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OMITTING TYPES IN KRIPKE MODELS

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ABSTRACT. When can a type be omitted in a Kripke model of some intuitionistic theory is investigated. As it is usual with intuitionistic systems, various classically equivalent formulations of the Omitting Types Theorem, become nonequivalent statements in the intuitionistic setting. Several such formulations are discussed in terms of whether they have the intended meaning in Kripke models, and several theorems are proved.

Classically, an Omitting Types Theorem states that an apparently weaker condition, concerning individual formulas from a type ("locally omitting"), suffices for the whole type to be omitted in some model. We will start by considering what meaning these expressions may have in the case of Kripke models of some intuitionistic theory T. A "type" should clearly be a type of an element of a Kripke model of T. If we restrict ourselves to Kripke models in which the frame, i.e., the partial ordering, has the least element the base node, this should be an element of the universe at the base node. A type for T can be defined as a set of formulas in the same language $\mathcal{L}(T)$ with one free variable, say x_0, consistent with T. Analogous definition may be given for n-types. If $\sum(x_0)$ is a type for T, we say that some Kripke model of T realizes $\sum$ if there is an element of the universe at the base node of this model, for which every formula from $\sum$ is forced. Dually, we say that some Kripke model of T omits $\sum$ if for every element of the universe at the base node of this model, there is some formula from $\sum$ which is not forced for this element. As for the "local omitting", we may consider the following four formulations:

(1) for any sentence $\exists x_0 \varphi(x_0)$ in $\mathcal{L}(T)$ consistent with T, there exists some formula $\sigma(x_0) \in \sum$ such that the sentence $\exists x_0(\varphi(x_0) \wedge \neg\sigma(x_0))$ is consistent with T;

(2) for any sentence $\exists x_0\varphi(x_0)$ in $\mathcal{L}(T)$ consistent with T, there exists some formula $\sigma(x_0) \in \sum$ such that the sentence $\exists x_0\neg(\varphi(x_0) \to \sigma(x_0))$ is consistent with T;

(3) for any sentence $\exists x_0\varphi(x_0)$ in $\mathcal{L}(T)$ consistent with T, there exists some formula $\sigma(x_0) \in \sum$ such that the sentence $\neg\forall x_0(\varphi(x_0) \to \sigma(x_0))$ is consistent with T;

(4) for any sentence $\exists x_0\varphi(x_0)$ in $\mathcal{L}(T)$ consistent with T, there exists some formula $\sigma(x_0) \in \sum$ such that $T \nvdash \forall x_0(\varphi(x_0) \to \sigma(x_0))$.

In intuitionistic predicate calculus it is easily provable that:

$$\exists x_0(\varphi(x_0) \land \neg\sigma(x_0)) \to \exists x_0\neg(\varphi(x_0) \to \sigma(x_0))$$

and

$$\exists x_0\neg(\varphi(x_0) \to \sigma(x_0)) \to \neg\forall x_0(\varphi(x_0) \to \sigma(x_0))$$

while neither of the reverse implications holds. Therefore, we have

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).$$

However, the whole statement (2), even intuitionisticaly, implies (1), so we may dismiss it. It is easy to show that for the remaining there statements none of the reverse implications holds intuitionisticaly. The statement (4) is the most interesting, not only because it is the weakest of the four, but also because it is strong enough to prove that in the Lindenbaum algebra of consequences of T, $\sum$ generates a nonprincipal filter. We shall also use (1), mainly for technical reasons, while (3) does not seem to deserve much attention.

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Let $\mathcal{L}$ be a countable first-order language, T a consistent (intuitionistic) theory in $\mathcal{L}$ and $\sum$ a set of formulas in $\mathcal{L}$ with at most x_0 free. In [3] the following theorem was proved.

Theorem 1. *If for any sentence $\exists x_0\varphi(x_0)$ in $\mathcal{L}$ consistent with T, there exists a formula $\sigma(x_0) \in \sum$ such that the sentence $\exists x_0(\varphi(x_0) \land \neg\sigma(x_0))$ is consistent with T then there exists a Kripke model of T with a countable universe at each node, such that for every element a of the universe at the base node, there exists a formula $\sigma(x_0) \in \sum$ such that $\neg\sigma[a]$ is forced at the base node.*

The proof is a Henkin-style argument along the lines of completeness proofs of [1] and [4]. T is gradually extended to an $\mathcal{L} \cup C$-saturated theory

(C being a countable set of new constants). At each stage three steps are made: for $n = 3k$ and $n = 3k + 1$ we work toward making the final theory saturated (we provide a "witness" from C for an existential consequence and add one of the disjuncts of a disjunction which is a consequence), while for $n = 3k + 2$ we add $\neg\sigma(c_k)$ for some appropriate $\sigma(x_0) \in \sum$.

It was observed later by Kripke that practically the same proof will prove the following dual theorem, which might be more useful for intuitionistic theories.

Theorem 2. (Intersecting Types Theorem) *If for any sentence $\exists x_0\varphi(x_0)$ in $\mathcal{L}$ consistent with T, there exists a formula $\sigma(x_0) \in \sum$ such that the sentence $\exists x_0(\varphi(x_0) \wedge \sigma(x_0))$ is consistent with T, then there exists a Kripke model of T with a countable universe at each node such that for every element a of the universe at the base node, there exists a formula $\sigma(x_0) \in \sum$ such that $\sigma[a]$ is forced at the base node.*

These results may be improved in two directions. One direction is to require T to be a saturated theory, i.e., a deductively closed consistent set of sentences satisfying the following two conditions:

- if $\exists x\varphi(x) \in T$ then $\varphi(c) \in T$ for some individual constant c from $\mathcal{L}(T)$
- If $\varphi \cup \psi \in T$ then $\varphi \in T$ or $\psi \in T$.

While Troelstra and Kreisel argue that we should not assume that all intuitionisticaly acceptable theories must be saturated (e.g. [5]), it is a fact that all major, naturally arising, examples of intuitionistic theories are saturated. Therefore, this is not an unreasonable requirement. In this case the condition (4) is sufficient for omitting.

Theorem 3. *If T is a saturated theory and for any sentence $\exists x_0\varphi(x_0)$ in $\mathcal{L}$ consistent with T there exists a formula $\sigma(x_0) \in \sum$ such that $T \nvdash \forall x_0(\varphi(x_0) \rightarrow \sigma(x_0))$ then there exists a Kripke model of T which omits $\sum$.*

Proof. Let $E = \{\exists x_0\varphi_0(x_0), \exists x_0\varphi_1(x_0), \ldots\}$ be an enumeration of all existential sentences in $\mathcal{L}$ consistent with T. By the hypothesis of the Theorem, for each $i \in \omega$ there exists a Kripke model $\mathfrak{M}_i = ((S_i, O_i, \leq_i); \mathfrak{A}_s\colon s \in S_i)$ of T, a formula $\tau_i(x_0) \in \sum$ and an element $a \in A_{O_i}$ such that

$$O_i \Vdash T, \qquad O_i \Vdash \varphi_i[a] \quad \text{and} \quad O_i \nVdash \sigma[a].$$

Let $\mathfrak{M} = (\sum \mathfrak{M}_i)'$ be the collection of models $\mathfrak{M}_i (i \in \omega)$ (c.f. [4]). We shall prove the following two claims:

1° $\mathfrak{M} \models T$

2° $\mathfrak{M}$ omits $\sum$, i.e., if $\mathfrak{M} = ((S, O, \leq); \mathfrak{A}_s\colon s \in S)$ then for every $a \in A_0$ there exists $\sigma(x_0) \in \sum$ such that $O \nVdash \sigma[a]$.

For 1° it is enough to note that T is saturated and is, therefore, preserved under the operation of collection (cf. [4]). For 2°, we note that A_0 of $\mathfrak{M}$ consists, by definition, of individual constants occurring in T. Therefore, if $c \in A_0$ the sentence $\exists x_0(x_0 = c)$ will be a sentence of $\mathcal{L}$ consistent with T, so for some $i \in \omega$, φ_i in our enumeration E will be $(x_0 = c)$. Then, for some $a \in A_{O_i}$ we will have $O_i \Vdash a = c$ and $O_i \nVdash \sigma_i[a]$ and so $O_i \nVdash \sigma_i(c)$. As $O \leq O_i$ in $\mathfrak{M}$, we obtain $O \nVdash \sigma_i(c)$. □

Another direction in which we can improve Theorem 1. is to put some restriction on elements of $\sum$. We will show that in two such cases we can obtain the omitting types theorem in full strength, i.e., using (4) as the "locally omitting" condition.

Theorem 4. *Let $\sum$ be a set of negated formulas in $\mathcal{L}$, with at most x_0 free. If for any sentence $\exists x_0\varphi(x_0)$ in $\mathcal{L}$ consistent with T, there exists a formula $\neg\sigma(x_0) \in \sum$ such that $T \nvdash \forall x_0(\varphi(x_0) \to \neg\sigma(x_0))$ then there exists a Kripke model of T omitting $\sum$.*

Proof. Consider $\sum' = \{\sigma : \neg\sigma \in \sum\}$. It is easy to prove that $T \nvdash \forall x_0(\varphi(x_0) \to \neg\sigma(x_0))$ if and only if $\exists x_0(\varphi(x_0) \wedge \sigma(x_0))$ is consistent with T. If $T \nvdash \forall x_0(\varphi(x_0) \to \neg\sigma(x_0))$, by completeness theorem (e.g. [1]) there exists a Kripke model $\mathfrak{M} = ((S, O, \leq); \mathfrak{A}_s : s \in S)$ of T in which for some $s \in S$ and $a \in A_s$ we have $s \Vdash \varphi[a]$ and $s \nVdash \neg\sigma[a]$ which means that for some $s' \in S$ we have $s \leq s'$ and $s' \Vdash \sigma[a]$. The truncation of $\mathfrak{M}$ at s', $\mathfrak{M}_{s'}$ will be a model of $T \cup \{\exists x_0(\varphi(x_0) \wedge \sigma(x_0))\}$. We may apply now the Intersecting Types Theorem (Theorem 4.) to $\sum'$ and obtain a Kripke model of T with a countable universe at each node which not only omits $\sum$ but in which actually for each element a of the universe at the base node there exists some $\neg\sigma(x_0) \in \sum$ such that $\sigma[a]$ is forced at the base node. □

Theorem 5. *Let $\sum$ be a set of formulas with at most x_0 free which are decidable in T, i.e., for each $\sigma(x_0) \in \sum$ we have $T \vdash \forall x_0(\sigma(x_0) \vee \neg\sigma(x_0))$. If for each sentence $\exists x_0\varphi(x_0)$ consistent with T there exists a formula $\sigma(x_0) \in \sum$ such that $T \nvdash \forall x_0(\varphi(x_0) \to \sigma(x_0))$ then there exists a Kripke model of T omitting $\sum$.*

Proof. As in the proof of Theorem 4, $T \nvdash \forall x_0(\varphi(x_0) \to \sigma(x_0))$ implies that in some Kripke model $\mathfrak{M}$ of T for some s and $a \in A_s$ we have $s \Vdash \varphi[a]$ and $s \nVdash \sigma[a]$. As $s \Vdash T$ we get $s \Vdash \neg\sigma[a]$ and the truncated model $\mathfrak{M}_s$ is a model of $T \cup \{\exists x_0(\varphi(x_0) \wedge \neg\sigma(x_0))\}$. We may then apply Theorem 1. and obtain the model of T omitting $\sum$. □

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