (95, 109), (97, 98), (99, 108), (106, 107), while the remaining 35 isomorphic classes are self-dual.

An abridged way to recognize duality of isomorphic classes would be to use dualities between $C-$ and $R-$, as well as between isotopy classes. Necessary data can be found in Tables 3, 4 and 9.

E.g., XII main class contains isotopy classes $11I$, $14I$ and $20I$. Consulting the numbers of included $C-$classes and $R-$classes, we conclude that $11I \sim 20I$ and that the isotopy class $14I$ is self-dual.

Let the class $14I$ be represented similarly as in Table 4. In addition, the isomorphic classes, as well as their cardinalities (in ( ) brackets) are listed in [ ] brackets after the corresponding $C(R)-$class:

$$\begin{aligned} 14I &= 21C[40(60), 73(120), 75(60), 106(120)] \\ &+ \quad 23C[42(30), 77(30), 107(120)] \\ &= 21R[40(60), 73(120), 75(60), 107(120)] \\ &+ \quad 22R[42(30), 77(30), 106(120)] \end{aligned}$$

Comparing the cardinalities of isomorphic classes included in distinct $C-$ and $R-$classes, it follows that $21C \sim 21R$ and $23C \sim 22R$. This implies (using also the cardinalities of isomorphic classes) that $\{40, 75\} \sim \{40, 75\}$, $\{42, 77\} \sim \{42, 77\}$ and $106 \sim 107$, which further gives that the isomorphic class 73 is self-dual.

It might be interesting to note that among the only six E-classes, which consist of two isomorphic classes each, there are two pairs of mutually dual $E-$classes: $(4, 79) \sim (47, 78)$ and $(8, 83) \sim (54, 82)$.

6. Parastrophic closures and $\Pi-$classes

Let r_A denote the permutation which produces the right inverse element of a loop A ($A[i, r_A(i)] = 1$ for each $i \in S(n)$). It can be proved that:

Theorem 2. *It is satisfed for each loop A from $L(n)$ that:*
$|PC(A)| \leq 6 \cdot order(r_A) \leq 6 \cdot max\ g.c.d.(s_1, \ldots, s_k)$,
where the maximum is taken over all the partitions $n - 1 = s_1 + \ldots + s_k$.

This is an analogue[1] to a statement ([3]) which claims that $PC(A)$ has non-empty intersections with at most six isomorphic classes for each loop A. Each loop from $PC(A)$ can be obtained from A by an application of transformations of the form $\lambda\rho\lambda\rho\ldots$, when the order of r_A is odd, respectively of the form $\lambda\rho\lambda\rho\ldots$ or $\tau\lambda\rho\lambda\rho\ldots$, when the order of r_A is even.

Among all the 9408 loops in $L(6)$, only 5650 reach the above upper bound $6\cdot$ order(r_A) for $|PC(A)|$. More precisely, the bound is reached with all those loops $A \in L(6)$, which satisfy that $|PC(A)| > 12$, and only with 150 loops with smaller $|PC(A)|$ (120 with $|PC(A)| = 12$ and 30 with $|PC(A)| = 6$). We conjecture that $PC(A) = 6 \cdot \text{order}(r_A)$ whenever $|PC(A)| > 12$.

[1] when non-isomorphic loops are replaced by non-identical loops

The loops $A \in L(6)$ with $|PC(A)| = 10$ seem to be particularly interesting. All of them have order$(r_A) = 5$. In addition, 10 is the largest length that we know of a minimal $\lambda\rho\lambda\rho\ldots$ cycle which maps A to A, which is less than the theoretical maximum.

On the basis of tests with random loops, we conjecture that the length of the parastrophic closure of a loop $A \in L(n)$ for a larger n almost always coincides with $6 \cdot$ order(r_A). The minimal value of $|PC(A)|$ is, however, equal to 1 for each n (e.g., when A is the multiplication table of the cyclic group).

The following lemma claims that the above considerations may be raised to the level of $\Pi-$classes:

Lemma 4. *Parastrophic closures within a* $\Pi-$*class have the same cardinality.*

Proof. Let PC_1 and PC_2 denote two parastrophic closures within a $\Pi-$class. There exist two isomorphic loops L_1 and L_2 belonging to PC_1 and PC_2 respectively.

A parastrophic closure is determined by its any incident loop. Using commutative diagrams which connect isomorphism and loop-parastrophy operators, one easily concludes that the parastrophic closuures corresponding to L_1 and L_2 have the same cardinality. □

The first two columns of the following Table 6. contain the cardinality of parastrophic closures and the total number of parastrophic closures within $L(6)$ of a fixed cardinality. The denotation $X : Y$ is associated to the isomorphic class X, which is included into the isotopy class Y. $\Pi-$classes correspond to the ( ) brackets. The number of parastrophic closures within each $\Pi-$class[2] is given in [ ] brackets after ( ) brackets:

In particular, Table 6, can be used for an illustration of Theorem 1. For example, data from Table 6 give the structure of isomorphic classes within XII main class, distributed w.r.t. isotopy classes and Π-classes, given in Table 7. Note that the type of isotopy classes within XII main class is $2 \cdot 30 + 2 \cdot 60 + 3 \cdot 120$ (this can be also found in Table 5).

The second, the third and the fourth row of Table 7 correspond to isotopy classes, while all the columns, except for the first, correspond to Π-classes. For each Π-class are given three additional data. The cardinalities and the number of the included parastrophic closures are given in the 5th and the 6-th row of the table respectively. On the other hand, the first row of the table contains the cardinalities of the included isomorphic classes (taken from

[2] all the parastrophic closures within a $\Pi-$class have equicardinal intersections with all the isomorphic classes within the same $\Pi-$class

1)	[60]	$(1:1I)$	[60]		
2)	[40]	$(2:2I)$	[30]	$(10:5I)$	[10]
3)	[96]	$(9:3I,\ 49:19I,\ 62:17I)$	[60]	$(15:8I,\ 39:13I,\ 64:21I)$	[6]
		$(36:11I,\ 42:14I,\ 70:20I)$	[30]		
4)	[25]	$(25:6I)$	[15]	$(80:22I)$	[10]
6)	[240]	$(4:3I,\ 47:17I,\ 72:19I)$	[10]	$(14:7I,\ 18:9I,\ 22:12I)$	[30]
		$(8:4I,\ 54:18I,\ 55:15I)$	[30]	$(21:11I,\ 40:14I,\ 57:20I)$	[30]
		$(29:11I,\ 67:20I,\ 77:14I)$	[15]	$(32:10I,\ 34:10I,\ 37:10I)$	[15]
		$(33:11I,\ 68:20I,\ 75:14I)$	[30]	$(38:8I,\ 69:21I,\ 76:13I)$	[15]
		$(43:15I,\ 58:4I,\ 63:18I)$	[60]	$(20:10I)$	[5]
8)	[30]	$(3:2I)$	[15]	$(46:16I)$	[15]
10)	[48]	$(93:16I,\ 96:16I)$	[24]	$(97:10I,\ 98:10I)$	[24]
12)	[50]	$(6:4I,\ 59:18I,\ 74:15I)$	[15]	$(78:17I,\ 79:3I,\ 81:19I)$	[10]
		$(28:7I,\ 31:9I,\ 35:12I)$	[15]	$(12:6I)$	[10]
18)	[80]	$(7:4I,\ 48:18I,\ 71:15I)$	[20]	$(11:6I,\ 13:6I,\ 17:6I)$	[20]
		$(16:7I,\ 19:9I,\ 23:12I)$	[20]	$(24:10I,\ 41:10I,\ 52:10I)$	[20]
24)	[75]	$(5:4I,\ 50:15I,\ 53:18I)$	[15]	$(30:10I,\ 45:10I,\ 66:10I)$	[15]
		$(26:9I,\ 27:12I,\ 65:7I)$	[15]	$(44:16I,\ 56:16I,\ 60:16I)$	[15]
		$(51:11I,\ 61:20I,\ 73:14I)$	[15]		
30)	[48]	$(85:4I,\ 86:18I,\ 91:4I,\ 95:15I,\ 105:18I,\ 109:15I)$			[24]
		$(89:11I,\ 90:11I,\ 101:20I,\ 102:20I,\ 106:14I,\ 107:14I)$			[24]
36)	[40]	$(82:18I,\ 83:4I,\ 84:15I)$	[10]	$(94:16I,\ 100:16I,\ 103:16I)$	[10]
		$(87:18I,\ 88:4I,\ 92:15I,\ 99:4I,\ 104:15I,\ 108:18I)$			[20]

Table 6.

	30	60	30	60	120	120
$11I$	36	21	29	33	51	89, 90
$14I$	42	40	77	75	73	106, 107
$20I$	70	57	67	68	61	101, 102
	3)	6)	6)	6)	24)	30)
	[30]	[30]	[15]	[30]	[15]	[24]

Table 7.

Table 9 of Appendix). For example, the last column of the table corresponds to a II-class having $3 \cdot 2 \cdot 120 = 30 \cdot 24 = 720$ loops.

7. Classifications on 5 and 4 elements

In this section are given the corresponding classifications of loops on 5 and 4 elements. Denotations in the tables are completely analogous to those on 6 elements, with the additional denotations ' and " for loops on 5 and 4 elements respectively.

$$n = 5$$

Table 2'

2 commutators		2 commutators	
1', 2'	$(3 \cdot 8 + 3 \cdot 8)$	6'	$(20 \cdot 6)$

Table 3'

$y = 1$	$y = 2$	$y = 3$	$y = 4$	$y = 5$	$y = 6$
$1R'$:$1C'$	$2R'$:$2C'$	$1R'$:$2C'$	$2R'$:$1C'$	$2R'$:$1C'$	$3R'$:$3C'$

Table 4'

$$1I' = 1C' + 2C' = 1R' + 2R' = \{1', 2', 3', 4', 5'\}$$
$$2I' = 3C' = 3R' = \{6'\}$$

Table 5'

I'	$1I'$	$1 \cdot 2 + 3 \cdot 8 + 1 \cdot 24$		II'	$2I'$	$1 \cdot 6$

Table 6'

1) [6]	(6':$2I'$)	[6]
2) [1]	(3':$1I'$)	[1]
6) [4]	(1':$1I'$, 2':$1I'$, 4':$1I'$)	[4]
8) [3]	(5':$1I'$)	[3]

$$n = 4$$

Table 2"

4 commutators			
1"	$(24 \cdot 1)$	2"	$(8 \cdot 3)$

Table 3"

$y = 1$	$y = 2$
$1R''$:$1C''$	$2R''$:$2C''$

Table 4"

$$1I'' = 1C'' = 1R'' = \{1''\}$$
$$2I'' = 2C'' = 2R'' = \{2''\}$$

Table 5"

I"	$1I''$	$1 \cdot 1$		II"	$2I''$	$1 \cdot 3$

Table 6"

1) [4]	(1":$1I''$)[1]	(2":$2I''$)[3]

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Appendix

The representatives of the 109 isomorphic classes within $L(6)$ are given in Table 8. Each one of these loops is represented by a 16-digit sequence; the four consecutive quadruples of the sequence contain the middle four elements of the 2nd, 3th, 4th and 5th row of the loop respectively:

1 = 1436456136526123	2 = 1436456136526124	3 = 1436456136526213
4 = 1436456156126123	5 = 1436456156236214	6 = 1436456161523624
7 = 1436456256136124	8 = 1436456256216213	9 = 1436456262513614
10 = 1436516262513614	11 = 1436516262513624	12 = 1436516262514613
13 = 1436516262514623	14 = 1436516262534621	15 = 1436516265134621
16 = 1436516462514623	17 = 1436516462533612	18 = 1436516462533621
19 = 1436516462534612	20 = 1436516465123621	21 = 1436516465124623
22 = 1436516465213612	23 = 1436516465233612	24 = 1436516465234612
25 = 1436526161523624	26 = 1436526161524623	27 = 1436526161533624
28 = 1436526165233614	29 = 1436526165234612	30 = 1436526461524613
31 = 1436526461533612	32 = 1436526465213612	33 = 1436526465214613
34 = 1436561261533264	35 = 1436561262533164	36 = 1436561461524263
37 = 1436562162533164	38 = 1436562462514163	39 = 1456416256136231
40 = 1456416256136234	41 = 1456416265233614	42 = 1456426156236132
43 = 1456426156236134	44 = 1456426165233612	45 = 1456426165233614
46 = 1456426165323614	47 = 1456456136126123	48 = 1456456136126234
49 = 1456456156326123	50 = 1456456156326213	51 = 1456456161323624
52 = 1456456162133624	53 = 1456456236216134	54 = 1456456236216213
55 = 1456456256316213	56 = 1456456261233614	57 = 1456456262133621
58 = 1456456262313614	59 = 1456461235616234	60 = 1456461265233164
61 = 1456461265313264	62 = 1456462135626134	63 = 1456462165323164
64 = 1456516462133621	65 = 1456526161323624	66 = 1456526165323614
67 = 1456526461233612	68 = 1456526465313612	69 = 1456561265313264
70 = 1456562165323164	71 = 1456612435614632	72 = 1456612435624631
73 = 1456621435614623	74 = 1456621435614632	75 = 1456651431624623
76 = 1456651432614623	77 = 1456652431624613	78 = 3156126456216432
79 = 3156126456236412	80 = 3156126465214632	81 = 3156126465234612
82 = 3156146256216234	83 = 3156146256236214	84 = 3156146265314623
85 = 3156426156236412	86 = 3156426156326413	87 = 3156426165231634
88 = 3156426165321624	89 = 3156456116236234	90 = 3156456116236432
91 = 3156456156236412	92 = 3156456256216413	93 = 3156456262311624
94 = 3156461215636234	95 = 3156461265231264	96 = 3156462115626234
97 = 3156462115636234	98 = 3156462115636432	99 = 3156462165321264
100 = 3156462165321463	101 = 3416156256216234	102 = 3416156456236132
103 = 3416156461524623	104 = 3416456216536124	105 = 3416456256316124
106 = 3416456261531624	107 = 3416526461521623	108 = 3416562115626234
109 = 3416562165321264		

Table 8.

Table 9. gives the number of loops within distinct isomorphic classes of $L(6)$. The set of (labels of) isomorphic classes which have cardinality c is

denoted by "S_c".

$$\begin{array}{lcl} S_6 & = & \{15,39,64\} \qquad S_{20}=\{4,10,47,72\} \\ S_{30} & = & \{20,29,32,34,36,37,38,42,67,69,70,76,77\} \qquad S_{40}=\{78,79,80,81\} \\ S_{60} & = & \{1,2,6,8,9,14,18,21,22,25,28,31,33,35,40,49,54,55,57,59,62,68,74,75\} \\ S_{120} & = & \{1,2,\dots,109\}-(S_6\cup S_{20}\cup S_{30}\cup S_{40}\cup S_{60}) \end{array}$$

Table 9.

The corresponding tables for $L(5)$ and $L(4)$ are:

Table 8.' $1' = 145451523$ $2' = 145452513$ $3' = 145512351$
$4' = 145521352$ $5' = 315451523$ $6' = 345451512$

Table 9.' $S_2' = \{3'\}$, $S_6' = \{6'\}$, $S'_8 = \{1', 2', 4'\}$, $S'_{24} = \{5'\}$

Table 8." 1" = 1441 2" = 1442
Table 9." $S"_1 = \{1"\}$, $S"_3 = \{2"\}$

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