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THEORY OF MULTIPLE ANTISYMMETRY

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ABSTRACT. Survey of problems in theory of multiple antisymmetry, which can be solved using antisymmetric characteristic method, is given.

0. Introduction and definitions

Originated from Speiser (1927) and realized by Weber (1929), the idea of representing symmetry groups of bands by black-white plane diagrams was the starting point for introducing the antisymmetry (Heesch, 1929). The color change white-black used as the possibility for the dimensional transition from the symmetry groups of friezes G_{21} to the symmetry groups of bands G_{321}, or from the plane groups G_2 to the layer groups G_{32}, applied on Fedorov space groups G_3 in order to derive the hyperlayer symmetry groups G_{43} (Heesch, 1930) was the beginning of the theory of antisymmetry. The further development of the theory of antisymmetry can be followed through the works by Shubnikov, Belov and Zamorzaev [1].

Its natural generalization, the multiple antisymmetry is suggested by Shubnikov (1945) and introduced by Zamorzaev (1957). Three months later, the different concept of the multiple antisymmetry is proposed by Mackay. During the next 30 years, mostly by the contribution of Kishinev school (Zamorzaev, Palistrant, Galyarskij...) the theory of multiple antisymmetry has become an integral part of mathematical crystallography and acquired the status of a complete theory extended to all categories of isometric symmetry groups of the space E^n ($n \leq 3$), different kinds of non-isometric symmetry groups (of similarity symmetry, conformal symmetry...) and P-symmetry groups [1,2,3,4]. On the other hand, investigation of Mackay approach to the multiple antisymmetry was not continued.

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Let the discrete symmetry group G with a set of generators $\{S_1, \dots, S_r\}$ be given by presentation [5]

$$g_n(S_1, \dots, S_r) = E, \quad n = \overline{1, s}$$

and let $e_1, \dots, e_l$ be antiidentities of the first,...,lth kind, satisfying the relations

$$e_i e_j = e_j e_i \quad e_i^2 = E \quad e_i S_q = S_q e_i, \quad i, j = \overline{1, l}, \;\; q = \overline{1, r} \qquad (1).$$

The group consisting of transformations $S' = e'S$, where e' is the identity, antiidentity, or some product of antiidentities, is caled the (multiple) antisymmetry group. In particular, for $l = i = j = 1$ we have the simple antisymmetry. From the point of view of the mathematical logic or discrete mathematics the system of antiidentities can be considered as l-dimensional Boolean space.

The groups of simple and multiple antisymmetry can be derived by Shubnikov-Zamorzaev method: by replacing the generators of G by antigenerators of one or several independent patterns of antisymmetry. Having in mind the theorem on dividing all groups of simple and multiple antisymmetry into groups of C^k $(1 \leq k \leq l)$, $C^k M^m$ $(1 \leq k, m;\, k+m \leq l)$ and M^m $(1 \leq m \leq l)$ types, and the derivation of the groups of C^k and $C^k M^m$ types directly from the generating group G and from the groups of M^m-type respectively, the only non-trivial problem is the derivation of the M^m-type groups [1].

In this paper we will consider only the junior multiple antisymmetry groups of the M^m-type, i.e. the multiple antisymmetry groups isomorphic with their generating symmetry group, that possess the independent system of antisymmetries.

Every junior multiple antisymmetry group G' of the M^m-type can be (uniquely) defined by the extended group/subgroup symbol

$$G/(H_1, \dots, H_m)/H,$$

where G is the generating group, H_i its subgroups of the index 2 satisfying the relationships $G/H_i \simeq C_2 = \{e_i\}$ $(1 \leq i \leq m)$, and H the subgroup of G of the index 2^m, the symmetry subgroup of G' $(G/H \simeq C_2^m = \{e_1\} \times \dots \times \{e_m\})$.

For the equality of multiple antisymmetry groups can be used three different criteria:

(1) "strong" equality criterion according to which the antiidentities e_i are noneq uivalent. Consequently, in the symbol $G/(H_1, \dots, H_m)/H$ the order of the subgroups $H_1, \dots, H_m$ is important. In the sense of interpretation,

this means that the bivalent changes e_i are physically different (nonequivalent) (e.g. (white black), (+ -), (S N), (0 1)...);

(2) "middle" equality criterion, where all e_i are treated as the equivalent ones (i.e. permutable), so the order of the subgroups mentioned it is not important; (3) "weak" equality criterion G/H.

Using the "strong" equality criterion, as the result we have Zamorzaev groups (Z-groups), and using the "middle" Mackay (or compound) multiple antisymmetry groups (M-groups) [6]. In this paper the consideration is restricted on Z-groups.

Theorem 1. (THE EXISTENTIAL CRITERION FOR M^m-TYPE GROUPS) *A Z-group G' will be of the M^m-type*

(a) *if all the relations (1) remain satisfied after replacing the generators by antigenerators; and*

(b) *if G' exausts all the antisymmetry patterns, for fixed m.*

For the derivation of Z-groups very efficiently used is the antisymmetric characteristic method [7,8,9].

Definition 1. Let all products of the generators of G, within which every generator participates once at the most, be formed and then subsets of transformations that are equivalent in the sense of symmetry with regard to the symmetry group G be separated. The resulting system is called the antisymmetric characteristic of group G ($AC(G)$).

The most of AC permit the reduction, i.e. a transformation into the simplest form; e.g., the AC of the plane symmetry group **pm** given by the presentation [5]

$$\{X,Y,R\} \quad XY=YX \quad R^2=(RX)^2=E \quad RY=YR$$

is $\{R,RX\}\{Y\}\{RY,RXY\}\{X\}\{XY\}$ and its reduced AC is $\{R,RX\}\{Y\}$.

Definition 2. Two or more Z-groups belong to a family iff they are derived from the same symmetry group G.

Theorem 2. *Two Z-groups G'_1 and G'_2 of the M^m-type for m fixed, with common generating group G, are equal iff they possess equal AC.*

Every $AC(G)$ completely defines the series $N_m(G)$, where by $N_m(G)$ is respectively denoted the number of Z-groups of the M^m-type derived from G, for m fixed ($1 \leq m \leq l$). For example, $N_1(\mathbf{pm})= 5$, $N_2(\mathbf{pm})= 24$, $N_3(\mathbf{pm})= 84$.

Theorem 3. *Symmetry groups that possess isomorphic AC generate the same number of Z-groups of the M^m-type for every fixed m ($1 \leq m \leq l$), which correspond to each other with regard to structure.*

Corolary. *The derivation of all Z-groups of the M^m-type can be completely reduced to the construction of all non-isomorphic AC and the derivation of the corresponding groups of the M^m-type from these AC.*

According to Theorem 3, it is possible to identify every AC with the corresponding isomorphic algebraic term, a representative of the equivalency class which consists of all isomorphic AC. For example, it is possible to identify $AC(\mathbf{pm})= \{R, RX\}\{Y\}$ with the term $\{A, B\}\{C\}$.

1. The derivation of (P,l)-symmetry groups from P-symmetry groups using AC

Let G^P be a junior group of P-symmetry derived from G [3]. By replacing in Definition 1 the term "transformations that are equivalent with respect to symmetry" with a more general notion "transformations that are equivalent with respect to P-symmetry", the transition from G to G^P induces the transition from $AC(G)$ to $AC(GP)$, which makes possible the derivation of groups of (P,l)-symmetry of the M^m-type using the metod of AC.

The said can be illustrated by the example of derivation of groups $G_2^{l,4}$ from groups G_2^4: $\{a, b^{(4)}\}(m)$ and $\{a^{(2)}, b^{(4)}\}(m)$.

In the first case, in the transition from $G = \mathbf{pm}$ to $G^4 = \{a, b^{(4)}\}(m)$ AC remains unchanged. In the second case, in the transition from $G = \mathbf{pm}$ to $G^4 = \{a^{(2)}, b^{(4)}\}(m)$, the equivalency of symmetry transformations is disturbed and the term $\{m, ma^{(2)}\}\{b^{(4)}\}$ is transformed into a new AC: $\{m\}\{ma\}\{b\}$. In accordance with the facts already mentioned, we have

$$\{a, b^{(4)}\}(m) \quad AC : \{m, ma\}\{b\} \simeq \{A, B\}\{C\} \quad N_1 = 5 \;\; N_2 = 24 \;\; N_3 = 84$$

$$\{a^{(2)}, b^{(4)}\}(m) \quad AC : \{m\}\{ma\}\{b\} \simeq \{A\}\{B\}\{C\} \quad N_1 = 7 \;\; N_2 = 42 \;\; N_3 = 168.$$

The given numbers N_m denote the number of groups of the M^m-type of the uncomplete $(4,l)$-symmetry. In a general case, besides the numbers N_m for p-even, we can discuss also the numbers (N_{m-1}) $(1 \leq m \leq l)$, where by (N_{m-1}) is denoted the number of groups of the complete (p,l)-symmetry of the M^m-type. For p-odd, the relationsip $N_m = (N_m)$ holds, and for p-even

$$(N_m) = N_m - (2^m - 1)(N_m - 1), \quad (N_0) = 1, \quad 1 \leq m \leq l.$$

One of the most important results obtained using the mentioned method, is the derivation of the groups $G_3^{l,p}$ from the groups G_3^p ($p = 3, 4, 6$, $P \simeq C_p$) [10] and calculation of the numbers N_m and (N_{m-1}):

$$N_1 = 4840 \;\; N_2 = 40996 \;\; N_3 = 453881 \;\; N_4 = 5706960 \;\; N_5 = 59996160$$

$$(N_1) = 4134 \ (N_2) = 29731 \ (N_3) = 260114 \ (N_4) = 2048760 \ (N_5) = 1249920.$$

By the same method, the crystallographic $(p2, l)$- and (p', l)-symmetry groups are derived from the P-symmetry groups G_3^{p2} and $G_3^{p'}$ $(p = 3, 4, 6, P \simeq D_n, D_{n(2n)})$ [11,12].

The derivation of (P, l)-symmetry groups of the M^m-type from P-symmetry groups using the AC-method can be reduced to a series of successive transitions

$$G \mapsto G^P \mapsto G^{P,1} \mapsto \ldots \mapsto G^{P,l}$$

and induced transitions

$$AC(G) \mapsto AC(G^P) \mapsto AC(G^{P,1}) \mapsto \ldots \mapsto AC(G^{P,l}).$$

Every induced AC consists of the same number of generators. Since every transition $G^{P,k-1} \mapsto G^{P,k}$, $(1 \le k \le l)$, is a derivation of simple antisymmetry groups using $AC(G^{P,k-1})$, for derivation of all multiple antisymmetry groups, the catalogue of all non-isomorphic AC formed by l generators and simple antisymmetry groups derived by these AC, is completely sufficient.

3. Reduction of multiple antisymmetry simple antisymmetry

The basis of this reduction is the idea already mentioned about the transition $G \mapsto G^P$ and induced transition $AC(G) \mapsto AC(G^P)$, where $AC(G)$ and $AC(G^P)$ consist of the same number of generators. This means that every step in the derivation of multiple antisymmetry groups

$$G \mapsto G^1 \mapsto G^2 \mapsto \ldots \mapsto G^{k-1} \mapsto G^k \mapsto \ldots \mapsto G^l,$$

i.e. the transition $G^{k-1} \mapsto G^k$, $(1 \le k \le l)$, is a derivation of simple antisymmetry groups using $AC(G^{k-1})$, followed by the induced transition $AC(G^{k-1}) \mapsto AC(G^k)$, $(1 \le k \le l-1)$. All the AC of induced series consist of the same number of generators.

The said can be illustrated by the example of derivation of multiple antisymmetry groups from the plane symmetry group **pm**:

pm $\{a, b\}(m)$ $AC: \{m, ma\}\{b\} \simeq \{A, B\}\{C\}$.

For $m = 1$ five groups of simple antisymmetry of the M^1-type are obtained:

$\{A, B\}\{C\}$

$\{E, E\}\{e_1\} \mapsto \{A, B\}\{C\}$.

$\{e_1, e_1\}\{E\} \mapsto \{A, B\}\{C\}$

$\{e_1, e_1\}\{e_1\} \mapsto \{A, B\}\{C\}$

$\{E, e_1\}\{E\} \mapsto \{A\}\{B\}\{C\}$

$\{E, e_1\}\{e_1\} \mapsto \{A\}\{B\}\{C\}$.

In the first three cases AC remains unchanged, but in two other cases AC is transformed into the new $AC : \{A\}\{B\}\{C\}$. To continue the derivation of multiple antisymmetry grups of the M^m-type from the symmetry group **pm**, only the derivation of simple antisymmetry groups from $AC : \{A\}\{B\}\{C\}$ is indispensable. This AC is trivial and gives seven groups of simple antisymmetry. If $AC : \{A, B\}\{C\}$ is denoted by 3.2 and $AC : \{A\}\{B\}\{C\}$ by 3.1, then the result obtained can be denoted in a symbolic form by $3.2 \mapsto 2(3.1) + 3(3.2)$. Then we have

$N_1(\mathbf{pm}) = N_1(3.2) = 5 \quad N_1(3.1) = 7$

$N_2(\mathbf{pm}) = N_2(3.2) = 2N_1(3.1) + 3N_1(3.2) - 5 \cdot 1 =$

$= 2(N_1(3.1) - 1) + 3(N_1(3.2) - 1) = 2\underline{6} + 3\underline{4} =$

$= 2N_1(3.1) + 3N_1(3.2) - N_1(3.2) = 2N_1(3.1) + 2N_1(3.2) = 24.$

The meaning of every step in the mentioned computation is:

1) substruction of the number $N_1(3.2)$, i.e. of the five groups of uncomplete multiple antisymmetry of the $2M$-type;

2) every group of the M^1-type gives exactly one of these $2M$-type groups, so we obtain $2\underline{6} + 3\underline{4}$ groups of complete multiple antisymmetry of the M^2-type [5,8,10]. This step contains also essential data for the calculation of the number N_3: 6 groups mentioned possess AC 3.1, two of 4 groups mentioned possess AC 3.1 and two AC 3.1. Among five groups of uncomplete multiple antisymmetry of the $2M$-type there are three groups with AC 3.2 and two with AC 3.1;

3) by substitution $5 = N_1(3.2)$ we obtain $N_2(3.2)$ expressed by $N_1(3.1)$ and $N_1(3.2)$, i.e. $2N_1(3.1)+2N_1(3.2)$. The sum of coefficients corresponding to the numbers N_1 in the last line gives $N_2(\mathbf{pm}) = 24$.

$N_3(\mathbf{pm}) = N_3(3.2) = 2 \cdot 6N_1(3.1) + 3 \cdot (2N_1(3.1) + 2N_1(3.2)) - 24 \cdot 3 =$

$= 18N_1(3.1) + 6N_1(3.2) - 24 \cdot 3 = 18(N_1(3.1) - 3) + 6(N_1(3.2) - 3) =$

$= 18\underline{4} + 6\underline{2} = 18N_1(3.1) + 6N_1(3.2) - 3(2N_1(3.1) + 2N_1(3.2)) =$

$= 12N_1(3.1) = 84 \quad (N_2(3.2)) = 12.$

Consequently, the method proposed makes possible complete reduction of the theory of multiple antisymmetry to the theory of simple antisymmetry. This refers not only to the possibility of computation of the numbers N_m and (N_{m-1}), but also to the possibility of applying the method of partial cataloguation of multiple antisymmetry groups of the M^m-type [8]. If we take the advantage of the suggested reduction, the use of this method is considerably simplified and demands only the catalogues of the simple antisymmetry groups of the M^1-type obtained from non-isomorphic AC.

4. Non-isomorphic AC formed by $1 \leq l \leq 4$ generators

As it is shown in §3 the theory of multiple antisymmetry can be reduced to the theory of simple antisymmetry. For that it is necessary to know all non-isomorphic AC formed by l generators. Non-isomorphic antisymmetry characteristics formed by $1 \leq l \leq 4$ generators are investigated in [9]. As the result of their study, the catalogue of that AC formed by $1 \leq l \leq 4$ generators, and the tables of the corresponding numbers N_m, are obtained. The completness of this catalogue is proved for $l \leq 2$, but for $l \geq 3$, having in mind a great number of possible cases which we must consider, the completness is not proved, and there is a possibility that some AC are not included into the catalogue.

In this catalogue for every AC is given a list of corresponding simple antisymmetry groups of the M^1-type, connections between AC in the case of transition from $m = 1$ to $m = 2$ and tables of the numbers N_m. The notation used and the method for obtaining results are the same as in the example of the symmetry group **pm** given in §3. In AC by parenthesis ( ) is denoted the obligation of cyclic permutation of appertaining elements, by [ ] the obligation of simultaneous commutation of elements; the elements in // parenthesis remain fixed on their places. AC obtained in all previous studies of the theory of simple and multiple antisymmetry for $1 \leq l \leq 4$ are included in this catalogue. The list is the following:

$\underline{l = 1}$

1.1 $\{A\}$.

$\underline{l = 2}$

2.1 $\{A\}\{B\}$;

2.2 $\{A, B\}$;

2.3 $\{A, B, AB\}$.

$\underline{l = 3}$

3.1 $\{A\}\{B\}\{C\}$;

3.2 $\{A, B\}\{C\}$;

3.3 $(A, B, C, AB, AC, BC, ABC)$;

3.4 $\{A, B\}\{C, ABC\}$;

3.5 (A, B, C);

3.6 (A, B, C, ABC);

3.7 $\{A, B, C\}$;

3.8 $\{A, B\}, \{C, ABC\}\}$;

3.9 $\{A, B, C, ABC\}$;

3.10 $\{A, B, C, AB, AC, BC, ABC\}$.

$\underline{l = 4}$

4.1 $\{A\}\{B\}\{C\}\{D\}$;

4.2 $\{A, B\}\{C\}\{D\}$;

4.3 $([A, B], [C, ABC], [D, ABD], [AC, BC], [AD, BD], [CD, ABCD], [ACD, BCD])$;

4.4 $\{A, B\}\{C, D\}\{AC, BD\}$;

4.5 $\{A\}\{B, C\}\{D, BCD\}$;

4.6 $\{A, B\}\{C, D\}$;

4.7 $\{B, AB\}\{C, AC\}\{D, AD\}$;

4.8 $\{A\}(B,C,D)$;
4.9 $(/A,B/,/C,ABC/,/D,ABD/,/ACD,BCD/)$;
4.10 $\{A,B,C\}\{D\}$;
4.11 $\{\{A,B,\{CA,CB\}\}\{D,CD\}$;
4.12 $\{[A,B],[C,D]\}$;
4.13 $\{\{B,AB\},\{C,AC\}\}\{D,AD\}$;
4.14 (A,B,C,D);
4.15 $(C,A,CA)\{(B,C,ABC),(BD,BCD,ABCD)\}$;
4.16 $\{\{A,B\},\{C,D\}\}$;
4.17 $(\{A,B\},\{C,ABC\},\{D,ABD\},\{AC,BC\},\{AD,BD\},\{CD,ABCD\},$
$\{ACD,BCD\})$;
4.18 $\{A,B,AB\}\{C,D\}$;
4.19 $\{A,B,C,ABC\}\{D\}$;
4.20 $\{\{A,B\},\{C,ABC\}\}\{\{D,ABD\},\{ACD,BCD\}\}$;
4.21 $(\{A,AD\},\{B,BD\},\{C,CD\})$;
4.22 $\{A,B,C,D\}$;
4.23 $(\{A,B\},\{C,ABC\}\}\{\{D,ABD\},\{ACD,BCD\})$;
4.24 $\{\{B,AB\}\{C,AC\},\{D,AD\}\}$;
4.25 $\{\{\{A,B\},\{C,ABC\}\},\{\{D,ABD\},\{ACD,BCD\}\}\}$;
4.26 $\{A,B,C,ABC\}\{D,ABD,ACD,BCD\}$;
4.27 $\{\{A,B\},\{C,D\},\{AC,BD\}\}$;
4.28 $\{\{A,B\},\{C,ABC\},\{D,ABD\},\{ACD,BCD\}\}$;
4.29 $\{A,B,C,D,ABC,ABD,ACD,BCD\}$;
4.30 $\{A,B,C,D,AB,AC,AD,BC,BD,CD,ABC,ABD,ACD,BCD,ABCD\}$.

Besides all AC found in practice during previous studies of the theory of simple and multiple antisymmetry for $1 \leq l \leq 4$, in this catalogue there are some AC which are not found before.

Conjecture 1. *Every abstract algebraic term formed in accordance with Definition 1 is AC of some symmetry group.*

Most of the AC given in this catalogue, which are not found in earlier practice, satisfy Conjecture 1. For example, AC 4.22 corresponds to the symmetry group **mmmm** of the category G_{40}, and AC 4.30 corresponds to the symmetry group **P1111** of the category G_4.

If Conjecture 1 is valid, AC 4.21 and 4.22 are counter-examples of the supposition [1, pp. 138] that equality of the first and last members of the series $N_m(G)$ and $N_m(G')$ implies equality of the second members of these series.

Conjecture 2. *Every series N_m obtained from AC_l formed by l generators is identical with some series (N_{m+1}) obtained from corresponding AC_{l+1} formed by $l+1$ generators.*

As the examples of AC_{l+1} and AC_l which satisfy the Conjecture 2 for $1 \leq l \leq 4$, it is possible to notice the pairs of AC: 2.2 and 1.1, 3.4 and 2.1, 3.8 and 2.2, 3.9 and 2.3, 4.7 and 3.1, 4.13 and 3.2, 4.17 and 3.3, 4.20 and 3.4, 4.21 and 3.5, 4.23 and 3.6, 4.24 and 3.7, 4.25 and 3.8, 4.28 and 3.9, 4.19 and 3.10.

Conjecture 3. *Let AC_l formed by generators $A_1, \dots, A_l$ be given. Then by the substitution $A'_i = A_i A_{l+1}$, $i = \overline{1,l}$, can be obtained a new AC_{l+1}, such that AC_l and AC_{l+1} satisfy Conjecture 2.*

The study of particular non-isomorphic AC for $l > 4$ is almost a technical problem. However, a proof of completness of the catalogue of non-isomorphic AC for $l > 2$ is immensely important and one of the aims of future studies of the theory of simple and multiple antisymmetry must be the construction of an algorythm, which makes possible direct derivation of all non-isomorphic AC formed by l generators.

In many cases, especially for AC with a large number of generators, for the computing of numbers N_m it is possible to use the direct product of AC.

5. Direct product of AC

Definition 2. Let AC' and AC'' with disjoint sets of generators be given. The new $AC = AC'AC''$ obtained by adding in writing AC'' to AC' is called the direct product of AC' and AC''.

Theorem 3. *Let N_m, N'_m, N''_m be the series of numbers defined by AC, AC', AC'' respectively. Then the relationship*

$$N_m = \sum_{\substack{k+l \geq m, \\ m \geq k, l \geq 0}} 2^{(m-k)(m-l)} C(m, m-k, m-l) N'_k N''_m$$

holds, where

$$C(l,k,m) = \frac{(2^l - 1)(2^{l-1} - 1)\dots(2^{l-k-m+1} - 1)}{(2^k - 1)(2^{k-1} - 1)\dots(2-1)(2^m - 1)(2^{m-1} - 1)\dots(2-1)}.$$

As an illustration of the AC which satisfy Theorem 3, we are giving the following example

$AC' = 2.2 = \{A, B\}$ $N_1(2.2) = 2$ $N_2(2.2) = 3$
$AC'' = 2.1 = \{C\}\{D\}$ $N_1(2.1) = 3$ $N_2(2.1) = 6$
$AC = \{A, B\}\{C\}\{D\} = 4.2.$

In accordance with Theorem 3,

$N_1(4.2) = 2{\cdot}3+2+3 = 11 N_2(4.2) = 3{\cdot}6+3+6+3{\cdot}2{\cdot}6+3{\cdot}3{\cdot}3+6{\cdot}2{\cdot}3 = 126$
$N_3(4.2) = 28 \cdot 2 \cdot 6 + 28 \cdot 3 \cdot 3 + 42 \cdot 3 \cdot 6 = 1344$
$N_4(4.2) = 560 \cdot 3 \cdot 6 = 10080.$

Other examples of $AC = AC'AC''$ from the catalogue of non-isomorphic AC for $1 \leq l \leq 4$ are $2.1 = (1.1)(1.1)$, $3.1 = (2.1)(1.1)$, $3.2 = (2.2)(1.1)$, $4.1 = (3.1)(1.1) = (2.1)(2.1)$, $4.2 = (3.2)(1.1)$, $4.6 = (2.2)(2.2)$, $4.8 = (3.5)(1.1)$, $4.10 = (3.7)(1.1)$, $4.18 = (2.3)(2.2)$, $4.19 = (3.9)(1.1)$.

6. Tables of numbers N_m

As the result we have the table survey of the numbers N_m for all nonisomorphic AC formed by $1 \leq l \leq 4$ generators:

$l = 1$

	N_1
1.1	1

$l = 2$

	N_1	N_2
2.1	3	6
2.2	2	3
2.3	1	1

$l = 3$

	N_1	N_2	N_3
3.1	7	42	168
3.2	5	24	84
3.3	4	24	96
3.4	4	15	42
3.5	3	14	56
3.6	3	12	42
3.7	3	10	28
3.8	3	9	21
3.9	2	4	7
3.10	1	1	1

$l = 4$

	N_1	N_2	N_3	N_4
4.1	15	210	2520	20160
4.2	11	126	1344	10080
4.3	9	120	1440	11520
4.4	9	108	1260	10080
4.5	9	84	756	5040
4.6	8	75	714	5040
4.7	8	63	462	2520
4.8	7	74	840	6720
4.9	7	66	672	5040
4.10	7	58	504	3360
4.11	7	54	420	2520
4.12	6	57	630	5040
4.13	6	39	252	1260
4.14	5	54	630	5040
4.15	5	44	448	3360

4.16	5	39	357	2520
4.17	5	36	264	1440
4.18	5	34	266	1680
4.19	5	28	168	840
4.20	5	27	147	630
4.21	4	23	154	840
4.22	4	22	147	840
4.23	4	21	126	630
4.24	4	19	98	420
4.25	4	18	84	315
4.26	4	16	63	210
4.27	3	21	210	1680
4.28	3	10	35	105
4.29	2	4	8	15
4.30	1	1	1	1

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