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IDENTITY, PERMUTATION AND BINARY TREES

Aleksandar Kron

ABSTRACT. Some extensions of the Anderson-Belnap (Dwyer-Powers) conjecture for $\mathbf{TW}_{\rightarrow}$ are applied to a set of binary trees.

A binary tree $\mathcal{T}$ is a tree with an origin and such that each node of $\mathcal{T}$ either has exactly two immediate successors or it is an end-node. A subtree $\mathcal{T}'$ of a tree $\mathcal{T}$ is a tree such that every node of $\mathcal{T}'$ is a node of $\mathcal{T}$ and the immediate successor relation in $\mathcal{T}'$ is the immediate successor relation in $\mathcal{T}$. A tree $\mathcal{T}$ is a *formula-like tree* (FLT) iff it has no proper subtree that is isomorphic to $\mathcal{T}$, and no proper subtree $\mathcal{T}'$ of $\mathcal{T}$ has a proper subtree $\mathcal{T}''$ isomorphic to $\mathcal{T}'$.

Every infinite FLT contains infinitely many (distinct) finite branches; every node of a FLT is a node of a finite branch. Hence, the maximal length of a branch of a FLT is ω.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are FLTs, then the tree obtained by adjoining a new origin $\mathcal{T}_1\mathcal{T}_2$ and such that $\mathcal{T}_1$ and $\mathcal{T}_2$ are immediate successors of $\mathcal{T}_1\mathcal{T}_2$, is a FLT.

With every node of a FLT $\mathcal{T}$ one of the numbers 0 or 1 is associated, as follows: 0 is associated with the origin of $\mathcal{T}$; if 0 (1) is associated with a node at level n, then 1 (0) is associated with its left hand immediate successor and 0 (1) is associated with its right hand immediate successor.

If 0 (1) is associated with the origin of a subtree $\mathcal{T}'$, then $\mathcal{T}'$ is a 0-subtree (1-subtree).

Let us consider the following operations on a FLT.

SU^0 Let $\mathcal{T}$ be a FLT and let $\mathcal{T}_1\mathcal{T}_2$ be one of its 0-subtrees; then the subtree $\mathcal{T}_1\mathcal{T}_2$ can be cut off and a subtree $(\mathcal{T}_2\mathcal{T}_3)(\mathcal{T}_1\mathcal{T}_3)$ can be inserted in $\mathcal{T}$ instead, where $\mathcal{T}_3$ is any FLT.

PR^0 Let $\mathcal{T}$ be a FLT and let $\mathcal{T}_2\mathcal{T}_3$ be one of its 0-subtrees; then the subtree $\mathcal{T}_2\mathcal{T}_3$ can be cut off and a subtree $(\mathcal{T}_1\mathcal{T}_2)(\mathcal{T}_1\mathcal{T}_3)$ can be inserted in $\mathcal{T}$ instead, where $\mathcal{T}_1$ is any FLT.

SUL^0 Let $\mathcal{T}$ be a FLT and let $\mathcal{T}_1$ be one of its 0-subtrees; then the subtree $\mathcal{T}_1$ can be cut off and a subtree $(\mathcal{T}_1\mathcal{T}_2)((\mathcal{T}_2\mathcal{T}_3)\mathcal{T}_3)$ can be inserted in $\mathcal{T}$ instead, where $\mathcal{T}_2$ and $\mathcal{T}_3$ are any FLTs.

SU^1 Let $\mathcal{T}$ be a FLT and let $(\mathcal{T}_2\mathcal{T}_3)(\mathcal{T}_1\mathcal{T}_3)$ be one of its 1-subtrees; then the subtree $(\mathcal{T}_2\mathcal{T}_3)(\mathcal{T}_1\mathcal{T}_3)$ can be cut off and the subtree $\mathcal{T}_1\mathcal{T}_2$ can be inserted in $\mathcal{T}$ instead.

PR^1 Let $\mathcal{T}$ be a FLT and let $(\mathcal{T}_1\mathcal{T}_2)(\mathcal{T}_1\mathcal{T}_3)$ be one of its 1-subtrees; then the subtree $(\mathcal{T}_1\mathcal{T}_2)(\mathcal{T}_1\mathcal{T}_3)$ can be cut off and the subtree $\mathcal{T}_2\mathcal{T}_3$ can be inserted in $\mathcal{T}$ instead.

SUL^1 Let $\mathcal{T}$ be a FLT and let $(\mathcal{T}_1\mathcal{T}_2)((\mathcal{T}_1\mathcal{T}_3)\mathcal{T}_3)$ be one of its 1-subtrees; then the subtree $(\mathcal{T}_1\mathcal{T}_2)((\mathcal{T}_2\mathcal{T}_3)\mathcal{T}_3)$ can be cut off and the subtree $\mathcal{T}_1$ can be inserted in $\mathcal{T}$ instead.

PERM^* Let $\mathcal{T}$ be a FLT and let $\mathcal{T}_1(\mathcal{T}_2\mathcal{T}_3)$ be one of its 0-or-1-subtrees; then the subtree $\mathcal{T}_1(\mathcal{T}_2\mathcal{T}_3)$ can be cut off and the subtree $\mathcal{T}_2(\mathcal{T}_1\mathcal{T}_3)$ can be inserted in $\mathcal{T}$ instead.

The main theorem: starting with a FLT $\mathcal{T}$ and successively performing the operations SU^0, PR^0, SUL^0, SU^1, PR^1, SUL^1, and PERM^* any finite number of times, in any order, and such that at least one of the first six rules is applied at least once, it is not possible to obtain $\mathcal{T}$ as a result.

1. Identity

When Alan Ross Anderson and Nuel D. Belnap were developing relevance logic, among numerous systems they have been considering there was an implicational fragment of a very weak logic called now **TW**. Since there is only one connective in such a fragment, namely $\rightarrow$, we omit it and we write AB for $A \rightarrow B$. Also, we omit parentheses, whenever this causes no confusion. ABC stands for $(AB)C$ and $A.BC$ for $A(BC)$. Under this proviso, the implicational fragment of **TW** has modus ponens (MP) as the sole rule of inference and the following axiom-schemata:

ID	AA
ASU	$AB.BC.AC$
APR	$BC.AB.AC$

This fragment is now called $\mathbf{TW}_{\rightarrow}$. Let us write $A \equiv B$ iff both AB and BA are provable in $\mathbf{TW}_{\rightarrow}$. Then Anderson and Belnap have conjectured (cf. [1], p. 95) that

$$A \equiv B \text{ if and only if } A \text{ and } B \text{ are the same formula.}$$

We shall call this conjecture *Anderson - Belnap conjecture* (A-B).

By A-B the identity of formulas in the language of $\mathbf{TW}_{\rightarrow}$ is determined by logical means only – by provability in the very weak theory of implication $\mathbf{TW}_{\rightarrow}$.

Let $\mathbf{TW}_{\rightarrow}$–ID be the system obtained from $\mathbf{TW}_{\rightarrow}$ by omitting the axiom schema ID. Dwyer and Powers have shown that A-B is equivalent to the following claim:

NOID In $\mathbf{TW}_{\rightarrow}$–ID there is no theorem of the form AA.

NOID is a very strong claim. ID is a paradigm of a logical truth and there is hardly a descent logical theory where ID is not true. Nevertheless, in $\mathbf{TW}_{\rightarrow}$–ID not only there is a formula A such that AA does not hold for it, but, moreover, AA holds for no formula A.

The systems $\mathbf{TW}_{\rightarrow}$ and $\mathbf{TW}_{\rightarrow}$–ID have other formulations as well.

Let us consider a theory that has ASU and APR as axiom-schemata, but instead of MP it has the following rules of inference:

SU	From AB to infer $BC.AC$
PR	From BC to infer $AB.AC$
TR	From AB and BC to infer AC

Let us call the new system $\mathbf{TRW}_{\rightarrow}$–ID. It is easy to see that the rules of $\mathbf{TRW}_{\rightarrow}$–ID are derived rules of $\mathbf{TW}_{\rightarrow}$–ID; hence, all theorems of $\mathbf{TRW}_{\rightarrow}$–ID are theorems of $\mathbf{TW}_{\rightarrow}$–ID. On the other hand, by an inductive argument it follows that $\mathbf{TRW}_{\rightarrow}$–ID is closed under MP (this was proved by Dwyer and Powers; cf. [4] and [2] for details. $\mathbf{TRW}_{\rightarrow}$–ID and $\mathbf{TRW}_{\rightarrow}$ are called in [4] **M** and **N**, respectively). Therefore, $\mathbf{TW}_{\rightarrow}$–ID and $\mathbf{TRW}_{\rightarrow}$–ID are equivalent.

Let us adjoin the axiom-scheme ID to $\mathbf{TRW}_{\rightarrow}$ – ID; the resulting theory is called $\mathbf{TRW}_{\rightarrow}$. It is clear that $\mathbf{TRW}_{\rightarrow}$ and $\mathbf{TW}_{\rightarrow}$ are equivalent.

Another equivalent formulation of NOID is the following one. Let us consider the theory $\mathbf{WTR}_{\rightarrow}$–ID, in the propositional language with $\rightarrow$ as the sole connective.

The rules of inference are SU, PR and TR, as in $\mathbf{TRW}_{\rightarrow}$–ID, but the axiom-schemata are

USA	$(BC.AC).AB$
RPA	$(AB.AC).BC$

The axioms of $\mathbf{WTR}_{\rightarrow}$–ID are not logical truths at all. By an inductive argument it can be shown that AB is a theorem of $\mathbf{TRW}_{\rightarrow}$–ID iff BA is a theorem of $\mathbf{WTR}_{\rightarrow}$–ID. Now NOID can be formulated as:

NOID' There is no formula provable both in $\mathbf{TRW}_{\rightarrow}$–ID and in $\mathbf{WTR}_{\rightarrow}$–ID.

Let $\mathbf{WTR}_{\rightarrow}$ be obtained from $\mathbf{WTR}_{\rightarrow}$–ID by adjoining ID; then, of course, NOID has the formulation

NOID" The only theorems common to $\mathbf{TRW}_{\rightarrow}$ and $\mathbf{WTR}_{\rightarrow}$ are all the formulas of the form AA.

A natural deduction formulation of $\mathbf{TRW}_{\rightarrow}$–ID

In the seventies some one-premiss natural deduction formulations of $\mathbf{TW}_{\rightarrow}$–ID and $\mathbf{TW}_{\rightarrow}$ have been elaborated in Belgrade by Božić, Došen and the present author.

Let us define *consequent* and *antecedent* occurrences of subformulas of a given formula A, as in [1], p. 93.

The formula A itself is a consequent occurrence of A in A.

If BC is a consequent (antecedent) occurrence of BC in A, then the displayed occurrence of B is an antecedent (consequent) occurrence of B in A, and the displayed occurrence of C is a consequent (antecedent) occurrence of C in A.

The logic $\mathbf{TRW}_{\rightarrow}$–ID has a neat formulation called $\mathbf{TRW}'_{\rightarrow}$–ID. There are no axioms in $\mathbf{TRW}'_{\rightarrow}$–ID and instead of SU, PR and TR we have the following four rules:

SU^0 Let AB have a consequent occurrence in a formula D; then we are allowed to substitute an occurrence of $BC.AC$ for that particular occurrence of AB in D, for any formula C;

PR^0 Let BC have a consequent occurrence in a formula D; then we are allowed to substitute an occurrence of $AB.AC$ for that particular occurrence of BC in D, for any formula C;

SU^1 Let $BC.AC$ have an antecedent occurrence in a formula D; then we are allowed to substitute an occurrence of AB for that particular occurrence of $BC.AC$ in D;

PR^1 Let $AB.AC$ have an antecedent occurrence in a formula D; then we are allowed to substitute an occurrence of BC for that particular occurrence of $AB.AC$ in D.

SU^0 and PR^0 are called *consequent* or 0-rules; SU^1 and PR^1 are called *antecedent* or 1-rules.

Let A and B be arbitrary formulas. Suppose that B is obtained from A by applying these four rules (at least one but not necessarily all of them) in any order; then we shall write $A \longrightarrow_{\mathbf{TRW'}\text{-ID}} B$ to denote this fact. Also, we shall write $A \longrightarrow_{\mathbf{TRW'}\text{-ID}} B \longrightarrow_{\mathbf{TRW'}\text{-ID}} C$ if $A \longrightarrow_{\mathbf{TRW'}\text{-ID}} B$ and $B \longrightarrow_{\mathbf{TRW'}\text{-ID}} C$. It is clear that the relation $\longrightarrow_{\mathbf{TRW'}\text{-ID}}$ is transitive.

If $A \longrightarrow_{\mathbf{TRW'}\text{-ID}} B$, then AB is called a *theorem* of $\mathbf{TRW}'_{\rightarrow}$-ID.

The theories $\mathbf{TRW}_{\rightarrow}$-ID and $\mathbf{TRW}'_{\rightarrow}$-ID are equivalent in the sense that they have the same set of theorems. This can be seen from the following considerations.

Let us define the *depth* of an occurrence of a subformula in a formula A as follows: A itself is at depth 0; if an occurrence of BC in A is at depth n, then the displayed occurrences of B and C in A are at depth $n+1$.

Theorem 1. *The theorems of* $\mathbf{TRW}'_{\rightarrow}$*-ID are theorems of* $\mathbf{TRW}_{\rightarrow}$*-ID.*

Proof. Suppose that $D \longrightarrow_{\mathbf{TRW'}\text{-ID}} E$. Proceed by induction on the number n of applications of 0-or-1-rules in the derivation $D \longrightarrow_{\mathbf{TRW'}\text{-ID}} E$ to show that DE is a theorem of $\mathbf{TRW}_{\rightarrow}$-ID.

Let $n = 1$. Suppose that E is obtained from D by one of the 0-or-1-rules. Proceed by another induction on depth at which the substitution takes place.

If the substitution takes place at depth 0, then DE is an instance of an axiom of $\mathbf{TRW}_{\rightarrow}$-ID.

Let $DE = D_1D_2.E_1E_2$. If $D \longrightarrow_{\mathbf{TRW'}\text{-ID}} E$ such that the substitution takes place at depth greater than 0, then either $D_1 = E_1$ and $D_2 \longrightarrow_{\mathbf{TRW'}\text{-ID}} E_2$ or $D_2 = E_2$ and $E_1 \longrightarrow_{\mathbf{TRW'}\text{-ID}} D_1$. In the first case, by induction hypothesis, D_2E_2 is a theorem of $\mathbf{TRW}_{\rightarrow}$. Hence, DE is obtained by PR. In the second case, by induction hypothesis, E_1D_1 is a theorem of $\mathbf{TRW}_{\rightarrow}$. Hence, DE is obtained by SU.

Let $n > 1$. Suppose that E' is obtained from D by $n-1$ applications of 0-or-1-rules, and that E is obtained from E' by a single application of a 0-or-1-rule; by induction hypothesis and the first part of the proof, DE' and $E'E$ are theorems of $\mathbf{TRW}_{\rightarrow}$-ID. Hence, by TR DE is a theorem of $\mathbf{TRW}_{\rightarrow}$-ID. □

Theorem 2. *The theorems of* $\mathbf{TRW}_{\rightarrow}$*-ID are theorems of* $\mathbf{TRW}'_{\rightarrow}$*-ID.*

Proof. It is easy to derive ASU and APR in $\mathbf{TRW}'_{\rightarrow}$-ID. Suppose that $A \longrightarrow_{\mathbf{TRW'}\text{-ID}} B$. This means that starting from A and applying the 0-or-1-rules we eventually obtain B. Let us start from BC; in this formula every consequent occurrence of a subformula in B is an antecedent occurrence in BC, and conversely, every antecedent occurrence of a subformula in B is a consequent occurrence in BC. It is easy to see that AC can be obtained from BC by applying the same rules that lead from A to B in reverse order. This means that AC is obtained from BC by applying a 0-rule instead of the corresponding 1-rule and a 1-rule instead of the corresponding 0-rule.

Hence, if AB is a theorem of $\mathbf{TRW}'_{\rightarrow}$-ID, so is $BC.AC$. In a similar way we can prove that $\mathbf{TRW}'_{\rightarrow}$-ID is closed under PR.

As to TR, it is trivial that if $A \longrightarrow_{\mathbf{TRW'}\text{-ID}} B$ and $B \longrightarrow_{\mathbf{TRW'}\text{-ID}} C$, then $A \longrightarrow_{\mathbf{TRW'}\text{-ID}} C$. Hence, the set of theorems of $\mathbf{TRW}'_{\rightarrow}$-ID is closed under transitivity and all theorems of $\mathbf{TRW}_{\to}$–ID are theorems of $\mathbf{TRW}'_{\to}$–ID. □

Many logicians have tried to prove or disprove A-B, but it turned out that this was a very difficult task.

NOID and hence A-B has been proved true by R.K. Meyer and E. Martin (cf. [6]) who used a semantics developed for this purpose. Thus, indeed, the graphical identity of two formulas in a language with $\to$ as the sole connective is determined by purely logical means defined in a logical calculus in the same language.

A purely constructive proof of NOID has been obtained in [4] (cf. also [2]).

2. Permutation

In $\mathbf{TW}_{\to}$–ID there is almost no rule of permutation admissible. The next theorem seems to give the maximum of permutation allowed in $\mathbf{TW}_{\to}$–ID.

Theorem 3. *If $AB.CD$ is a theorem, then either (a) $A = C$ and BD is a theorem or else (b) $B = D$ and CA is a theorem or else (c) CA and BD are theorems or else (d) $C.ABD$ is a theorem.*

Proof. Consider $\mathbf{TRW}_{\to}$–ID and proceed by induction on theorems. □

Let $C.DE$ be a subformula of A; suppose that B is obtained from A by substitution of $D.CE$ for $C.DE$, at a single occurrence of $C.DE$ and let us write $A \sim B$ iff B can be obtained from A by a finite (possibly zero) number of substitutions of this kind. It is clear that $\sim$ is an equivalence relation. For any A by A^* we shall denote any formula B such that $A \sim B$.

Let us consider the following permutation rules.

PERM* From A to infer A^*.

RPERM If AB is a theorem, so is A^*B^*.

PERM If A is a theorem, so is A^*.

Here 'theorem' means 'theorem of the system under consideration'.

Let us adjoin RPERM to $\mathbf{TRW}_{\to}$–ID and let the resulting system be called $\mathbf{PTW}_{\to}$–ID.

If PERM is adjoined to $\mathbf{TRW}_{\to}$–ID, the resulting system is called $\mathbf{L}$; APR is then redundant.

Obviously, the theorems of $\mathbf{PTW}_{\to}$–ID are theorems of $\mathbf{L}$.

Theorem 4. $\mathbf{L} = \mathbf{PTW}_{\to}$*–ID + PASU + SUP + PRP, where* PASU *is the following axiom scheme* (ASU *with permutation)*

$$\text{PASU} \qquad A.AB.BCC$$

and SUP *and* PRP *are the following rules* (SU *and* PR *with permutation):*

SUP From AB to infer $A.BCC$

PRP From BC to infer $A.ABC$

Proof. It is clear that the theorems of $\mathbf{PTW}_{\rightarrow}$–ID + PASU + SUP + PRP are theorems of $\mathbf{L}$, for PASU, SUP and PRP are obtained from ASU, SU, and PR by PERM, respectively.

On the other hand, by induction on theorems it can be shown that $\mathbf{PTW}_{\rightarrow}$–ID + PASU + SUP + PRP is closed under PERM. The only place in this proof that requires a little care is TR. Suppose that $A.CD$ is obtained in $\mathbf{L}$ from AB and $B.CD$ by TR, and that then $C.AD$ is obtained by PERM. By induction hypothesis, $C.BD$ is a theorem of $\mathbf{PTW}_{\rightarrow}$–ID + PASU + SUP + PRP. On the other hand, from AB we obtain $BD.AD$ by SU; hence, $C.AD$ is a theorem, by TR. Therefore, the theorems of $\mathbf{L}$ are theorems of $\mathbf{PTW}_{\rightarrow}$–ID + PASU + SUP + PRP. □

It has been proved in [4] that NOID holds for $\mathbf{L}$ as well:

NOID(L) there is no theorem of $\mathbf{L}$ either of the form AA or of the form ABB or of the form $ABBA$ or of the form $A.ABB$.

This result was obtained by constructing a cut-free Gentzen-style formulation of $\mathbf{L}$ also called $\mathbf{L}$. The structure of the proof is the following: it was obvious that pp is not derivable in $\mathbf{L}$; if we assume that AA is derivable for some formula A, then there is a formula B of smallest degree such that BB is derivable. In considering the possible derivations of BB, there always was a formula C of degree smaller than the degree of B such that CC was derivable.

Neither $\mathbf{PTW}_{\rightarrow}$–ID nor $\mathbf{L}$ is closed under modus ponens. A counterexample provided in [5] can be used here as well. Let $A = pp.pp.pp$ and $B = (pp.pp)p.ppp$; AB is an instance of ASU. If $\mathbf{PTW}_{\rightarrow}$–ID were closed under MP, applying RPERM to $AB.Bp.Ap$ we would obtain $AB.A.Bpp$; hence, by MP applied twice, Bpp would be obtained in $\mathbf{PTW}_{\rightarrow}$–ID, contrary to NOID(L).

There are proper extensions of $\mathbf{L}$ closed under MP such that NOID still holds for them. Let $\mathbf{K}$ be the system defined by ASU, MP, PERM and the following assertion rule

ASS1 . If A is a theorem of $\mathbf{K}$, so is ABB.

There is a Gentzen-style formulation of $\mathbf{K}$ called in [5] $\mathbf{J}$; it has been proved that

NOID(J) there is no theorem of **J** either of the form AA or else of the form $A.ABB$ or else of the form $ABBA$.

By ASU and MP **K** is closed under another assertion rule as well:

ASS2 If A and BC are theorems of **K**, so is ABC.

The connection between **K** and **L** is given in the next theorem.

Theorem 5. $\mathbf{K} = \mathbf{L} + ASS1$.

Proof. The rules of $\mathbf{L}$ + ASS1 are derived in **K**. We have to prove that $\mathbf{L}$ + ASS1 is closed under MP.

Suppose that (a) A_i and (b) $A_1 \ldots .A_{i-1}.A_i.A_{i+1} \ldots .A_np$ are theorems of $\mathbf{L}$ + ASS1; we want to prove that (c) $A_1. \ldots .A_{i-1}.A_{i+1} \ldots .A_np$ is a theorem of $\mathbf{L}$ + ASS1. Proceed by induction on the combined weight of (a) and (b) (for the definition of combined weight cf. [7], p. 113).

Let (b) be an instance $A_3C.CD.A_3D$ of ASU, where $A_1 = A_3C$, $A_2 = CD$, and $D = A_4 \ldots .A_np$.

If $i < 3$, then (c) is obtained from A_i either by SU or by PR.

If $i = 3$, then from (a) we obtain (c') A_3CC by ASS1; hence, by (c') and SU we have $CD.A_3CD$; eventually, by PERM we derive (c).

Let $i > 3$ and $E = A_{i+1} \ldots .A_np$; from A_i we obtain A_iEE by ASS1. Then we apply PR to derive $A_3D.A_3.A_4 \ldots A_{i-1}E$ and (c') $(CD.A_3D).CD .A_3.A_4 \ldots .A_{i-1}E$. But as an instance of ASU we have $A_3C.CD.A_3D$; hence, by using (c') and TR we obtain (c).

Let (b) be obtained by SU; if $i = 1$, then $A_1 = A_1'C$, where $C = A_3 \ldots .A_np$ and $A_2 = A_2'C$, and (b) is obtained from (b') $A_2'A_1'$. From (a) and (b') we obtain $A_2'C$ by TR, as required.

If $i = 2$, then (b) is $A_1'C.A_2C$ and it is obtained from (b') A_2A_1'. By (a), (b') and the induction hypothesis we have (c') A_1'; by (c') and ASS1 we obtain (c) $A_1'CC$.

Let $i \geq 3$ and let E be as before; then (b) is $A_1'(A_3 \ldots .A_{i-1}.A_iE).A_2.A_3 \ldots .A_{i-1}.A_iE$, and it is obtained from (b') A_2A_1', where, obviously, $A_1 = A_1'.A_3 \ldots .A_{i-1}.A_iE$.

From (a) A_i we obtain A_iEE by ASS1, and then (c') $A_1'(A_3 \ldots .A_{i-1}.A_iE) .A_1'.A_3 \ldots .A_{i-1}E$ by PR. By using PERM we have (c") $A_1'.A_1'(A_3 \ldots .A_{i-1} .A_iE).A_3 \ldots .A_{i-1}E$. Hence, by (b'), (c") and TR we obtain $A_2.A_1'(A_3 \ldots .A_{i-1}.A_iE).A_3 \ldots .A_{i-1}E$; by using PERM again, we obtain (c).

Let (b) be obtained by PR; if $i = 1$, then $A_1 = A_2C$ and (b) is obtained from (b') $C.A_3 \ldots .A_np$. From (a) and (b') we obtain $A_2.A_3 \ldots .A_np$ by TR, as required.

If $i = 2$, then (b) is $A_2C.A_2.A_3 \ldots .A_np$ and it is obtained from (b') $C.A_3 \ldots .A_np$. By (a) and ASS1 we obtain A_2CC, and then, by using (b') and TR we obtain (c).

Let $i \geq 3$ and let E be as before; then (b) is $A_2C.A_2.A_3 \ \ldots \ .A_{i-1}.A_iE$, and it is obtained from (b') $C.A_3 \ \ldots \ .A_{i-1}.A_iE.$, where, obviously, $A_1 = A_2C$. By induction hypothesis, $A_2C.A_2.A_3 \ \ldots \ .A_{i-1}E$ is a theorem; now (c) is obtained by PR.

If (b) is obtained by PERM, the use of induction hypothesis is straightforward.

Let (b) be obtained by ASS1; then $A_1 = A_1'.A_2 \ \ldots \ .A_np$ and (b) is obtained from (b') A_1'. If $i = 1$, then by (b'), (a) and the induction hypothesis, (c) $A_2 \ \ldots \ .A_np$ is a theorem.

If $i > 1$, let E be as before; from (b') we obtain (c') $A_1'(A_2 \ \ldots \ .A_{i-1}E).A_2 \ \ldots \ .A_{i-1}E$ by ASS1. On the other hand, from (a) we derive (c") $A_i(A_1'.A_2 \ \ldots \ .A_{i-1}E).A_1'.A_2 \ \ldots \ .A_{i-1}E$ by ASS1. Hence, by (c"), (c'), TR and PERM we obtain (c).

Let (b) be obtained from (b') A_1C and (b") $C.A_2 \ \ldots \ .A_np$ by TR. If $i = 1$, then by (a), (b') and the induction hypothesis we obtain C; hence, by C, (b") and the induction hypothesis we obtain (c).

If $i > 1$, let E be as before; by induction hypothesis, (c') $C.A_2 \ \ldots \ .A_{i-1}E$ is a theorem; hence, by (b'), (c') and TR we obtain (c).

This completes the proof of the theorem. □

The system **K** has an interesting property called NOE. It has been proved in [5] that **J** and hence **K** is closed under the following rule:

NOE $(A_1 \ \ldots \ .A_nB)B$ is a theorem of **K** iff so are $A_1, \ldots, A_n$.

In particular, there is no theorem of **K** of the form $AABB$.

Natural deduction systems $\mathbf{L}'$, $\mathbf{L}''$ and $\mathbf{L}'''$

Let $\mathbf{L}'$ be the one-premiss natural deduction system obtained from $\mathbf{TRW}'_{\rightarrow}$ –ID by adjoining the rule PERM*.

Let A and B be arbitrary formulas. Suppose that B is obtained from A by applying SU^0, SU^1, PR^0, PR^1, or PERM* a finite number of times; then we shall write $A \longrightarrow_{\mathbf{L}'} B$ to denote this fact. If $A \longrightarrow_{\mathbf{L}'} B$ and one of the first four rules is applied at least once, then AB is called a theorem of $\mathbf{L}'$.

The restriction in the definition of theorems of $\mathbf{L}'$ is obvious; without it we have the following derivation: starting from $A.BC$ we apply PERM* twice and we obtain $A.BC$ again; hence, without the restriction, $A(BC).A.BC$ would be a theorem of $\mathbf{L}'$.

Theorem 6. **PTW**$_{\rightarrow}$*-ID and* $\mathbf{L}'$ *have the same set of theorems.*

Proof. Let A^*B^* be a theorem of $\mathbf{L}'$ obtained from AB by RPERM; by induction hypothesis, $A \longrightarrow_{\mathbf{L}'} B$. Hence, $A^* \longrightarrow_{\mathbf{L}'} A \longrightarrow_{\mathbf{L}'} B \longrightarrow_{\mathbf{L}'} B^*$.

This shows that $\mathbf{L}'$ is closed under RPERM and it is easy to see that the theorems of $\mathbf{PTW}_{\rightarrow}$–ID are theorems of $\mathbf{L}'$.

Suppose that $D \longrightarrow_{\mathbf{L}'} E$ and proceed by induction on the number n of applications of 0-or-1-rules. Let $n = 1$ and let E be obtained from E' by an application of such a rule and proceed by another induction on depth. If the rule is applied at depth 0, then $E'E$ is a theorem of $\mathbf{PTW}_{\rightarrow}$–ID by axioms. Obviously, $D \sim E'$ and DE is obtained by RPERM only.

Let $E'E = E_1'E_2'.E_1E_2$. If E is obtained from E' such that E_2 is obtained from E_2' by a 0-or-1-rule, then $E_1' = E_1$ and, by the second induction hypothesis, $E_2'E_2$ is a theorem of $\mathbf{PTW}_{\rightarrow}$–ID. Hence, $E'E$ is obtained by PR. Again, we have $D \sim E'$ and we obtain DE.

If E is obtained from E' such that E_1' is obtained from E_1 by a 0-or-1-rule, then $E_2' = E_2$ and, by the second induction hypothesis, E_1E_1' is a theorem of $\mathbf{PTW}_{\rightarrow}$–ID. Hence, $E'E$ is obtained by SU. Again, $D \sim E'$ and DE is a theorem of $\mathbf{PTW}_{\rightarrow}$–ID.

Let $n > 1$. Suppose that E' is obtained from D by $n-1$ applications of 0-or-1-rules, and that E is obtained from E' either by a single application of a 0-or-1-rule or by PERM*; by the first induction hypothesis DE' is a theorem of $\mathbf{PTW}_{\rightarrow}$–ID. If E is obtained from E' by an application of a 0-or-1-rule, then $E'E$ is a theorem of $\mathbf{PTW}_{\rightarrow}$–ID by the first part of this proof; hence, DE is a theorem of $\mathbf{PTW}_{\rightarrow}$–ID by TR. If E is obtained from E' by PERM*, then DE is obtained by RPERM from DE'.

This completes the proof of the theorem. □

Let $\mathbf{L}''$ be the one-premiss natural deduction system obtained from $\mathbf{L}'$ by adjoining the following two new 0-or-1-rules

SUL^0 Let A have a consequent occurrence in a formula D; then we are allowed to substitute an occurrence of $AB.BCC$ for that particular occurrence of A in D, for any formulas B and C;

SUL^1 Let $AB.BCC$ have an antecedent occurrence in a formula D; then we are allowed to substitute an occurrence of A for that particular occurrence of $AB.BCC$ in D.

Theorem 7. *The theorems of* $\mathbf{L}''$ *are theorems of* $\mathbf{L}$.

Proof. The proof of Theorem 6 can be extended in the case when either SUL^0 or SUL^1 is applied to E' at depth 0; then in $\mathbf{L}$ we can apply PERM to an instance of ASU. □

It is easy to derive ASU in $\mathbf{L}''$. Also, we can show that $\mathbf{L}''$ is closed under SU, PR, and TR. Hence, $\mathbf{L}''$ contains $\mathbf{TRW}_{\rightarrow}$–ID. However, there are theorems of $\mathbf{L}$ that are not theorems of $\mathbf{L}''$. In particular, $\mathbf{L}''$ is not closed under PERM: there is a theorem of $\mathbf{L}''$ of the form $A.BC$ such that $B.AC$ is not a theorem of $\mathbf{L}''$, as in the following example.

We have $(pp.ppp)p \longrightarrow_{\mathbf{L}'} pp$ by SUL[1], but not $p \longrightarrow_{\mathbf{L}'} (pp.ppp)pp$; the latter derivation is impossible in $\mathbf{L}''$. On the other hand, in $\mathbf{L}$ from the instance $pp.pp.pp$ of ASU we obtain $p.pp.ppp$ by PERM; then we apply SU to obtain $(pp.ppp)p.pp$; eventually, we use PERM to prove $p.(pp.ppp)pp$.

Let $\mathbf{L}'''$ be obtained from $\mathbf{L}''$ by adjoining PERM. This means that the set of theorems of $\mathbf{L}'''$ is defined as the smallest set of formulas satisfying the following two clauses: (1) if $A \longrightarrow_{\mathbf{L}'} B$ in $\mathbf{L}''$, then AB is a theorem of $\mathbf{L}'''$ and (2) if $A.BC$ is a theorem of $\mathbf{L}'''$, then $B.AC$ is a theorem of $\mathbf{L}'''$.

The definition of a theorem of $\mathbf{L}'''$ can be given by (1) and (2'): if $A \longrightarrow_{\mathbf{L}'} BC$ in $\mathbf{L}''$, then $B.AC$ is a theorem of $\mathbf{L}'''$.

It is not difficult to see that the definition using (1) and (2) is equivalent to the definition using (1) and (2'). Of course, $\mathbf{L}$ and $\mathbf{L}'''$ have the same set of theorems.

A natural deduction system $\mathbf{K}'$

Let us adjoin ASS1 to $\mathbf{L}'''$ and let the resulting system be called $\mathbf{K}'$. Hence, the set of theorems of $\mathbf{K}'$ is the smallest set satisfying the following conditions: (i) if AB is a theorem of $\mathbf{L}'''$, then AB is a theorem of $\mathbf{K}'$ and (ii) if A is a theorem of $\mathbf{K}'$, then ABB is a theorem of $\mathbf{K}'$.

The definition of a theorem of $\mathbf{K}'$ can be given by (i) and (ii'): if $A.BC$ is a theorem of $\mathbf{L}'''$, then $B.AC$ is a theorem of $\mathbf{K}'$.

It is easy to prove

Theorem 8. $\mathbf{K}$ *and* $\mathbf{K}'$ *have the same set of theorems.*

3. Binary trees

In [3] a connection between $\mathbf{TRW}'_{\rightarrow}$–ID and binary trees has been established.

By a *binary tree* we understand a tree such that (1) there is a unique element at level 0 called the *origin* of $\mathcal{T}$ and (2) each node of $\mathcal{T}$ is either an end-node or has exactly two immediate successors.

By a *subtree* $\mathcal{T}'$ of a binary tree $\mathcal{T}$ we understand a subset $\mathcal{T}'$ of nodes of $\mathcal{T}$ such that $\mathcal{T}'$ is a binary tree and the immediate successor relation in $\mathcal{T}'$ is the immediate successor relation in $\mathcal{T}$.

A subtree $\mathcal{T}'$ of $\mathcal{T}$ is *proper* iff $\mathcal{T}'$ is a subtree of $\mathcal{T}$, and $\mathcal{T}'$ and $\mathcal{T}$ are not identical. Obviously, a subtree $\mathcal{T}'$ of $\mathcal{T}$ is proper iff the origin of $\mathcal{T}'$ is distinct from the origin of $\mathcal{T}$.

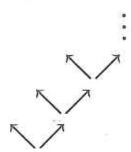
A binary tree $\mathcal{T}$ is finite (infinite) iff the set of nodes of $\mathcal{T}$ is finite (infinite).

After several conversations with Ilijas Farah in the period 1990 - 92 the concept of a *formula-like tree* (FLT) has emerged and the present author was able to represent the one-premiss natural deduction systems considered above as systems of operations on FLTs.

Let $\mathcal{T}'$ and $\mathcal{T}''$ be two binary trees. We shall say that they are *isomorphic* iff there is a mapping h from $\mathcal{T}'$ onto $\mathcal{T}''$ such that the following conditions are satisfied: (1) if x is the origin of $\mathcal{T}'$, then $h(x)$ is the origin of $\mathcal{T}''$ and (2) if the nodes y and z of $\mathcal{T}'$ are the left and the right immediate successor, respectively, of a node x in $\mathcal{T}'$, then $h(y)$ and $h(z)$ in $\mathcal{T}''$ are the left and the right immediate successor, respectively, of $h(x)$ in $\mathcal{T}''$.

If $\mathcal{T}'$ and $\mathcal{T}''$ are finite trees and one of them is a proper subtree of the other, they cannot be isomorphic. However, if they are infinite, it is possible that they are isomorphic and yet that one of them is a proper subtree of the other.

Let the *full binary tree* (FBT) be the infinite binary tree with no finite branch. In FBT every subtree is FBT. There are examples of binary trees that have isomorphic proper subtrees and are different from FBT. Here is one:



This is an infinite tree; each node in the infinite (rightmost) branch has an end-node as the left successor and a node in the infinite branch as the right successor. Any proper subtree with the origin in the infinite branch is isomorphic to the whole tree.

Let us call a tree $\mathcal{T}$ *formula-like tree* (FLT) iff (1) it has no proper subtree that is isomorphic to $\mathcal{T}$ and (2) no proper subtree $\mathcal{T}'$ of $\mathcal{T}$ has a proper subtree isomorphic to $\mathcal{T}'$.

There is a trivial consequence of the above definition and the fact that being a subtree is a transitive relation.

Theorem 9. *A subtree of a* FLT *is a* FLT.

Every finite binary tree is a FLT, but there are infinite FLTs as well. For example, take the above infinite tree and extend each end-node by a finite tree that is different from all finite trees adjoined to previous end-nodes.

Every infinite FLT contains infinitely many (distinct) finite branches; every node of a FLT is a node of a finite branch. Hence, any branch of a FLT is at most of length ω. Therefore, the nodes of a FLT are arranged in levels and to each level there is attached a natural number. The number 0 is attached to the origin.

Theorem 10. *If $\mathcal{T}_1$ and $\mathcal{T}_2$ are FLTs, then*

$$\begin{array}{ccc} \mathcal{T}_1 & & \mathcal{T}_2 \\ & \nwarrow\nearrow & \\ & \mathcal{T}_1\mathcal{T}_2 & \end{array}$$

is a FLT.

Proof. If the contrary is the case, then there is a proper subtree $h(\mathcal{T}_1\mathcal{T}_2)$ of $\mathcal{T}_1\mathcal{T}_2$ isomorphic to $\mathcal{T}_1\mathcal{T}_2$. By definition of isomorphism, $h(\mathcal{T}_1\mathcal{T}_2)=h(\mathcal{T}_1)h(\mathcal{T}_2)$ and $h(\mathcal{T}_1)$ and $h(\mathcal{T}_2)$ are isomorphic to $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. The origin of $h(\mathcal{T}_1\mathcal{T}_2)$ is either in the subtree $\mathcal{T}_1$ or in the subtree $\mathcal{T}_2$, say in $\mathcal{T}_1$. Now the origin of $h(\mathcal{T}_1\mathcal{T}_2)$ coincides either with the origin of $\mathcal{T}_1$ or with another node of $\mathcal{T}_1$. Since $h(\mathcal{T}_1\mathcal{T}_2) = h(\mathcal{T}_1)h(\mathcal{T}_2)$, the left successor of the node $h(\mathcal{T}_1\mathcal{T}_2)$ is $h(\mathcal{T}_1)$. But $\mathcal{T}_1$ and $h(\mathcal{T}_1)$ are isomorphic and $h(\mathcal{T}_1)$ is a proper subtree of $\mathcal{T}_1$, contrary to the assumption that $\mathcal{T}_1$ is a FLT.

We proceed similarly if the origin of $h(\mathcal{T}_1\mathcal{T}_2)$ is in the subtree $\mathcal{T}_2$. □

Formulas and binary trees

Formulas of the propositional language with $\rightarrow$ as the sole connective are naturally connected with finite binary trees. The nodes of such a tree are subformulas of the formula A we are constructing the tree for. Thus, the formula A itself is at the origin of the tree. If an occurrence of a subformula BC of A is at a node at level n, then, at level $n+1$, the displayed occurrence of B is the left and the displayed occurrence of C is the right successor of the displayed occurrence of BC. The end-nodes of such formula trees are occurrences of propositional variables.

Suppose that in the propositional language that we are considering there is only one propositional variable, say p (this is sufficient to prove NOID, NOID(L) NOID(J) and NOE); then we can identify formulas and finite binary trees. Let p be the tree consisting of a single node. If A and B are finite binary trees, then AB is the tree obtained by taking a node as the origin of the tree such that A and B are the left and the right immediate successor of the origin.

In the sequel we shall interpret formulas as FLTs. For any propositional variable p we choose a FLT $\mathcal{T}$ and we interpret p as $\mathcal{T}$. Let A and B be any formulas and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the FLTs such that A and B are interpreted as FLTs $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively; let us choose a new node called $\mathcal{T}_1\mathcal{T}_2$ as the origin of a new tree and take $\mathcal{T}_1$ and $\mathcal{T}_2$ to be the only immediate successors of $\mathcal{T}_1\mathcal{T}_2$, thus:

$$\begin{array}{ccc} \mathcal{T}_1 & & \mathcal{T}_2 \\ & \nwarrow\nearrow & \\ & \mathcal{T}_1\mathcal{T}_2 & \end{array}$$

By Theorem 10, $\mathcal{T}_1\mathcal{T}_2$ is a FLT and we interpret AB as $\mathcal{T}_1\mathcal{T}_2$.

In denoting trees we shall use the conventions adopted in writing formulas.

Propositional formulas have a property called *substitution*; let us show that FLTs enjoy the same property. Suppose that $\mathcal{T}_1$ is a FLT and $\mathcal{T}_2$ a subtree of $\mathcal{T}_1$:

$$\begin{array}{c}\mathcal{T}_2\\ \vdots\\ \mathcal{T}_1\end{array}$$

then this occurrences of the FLT $\mathcal{T}_2$ in $\mathcal{T}_1$ can be cut off and a FLT $\mathcal{T}_3$ can be inserted instead:

$$\begin{array}{c}\mathcal{T}_3\\ \vdots\\ \mathcal{T}_1\end{array}$$

Theorem 11. *Let $\mathcal{T}_1$ be a FLT, let $\mathcal{T}_2$ be a subtree of $\mathcal{T}_1$ and let $\mathcal{T}_4$ be the tree obtained from $\mathcal{T}_1$ by cutting off $\mathcal{T}_2$ and by inserting a* FLT *$\mathcal{T}_3$ instead; then $\mathcal{T}_4$ is a* FLT.

Proof. Proceed by induction on levels. Let $\mathcal{T}_2$ be $\mathcal{T}_1$; then, obviously, $\mathcal{T}_4$ is $\mathcal{T}_3$. If $\mathcal{T}_1$ is $\mathcal{T}_1'\mathcal{T}_1''$, then the origin of $\mathcal{T}_2$ is at a certain level n in $\mathcal{T}_1$. If it is in, say, $\mathcal{T}_1'$, then in $\mathcal{T}_1'$ it is at level smaller than n and by induction hypothesis the result $\mathcal{T}_4'$ of substitution of $\mathcal{T}_3$ for $\mathcal{T}_2$ in $\mathcal{T}_1'$ is a FLT. By Theorem 9, $\mathcal{T}_1''$ is a FLT; hence, by Theorem 10, so is $\mathcal{T}_4'\mathcal{T}_1''$, i.e. $\mathcal{T}_4$. □

Natural deduction and FLTs

There is a connection between derivations in one-premiss natural deduction systems $\mathbf{TW}'_{\to}$–ID, $\mathbf{L}'$, and $\mathbf{L}''$ and FLTs. In order to explain this connection, let us show how the rules of $\mathbf{L}''$ can be interpreted as operations on FLTs.

To every node of a FLT $\mathcal{T}$ we associate one of the numbers 0 or 1, as follows: 0 is associated with the origin of $\mathcal{T}$; if 0 (1) is associated with a node at level n, then 1 (0) is associated with its left hand successor and 0 (1) is associated with its right hand successor at level $n+1$.

If 0 (1) is associated with a node of a tree, then we shall call it a 0-node (1-node).

Now the rules SU^0, PR^0, SU^1, PR^1, SUL^0, SUL^1, and PERM* can be represented as operations FLTs as follows.

SU^0 Let $\mathcal{T}$ be a FLT and let $\mathcal{T}_1\mathcal{T}_2$ be one of its 0-nodes:

$$\begin{tikzcd}
\mathcal{T}_1 & & \mathcal{T}_2 \\
& \mathcal{T}_1\mathcal{T}_2 \arrow[ul] \arrow[ur] & \\
& \vdots &
\end{tikzcd}$$

Then the subtree $\mathcal{T}_1\mathcal{T}_2$ can be cut off and a subtree $\mathcal{T}_2\mathcal{T}_3.\mathcal{T}_1\mathcal{T}_3$ can be inserted in $\mathcal{T}$ instead:

$$\begin{tikzcd}
\mathcal{T}_2 & & \mathcal{T}_3 & \mathcal{T}_1 & & \mathcal{T}_3 \\
& \mathcal{T}_2\mathcal{T}_3 \arrow[ul] \arrow[ur] & & & \mathcal{T}_1\mathcal{T}_3 \arrow[ul] \arrow[ur] & \\
& & \mathcal{T}_2\mathcal{T}_3.\mathcal{T}_1\mathcal{T}_3 \arrow[ul] \arrow[urr] & & & \\
& & \vdots & & &
\end{tikzcd}$$

where $\mathcal{T}_3$ is any FLT. Let us call the new tree $\mathcal{T}'$.

In a similar way we can represent the remaining 0-rules.

As to the 1-rules, let us represent SU^1. Suppose that $\mathcal{T}'$ is a FLT and let $\mathcal{T}_2\mathcal{T}_3.\mathcal{T}_1\mathcal{T}_3$ be one of its 1-nodes; then the subtree $\mathcal{T}_2\mathcal{T}_3.\mathcal{T}_1\mathcal{T}_3$ can be cut off and the subtree $\mathcal{T}_1\mathcal{T}_2$ can be inserted instead, producing thus the tree $\mathcal{T}$.

In a similar way we may represent the remaining 1-rules.

Now we represent PERM*.

PERM* Let $\mathcal{T}$ be a FLT and let $\mathcal{T}_1.\mathcal{T}_2\mathcal{T}_3$ be one of its 0-or-1-nodes:

$$\begin{tikzcd}
& \mathcal{T}_2 & & \mathcal{T}_3 \\
\mathcal{T}_1 & & \mathcal{T}_2\mathcal{T}_3 \arrow[ul] \arrow[ur] & \\
& \mathcal{T}_1.\mathcal{T}_2\mathcal{T}_3 \arrow[ul] \arrow[ur] & & \\
& \vdots & &
\end{tikzcd}$$

Then the subtree $\mathcal{T}_1.\mathcal{T}_2\mathcal{T}_3$ can be cut off and the subtree $\mathcal{T}_2.\mathcal{T}_1\mathcal{T}_3$ can be inserted in $\mathcal{T}$ instead:

$$\begin{tikzcd}
& \mathcal{T}_1 & & \mathcal{T}_3 \\
\mathcal{T}_2 & & \mathcal{T}_1\mathcal{T}_3 \arrow[ul] \arrow[ur] & \\
& \mathcal{T}_2.\mathcal{T}_1\mathcal{T}_3 \arrow[ul] \arrow[ur] & & \\
& \vdots & &
\end{tikzcd}$$

By Theorem 11, the result of an application of a 0-or-1-rule or PERM* to a FLT is a FLT. Suppose that these rules are applied to a finite binary tree; then NOID can be stated as follows:

NOID($\mathcal{T}$) (1) Starting from a FLT $\mathcal{T}$ and successively performing the operations SU^0, PR^0, SU^1, PR^1, SUL^0, SUL^1, and PERM* any finite number of times and in any order, and such that one of the first six operation is performed at least once, it is not possible to obtain $\mathcal{T}'$ as a result, where $\mathcal{T}'$ is either $\mathcal{T}$ or $\mathcal{T}\mathcal{T}_1\mathcal{T}_1$;
(2) starting with a FLT $\mathcal{T}\mathcal{T}_1$ and successively performing the operations SU^0, PR^0, SU^1, PR^1, SUL^0, SUL^1, and PERM* any finite number of times and in any order, and such that one of the first six operation is performed at least once it is not possible to obtain $\mathcal{T}_1$ as a result;
(3) starting with a FLT $\mathcal{T}\mathcal{T}_1\mathcal{T}_1$ and successively performing the operations SU^0, PR^0, SU^1, PR^1, SUL^0, SUL^1, and PERM* any finite number of times and in any order, and such that one of the first six operation is performed at least once it is not possible to obtain $\mathcal{T}$ as a result.

If the rules are applied to a finite FLT $\mathcal{T}$, then NOID($\mathcal{T}$) is true, since we can identify formulas and finite binary trees.

Theorem 12. *NOID($\mathcal{T}$) is true for any FLT $\mathcal{T}$.*

Proof. An interpretation of a theorem of **L** in the set of all FLTs is a homomorphic image of a theorem of **L**; hence, it has a form of a theorem of **L**. By NOID(J) there is no theorem of **L** either of the form AA or of the form $A.ABB$ or of the form ABB; hence, there is no FLT either of the form $\mathcal{T}\mathcal{T}$ or of the form $\mathcal{T}_1.\mathcal{T}_1\mathcal{T}_2\mathcal{T}_2$ or of the form $\mathcal{T}_1\mathcal{T}_2\mathcal{T}_2$. If a FLT $\mathcal{T}_2$ can be obtained from a FLT $\mathcal{T}_1$ by a finite number of applications of 0-or-1-rules and PERM*, then $\mathcal{T}_1\mathcal{T}_2$ is a homomorphic image of a theorem of **L**. Hence, NOID($\mathcal{T}$) is true. □

There is no natural interpretation of one-premiss natural deduction systems $\mathbf{L}'''$ or $\mathbf{K}'$ in terms of operations on FLTs, for there are theorems of these systems that cannot be obtained by performing such operations only.

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ALEKSANDAR KRON, FILOZOFSKI FAKULTET, 11000 BEOGRAD, YUGOSLAVIA