FILOMAT (Niš) 9:3 (1995), 675-687

Algebra, Logic & Discrete Mathematics
Niš, April 14–16, 1995.

SEMIGROUPS OF INTEGRAL FUNCTIONS IN VALUED FIELDS

Ghiocel Groza

ABSTRACT. Let K be a valued field and IK[[X]] the commutative algebra of integral functions over K. This paper is devoted to study some semigroups S of $(IK[[X]], \circ)$, where $f \circ g$ is the composite function of $f, g \in IK[[X]]$. In the first section we define a topology $Inv_K S$ on K and we extend to integral functions some notions used for polynomials (see [5] and [6]). Here we study some connections between the subsemigroups $(S, \circ)$ of $(IK[[X]], \circ)$ and the topologies $Inv_K S$ on K. In the second section we study when a particular subset of K is an open set in the topology defined on K by some semigroup of integral functions.

1. Semigroups and topologies

Let K be a field admitting a rank 1 nontrivial valuation $| \, |$ (see [2] or [3]), this is a mapping from K into $\in R$ such that for all $x, y \in K$

i) $| x | \geq 0$ and $| x | = 0$ iff $x = 0$;

ii) $| xy | = | x || y |$;

iii) $| x + y | \leq | x | + | y |$;

iv) there exists an element $z \in K \setminus \{0\}$ such that $| z | \neq 1$.

For $x, y \in K$, define $d(x, y) = | x - y |$. Thus (K, d) is a metric space and we can, therefore, introduce the customary topological concepts into such a space in terms of the metric.

A formal power series

$$f(X) = \sum_{k=0}^{\infty} a_k X^k \in K[[X]] \tag{1}$$

is called an integral function over K if for every $x \in K$ the sequence

$$S_n(X) = \sum_{k=0}^{n} a_k X^k \tag{2}$$

is a Cauchy sequence. We denote by $IK[[X]]$ the commutative algebra of integral functions over K. If $f, g \in IK[[X]]$ we consider $f \circ g \in I\hat{K}[[X]]$ the composite function of f and g, where $\hat{K}$ is a completion of K. We consider $(S, \circ)$ a semigroup of integral functions over K and we denote by

$$Inv_K S = \{D \subset K;\ f(D) \subset D, \forall f \in S\}.$$

Obviously, if K is a complete field, then $(IK[[X]], \circ)$ is a semigroup and for every subsemigroup $S \subset IK[[X\}]$, $K \in Inv_K S$.

Proposition 1. *Let K be a valued field and let $(S, \circ)$ be a semigroup of integral functions over K. If $K \in Inv_K S$, then $Inv_K S$ defines a topology on K such that K is a locally quasi-compact and locally connected topological space. Furthermore for every $a \in K$ there exists $D_a \in Inv_K S$ such that D_a is the smallest open set from $Inv_K S$ which contains a.*

Proof. Suppose that $\{D_i\}, i \in I$ is a family of sets from $Inv_K S$. It is easily to see that

$$\bigcup_{i \in I} D_i \in Inv_K S \text{ and } \bigcap_{i \in I} D_i \in Inv_K S.$$

Thus $Inv_K S$ is a topology on K. If $a \in K$ we consider

$$D_a = \bigcup_{f \in S} \{f(a)\} \bigcup \{a\}.$$

Then $D_a \in Inv_K S$ and D_a is the smallest open set from $Inv_K S$ which contains a. Since D_a is a quasi-compact and connected subspace of K (see [4]) it follows that $(K, Inv_K S)$ is a locally quasi-compact and locally connected topological space. □

Remark 1. If

$$(S_1, \circ), (S_2, \circ)$$

are two semigroups of integral functions over K, then $Inv_K S_1$ is not necessarily different from $Inv_K S_2$. For example we consider

$$K = \in C,\ S_1 = I \in C[[X]] \text{ and } S_2 = \in C[X].$$

Then

$$Inv_K S_1 = Inv_K S_2$$

is the coarsest topology on $\in C$. However for cyclic semigroups we have:

Proposition 2. *Let K be a valued field of characteristic zero and let $(S_1, \circ)$, $(S_2, \circ)$ be two cyclic semigroups of integral functions over K. If $S_1 \neq S_2$, then*

$$Inv_K S_1 \neq Inv_K S_2.$$

Proof. Let f_i be a generator of $S_i, i = 1, 2$. Since the set of zeros from K of an integral function over K is countable (see [1], p. 144 for a non-archimedean valuation), we consider the countable set M of zeros of the integral functions

$$f_1^j(X) - f_2^k(X), j, k \in \in N, j^2 + k^2 \neq 0.$$

There exists then $a \in K \setminus M$ and we denote

$$D_a^i = \bigcup_{k \in \in N} \{f_i^k(a)\} \bigcup \{a\}, i = 1, 2.$$

Hence it follows that

$$D_a^1 \neq D_a^2 \text{and} Inv_K S_1 \neq Inv_K S_2. \quad \Box$$

We now raise the question as to when the topological space $(K, Inv_K S)$ is separable. Since $h(a) \in D_a$, for every $h \in S$, from Proposition 1 it follows immediately:

Proposition 3. *Let K be a valued field and let $(S, \circ)$ be a semigroup of integral functions over K. If $K \in Inv_K S$, the following conditions are equivalent:*

a) *$(K, Inv_K S)$ is a Hausdorff space.*
b) *$S = \{X\}$.*
c) *$Inv_K S$ is the discrete topology on K.*

We recall that the assertion that for every two distinct points at least one of them has a neighbourhood that does not contain the other is called axiom T_0.

Proposition 4. *Let be K a valued field and let $(S, \circ)$ be a semigroup of integral functions over K. If $K \in Inv_K S$, then $(K, Inv_K S)$ is a T_0- (Kolmogoroff) space if and only if for every $a \in K$, either a is a fixed point of S, that is $h(a) = a$, for all $h \in S$, or, if there exists $h_1 \in S$ such that $h_1(a) \neq a$, then $h_2 h_1(a) \neq a$, for all $h_2 \in S$.*

Proof. If $(K, Inv_K S)$ is a T_0-space, then we consider $a \in K$ such that there exists $h_1 \in S$ for which $h_1(a) \neq a$. Suppose there exists $h_2 \in S$, such that

$h_2h_1(a) = a$. By Proposition 1, either $h_1(a) \notin D_a$, or $a \notin D_{h_1(a)}$, which is absurd since $h_1(a) \in D_a$ and $a = h_2h_1(a) \in D_{h_1(a)}$.

Conversly, let $a, a' \in K, a \neq a'$. If a, for example, is a fixed point of S, then $a' \notin D_a = \{a\}$, otherwise suppose $a' \in D_a$ and $a \in D_{a'}$. Hence there exist $h_1, h_2 \in S$ such that $a' = h_1(a)$ and $a = h_2(a')$. Since $h_1(a) \neq a$ it follows that $h_2h_1(a) \neq a$, which is absurd since $a = h_2(a') = h_2h_1(a)$. This shows that $(K, Inv_K S)$ is a Kolmogoroff space. □

Corollary. *Let K be a valued field, $f(X) \in IK[[X]]$ and let $S = (f)$ a cyclic semigroup of integral functions over K. If $K \in Inv_K S$, then $(K, Inv_K S)$ is a Kolmogoroff space if and only if for all $a \in K$ either a is a fixed point of $f(X)$ or for all $k \in \in N, k \geq 2$, there exists an integral functions $g_k(X)$ over K such that $g_k(a) \neq 0$ and $f^k(X) = g_k(X) + X$.*

The proof follows directly from Proposition 4.

Example 1. Suppose that $K_1 = \in R, K_2 = \in C$ and $| \, |$ is the usual archimedean valuation. Let

$$f(X) = e^X + X.$$

If $S = (f)$, then by Corollary it follows that $(\in R, Inv_{\in R} S)$ is a Kolmogoroff space and $(\in C, Inv_{\in C} S)$ is not a Kolmogoroff space.

Remark 2. If $(K, Inv_K S)$ is a Kolmogoroff space, we define a partial ordering $\leq$ on K such that $a \leq a'$ if and only if a belongs to the closure of $\{a'\}$ in $Inv_K S$ (see [4], Ch. 1). Then the open intervals of $(K, \leq)$ form a basis for the topology $Inv_K S$. The assertion follows by Proposition 1 and by [4], Ch. 1.

2. Invariant sets and semigroups

In this section we study the connection between particular subsets of K and particular semigroups of integral functions. We shall use the terminology and notation introduced in Section 1. We shall need the following result from [7].

Theorem 1. *Let K be a complete valued field, $\{x_n\}_{n \geq 1}$ an infinite sequence of distinct elements in K such that*

$$\lim_{n \to \infty} | x_n | = \infty \tag{3}$$

and $\{y_n\}_{n \geq 1}$ an arbitrary infinite sequence of elements in K. Then there exists a function $f(X) \in IK[[X]]$ such that

$$f(x_j) = y_j, \ \forall j \geq 1. \tag{4}$$

Theorem 2. *Let K be a complete valued field and let $M = \{x_n\}_{n\geq 1}$ be a countable subset of K which satisfies (3). Then there exists an infinite cyclic semigroup S of integral functions over K such that $M \in Inv_K S$.*

The proof follows immediately from Theorem 1.

We shall now study some particular cases when K is not necessarily a complete field. We begin with a lemma on a determinant which is a generalization of the Vandermonde determinant.

Lemma 1. *Let K be a field of characteristic zero, $m, n, k \in \in N$ and $m \geq k$. We consider the polynomial $D_{m,n,k}(X_0, X_1, \ldots, X_n) \in K[X_0, X_1, \ldots, X_n]$ defined by the determinant which has the order $(k+1)(n+1)$, its j-th row, $j = 1, \ldots, n+1$ has the form*

$$(X_{j-1}^{m}, X_{j-1}^{m+1}, \ldots, X_{j-1}^{m-1+(k+1)(n+1)})$$

and the following rows are their derivatives up to order k inclusive. Then there exists $C \in K \setminus \{0\}$ such that

$$D_{m,n,k}(X_0, X_1, \ldots, X_n) = C \prod_{i=0}^{n} X_i^{m(k+1)} \prod_{0\leq i<j\leq n} (X_j - X_i)^{(k+1)^2} \tag{5}$$

Proof. By induction on k, using Laplace's theorem, it is easily verified that the (total) degree of $D_{m,n,k}$ is

$$\deg D_{m,n,k}(X_0, X_1, \ldots, X_n) = \frac{(n+1)(k+1)}{2}(2m + n + kn) \tag{6}$$

We shall denote $D_{m,n,k}$ by D, for simplicity. Let $(X_1 - X_0)^q$ be the highest power of $X_1 - X_0$ which divides D in $K[X_0, X_1, \ldots, X_n]$. Then

$$\frac{\partial^j D}{\partial X_0^j}(X_1, X_1, X_2, \ldots, X_n) \equiv 0, \; j = 0, 1, 2, \ldots, q-1 \tag{7}$$

and

$$\frac{\partial^q D}{\partial X_0^q}(X_1, X_1, X_2, \ldots, X_n) \tag{8}$$

is not identically equal to zero. Since the derivative of a determinant Δ of order N is the sum of N determinants Δ_s in which all rows (except the s-th) are the same as in Δ and the s-th row in Δ_s is the derivative of s-th row in Δ, it follows that

$$\frac{\partial D}{\partial X_0}(X_0, X_1, \dots, X_n)$$

is a sum of such determinants in which all rows (excepts the i-th rows, $i = 1, n+2, 2n+3, \dots, kn+k+1$) are the same as in D and i-th rows are the i-th rows in D or a derivative of the i-th rows in D.

On the other hand, by using suitable derivative of D, it follows that D is not identically equal to zero. To obtain (7) it is enough to prove that

$$\frac{\partial^j D}{\partial X_0^j}(X_0, X_1, \dots, X_n) \tag{9}$$

is a sum of such determinants in which there exists a row which is equal to the first row of D or is equal to a derivative up to order k inclusive of the first row of D. If q_1 is the smallest value of j such (9) has not this property, it follows that

$$q \geq q_1 \geq (k+1)^2 \tag{10}$$

Since D is a homogeneous polynomial and it remains unchanged, to within sign, under any transposition of two unknows, it follows that the degree of the product of all the factors $(X_j - X_i)^q$, $j > i$, where $(X_j - X_i)^q$ is the highest power of $X_j - X_i$ which divides D is equal to

$$C_{n+1}^2 q \geq \frac{n(n+1)}{2}(k+1)^2 \tag{11}$$

Similarly, if we denote by X_i^p the highest power of X_i which divides D, it follows that the degree of the product of all the factors

$$X_i^p,\ i = 0, 1, \dots, n, \ \text{ is equal to } \ (n+1)p \geq (n+1)(k+1)m \tag{12}$$

Since

$$\frac{n(n+1)}{2}(k+1)^2 + (n+1)(k+1)m = \frac{(n+1)(k+1)}{2}(2m+n+kn) = \deg D,$$

by (11) and (12) it follows that $q = (k+1)^2$ and $p = m(k+1)$, which gives the assertion. □

Theorem 3. *Let K be a field of characteristic zero and $|\,|$ a rank 1 nontrivial valuation of K. We denote by $\hat{K}$ a completion of K for its topology defined by $|\,|$. We consider K_1 a countable subset of $\hat{K} \setminus \{0\}$ and K_2 a dense subset of $\hat{K}$. If $\{L_n\}_{n \in \in N}$ is a family of dense subsets of $\hat{K}$, then there exists a function*

$$f(X) = \sum_{n=0}^{\infty} a_n X^n \in I\hat{K}[[X]] \setminus \hat{K}[[X]] \tag{13}$$

such that

a) $a_n \in L_n$ *for all* $n \in \in N$;
b) $f^{(k)}(x) \in K_2$ *for all* $x \in K_1$ *and* $k \in \in N$.

Proof. Let $\{x_i\}_{i \in \in N}$ be the elements of K_1 and we denote by

$$S_n(X) = \sum_{k=0}^{n} a_k X^k \tag{14}$$

We consider the sequences $u_n = 1 \cdot 2 + 2 \cdot 3 + \ldots + n(n+1) - 1$,

$$v_n = u_n^{-u_n}.$$

Because K_2 is a dense subset of $\hat{K}$ and the polynomials are continuous functions we can find $y_{0,0}, y_{1,0} \in K_2$ such that the system

$$\begin{cases} b_0 + b_1 x_0 = y_{0,0} \\ b_0 + b_1 x_1 = y_{1,0} \end{cases} \quad : \tag{15}$$

has the solutions $b_0, b_1 \in \hat{K}$ with the following property

$$|\, b_i \,| < v_2,\ i = 0, 1. \tag{16}$$

With the notations of Lemma 1 we have

$$D_{2,2,1}(x_0, x_1, x_2) \neq 0.$$

Let F_2 be the finite set of the cofactors of the elements in $D_{2,2,1}(x_0, x_1, x_2)$. Since L_0, L_1 are dense subsets in $\hat{K}$, there exist $a_i \in L_i$, $i = 0, 1$, such that

$$\begin{cases} |\, a_i \,| < v_2, \\ |\, S_1(x_j) - y_{j,0} \,| < v_2 \\ \left| \dfrac{c}{D_{2,2,1}} (S_1(x_j) - y_{j,0}) \right| < \dfrac{1}{2 \cdot 3} v_3,\ \forall c \in F_2,\ j = 0, 1. \end{cases} \tag{17}$$

Because K_2 is a dense subset of $\hat{K}$ there exist the elements $y_{2,0}, y_{0,1}, y_{1,1}, y_{2,1} \in K_2$ such that

$$(18)\qquad \begin{cases} \left|\dfrac{c}{D_{2,2,1}}(S_1(x_2) - y_{2,0})\right| < \dfrac{1}{2\cdot 3}v_3, \\ \left|\dfrac{c}{D_{2,2,1}}(a_1 - y_{j,1})\right| < \dfrac{1}{2\cdot 3}v_3, \ \forall c \in F_2, \ j = 0,1,2. \end{cases}$$

Applying Cramer's rule it follows that the system

$$(19)\qquad \begin{cases} a_0 + a_1x_0 + b_2x_0^2 + b_3x_0^3 + b_4x_0^4 + b_5x_0^5 + b_6x_0^6 + b_7x_0^7 = y_{0,0} \\ a_0 + a_1x_1 + b_2x_1^2 + b_3x_1^3 + b_4x_1^4 + b_5x_1^5 + b_6x_1^6 + b_7x_1^7 = y_{1,0} \\ a_0 + a_1x_2 + b_2x_2^2 + b_3x_2^3 + b_4x_2^4 + b_5x_2^5 + b_6x_2^6 + b_7x_2^7 = y_{2,0} \\ a_1 + 2b_2x_0 + 3b_3x_0^2 + 4b_4x_0^3 + 5b_5x_0^4 + 6b_6x_0^5 + 7b_7x_0^6 = y_{0,1} \\ a_1 + 2b_2x_1 + 3b_3x_1^2 + 4b_4x_1^3 + 5b_5x_1^4 + 6b_6x_1^5 + 7b_7x_1^6 = y_{1,1} \\ a_1 + 2b_2x_2 + 3b_3x_2^2 + 4b_4x_2^3 + 5b_5x_2^4 + 6b_6x_2^5 + 7b_7x_2^6 = y_{2,1} \end{cases}$$

in the unknowns b_i, has solutions with the following property

$$(20)\qquad | b_i |< v_3, \ i = u_1 + 1, \dots , u_2.$$

We now consider

$$D_{8,3,2}(x_0, x_1, x_2, x_3) \neq 0$$

and we denote by F_3 the set of the cofactors of the elements in $D_{8,3,2}$. Since L_i, $i = u_1 + 1, \dots , u_2$, are dense subsets in $\hat{K}$, by (19) and (20) it follows that there exist $a_i \in L_i$, $i = u_1 + 1, \dots , u_2$, such that

$$(21)\qquad \begin{cases} | a_i |< v_3, \ i = u_1 + 1, \dots , u_2, \\ | S_{u_2}(x_j) - y_{j,0} |< v_3, \\ | S'_{u_2}(x_j) - y_{j,1} |< v_3, \\ \left|\dfrac{c}{D_{8,3,2}}(S_{u_2}(x_j) - y_{j,0})\right| < \dfrac{1}{3\cdot 4}v_4, \\ \left|\dfrac{c}{D_{8,3,2}}(S'_{u_2}(x_j) - y_{j,1})\right| < \dfrac{1}{3\cdot 4}v_4, \ \forall c \in F_3, \ j = 0,1,2. \end{cases}$$

Now by induction on r, we consider

$$D_{u_r+1,r+1,r}(x_0, x_1, \dots , x_{r+1}) \neq 0$$

and we denote by F_{r+1} the set of the cofactors of its elements.

We suppose that we have found $y_{j,k} \in K_2$, $j = 0, 1, \dots, r, k = 0, 1, \dots, r-1$ and $a_i \in L_i$, $i = 0, 1, \dots, u_r$, such that

$$(22) \qquad | a_i | < v_{t+1},\ i = u_{t-1} + 1, \dots, u_t,\ \forall t = 1, \dots, r$$

$$(23) \qquad | S_{u_r}^{(k)}(x_j) - y_{j,k} | < v_{r+1},\ \forall j = 0, \dots, r,\ k = 0, \dots, r-1$$

$$(24) \qquad \left| \frac{c}{D_{u_r+1,r+1,r}} (S_{u_r}^{(k)}(x_j) - y_{j,k}) \right| < \frac{1}{(r+1)(r+2)} v_{r+2},$$
$$\forall c \in F_{r+1}, j = 0, .., r,\ k = 0, \dots, r-1.$$

Since K_2 is a dense subset in $\hat{K}$ there exist the elements $y_{r+1,0}, y_{r+1,1}, \dots, y_{r+1,r}, y_{0,r}, \dots, y_{r+1,r} \in K_2$ such that the condition (24) hold true for all $j = 0, 1, \dots, r+1$ and $k = 0, 1, \dots, r$. Then the system

$$(25) \qquad S_{u_r}^{(k)}(x_j) + (b_{u_r+1} X^{u_r+1} + \dots + b_{u_{r+1}} X^{u_{r+1}})^{(k)}_{X=x_j} = y_{j,k},$$
$$0 \leq j \leq r+1,\ 0 \leq k \leq r$$

in the unknowns b_i, which for $r = 1$ coincides with the system (19), has the solutions b_i with the following property

$$(26) \qquad | b_i | < v_{r+2},\ i = u_r + 1, \dots, u_{r+1}$$

Since $L_i, i = u_r + 1, \dots, u_{r+1}$ are dense subsets in $\hat{K}$, by (25) and (26) it follows that there exist $a_i \in L_i$ such that the conditions (22) - (24) are satisfied for $r+1$. This proves (22) - (24) for every r.

We consider now $n \in \in N$. Then there exists $r \in N$ such that

$$u_r < n \leq u_{r+1}$$

and by (22) it follows that

$$| a_n |^{\frac{1}{n}} < v_{r+2}^{\frac{1}{n}} < u_{r+2}^{-u_r+2/n} < \frac{1}{u_{r+2}}.$$

Hence

$$\lim_{n \to \infty} | a_n |^{\frac{1}{n}} = 0$$

and

$$f(X) = \sum_{n=0}^{\infty} a_n X^n \in I\hat{K}[[X]].$$

We remark that we can find $a_n \neq 0$. To prove b) we consider $k, j \in \in N$ and we chose $r > k$ and $r > j$. Then by (23) it follows that

$$f^{(k)}(x_j) = y_{j,k} \in K_2$$

and this establishes the theorem. □

Corollary. *Let K be a countable field of characteristic zero and $||$ a rank 1 nontrivial valuation of K. We denote by*

$$(27) \qquad S_K = \{f(X) \in IK[[X]],\ f^{(k)}(x) \in K, forall x \in K\ and k \in \in N\}.$$

Then S_K is a semigroup which contains some integral functions which are not polynomials.

The assertion follows from Theorem 3 by taking $K_2 = L_n = K$ for all $n \in \in N$ and $K_1 = K \setminus \{0\}$.

Let

$$S_\infty = \{S_1; \exists f(X) \in IK[[X]] \setminus K[X],\ f \in S_1\},$$

where $(S_1, \circ)$ is a subsemigroup of $(S_K, \circ)$. In the last part of this paper we shall prove that we can find an infinite subset D of $\in Q$ such that, for all $S_1 \in S_\infty$, the topology $Inv \in_Q S_1$ does not contain the set D. More precise we have the following assertion:

Theorem 4. *Suppose $K = \in Q$ and $||$ is the usual absolute value function. Let $D = \{1/n\}_{n \in \in N^*}$ and let $f(X) \in S_{\in Q}$ such that*

$$(28) \qquad f(D) \subset D$$

then $f(X)$ is a polynomial which is of the form

$$(29) \qquad f(X) = \frac{1}{r} X^s,\ r \in \in N^*,\ s \in \in N$$

Proof. If

$$(30) \qquad f(X) = \sum_{j=0}^{\infty} a_j X^j,\ a_j = \frac{\alpha_j}{\beta_j},\ \alpha_j, \beta_j \in \in Z,\ \beta_j \neq 0$$

we may assume that $\beta_j > 0$ and $\beta_j \mid \beta_{j+1}$ for all $j \in \in N$. We denote

$$(31) \qquad f(\frac{1}{n}) = \frac{1}{k_n},\ k_n \in \in N^*.$$

Since $f(X)$ is a continuous function it follows that

$$(32) \qquad \lim_{n \to \infty} f(\frac{1}{n}) = a_0 = \lim_{n \to \infty} \frac{1}{k_n}.$$

We may assume that $f(X) \notin K$ and because the zeros of an integral function which does not vanish identically are isolated, it follows that

$$\lim_{n \to \infty} \frac{1}{k_n} = 0 = a_0.$$

Let a_i be the first coefficient which is not equal to zero. Since $f(X)$ is an integral function we have

$$\lim_{m \to \infty} | a_m |^{\frac{1}{m}} = 0$$

and then there exists $m_0 \in \in N, m_0 \geq i$ such that for all $m \geq m_0$

$$(33) \qquad | f(x) - \sum_{j=1}^{m} a_j x^j | \leq x^{m+1}, \forall x \in [0,1].$$

By (31) and (33) it follows that, for all $m \geq m_0$ and $n \in \in N^*$,

$$(34) \qquad \left| \frac{1}{k_n} - (\frac{\alpha_i}{\beta_i} \cdot \frac{1}{n^i} + \ldots + \frac{\alpha_m}{\beta_m} \cdot \frac{1}{n^m}) \right| \leq \frac{1}{n^{m+1}}.$$

Hence

$$(35) \qquad \lim_{n \to \infty} \frac{n^i}{k_n} = \frac{\alpha_i}{\beta_i}$$

and for all $n \in \in N^*$ and $m \geq m_0$

$$(36) \qquad \left| \beta_m n^m - k_n(\alpha_i \beta_m \beta_i^{-1} n^{m-i} + \ldots + \alpha_m) \right| \leq \beta_m \frac{k_n}{n}.$$

Suppose that there exists a fixed

$$(37) \qquad m \geq m_0, \; m \geq 2i \, suchthatalpha_m \neq 0.$$

Then by (35) and (36) there exists $r_m \in \in Z$ such that for all $n \in \in N^*$

$$(38) \qquad \beta_m n^m - k_n(\alpha_i \beta_m \beta_i^{-1} n^{m-i} + \ldots + \alpha_m) - r_m = 0,$$

where $r_m = O(n^{i-1})$. We consider the polynomials

$$P_1(X) = \beta_m X^m,$$
$$P_2(X) = \alpha_i \beta_m \beta_i^{-1} X^{m-i} + \ldots + \alpha_m.$$

Then there exist $R_1(X), Q_1(X) \in \in Q[X]$ such that

$$P_1(X) = Q_1(X)P_2(X) + R_1(X), \tag{39}$$

where $degR_1(X) < m - i$ and $degQ_1(X) = i$. By (38) and (39) it follows that

$$k_n = Q_1(n) + \frac{R_1(n) - r_m}{P_2(n)} = 0. \tag{40}$$

Since

$$\lim_{n\to\infty} \frac{R_1(n) - r_m}{P_2(n)} = 0$$

there exists $n_0 \in \in N^*$ such that

$$\left| \frac{R_1(n) - r_m}{P_2(n)} \right| < \frac{1}{d+1}, \forall n \geq n_0, \tag{41}$$

where d is the least common multiple of the denominators of the coefficients of $Q_1(X)$. Because $k_n \in \in N$, by (40), it follows that there exists $n_1 \in \in N^*$ such that

$$k_n = Q_1(n), \forall n \geq n_1.$$

Hence

$$f(\frac{1}{n}) = \frac{1}{Q_1(n)} = \frac{n^{-i}}{Q_2(n^{-1})}, \tag{42}$$

where

$$Q_2(X) = X^i Q_1(\frac{1}{X}).$$

Since D has a limit point, by (42), it follows that

$$f(X) = \frac{X^i}{Q_2(X)}.$$

Since also $f(X)$ is an integral function we must have $Q_2(X) \in \in Q$ and $i = 0$. Then there exists $m \in \in N$ such that $\alpha_m \neq 0$ and for all $m_1 > m$, $\alpha_{m_1} = 0$. Thus $f(X)$ is a polynomial and by (34)

$$| \beta_m n^m - k_n(\alpha_i \beta_m \beta_i^{-1} n^{m-i} + \ldots + \alpha_m) | < \frac{\beta_m}{n}.$$

Hence there exists $n_2 \in \in N^*$ such that for all $n \geq n_2$

$$\beta_m n^m = k_n(\alpha_i \beta_m \beta_i^{-1} n^{m-i} + \ldots + \alpha_m). \tag{43}$$

We denote $(n, \alpha_m) = d_m$ and $n = d_n v_n$. Then $\lim v_n = \infty$ and by (43) $v_n^m \mid k_n$. Hence, if $m > i$, then

$$\lim_{n\to\infty} \frac{k_n}{n^i} = \infty,$$

which is absurd. Then $m = i$ and

$$f(X) = \frac{\alpha_i}{\beta_i} X^i.$$

Hence by (28) it follows (29). □

References

[1] Y.Amice, *Les nombres p-adiques*, Presses Universitaires de France, 1975.

[2] E. Artin, *Algebraic numbers and algebraic functions*, Gordon and Breach, 1967.

[3] G. Bachman, *Introduction to p-adic number and valuation theory*, Academic Press, New-York-London, 1964.

[4] N. Bourbaki, *Topologie générale*, Chap. 1 et 2, 3rd ed., Hermann, Paris, 1961.

[5] P. J. Cahen, *Integer-valued polynomials on a subset*, Proc. Amer. Math. Soc. **117** (1993), 919–929.

[6] J. L. Chabert, *Sur le théorème de Stone-Weierstrass en algèbre commutative*, Rend. Circ. Mat. Palermo, serie II XVII (1994), 51–70.

[7] G. Groza, *Construction of integral functions in valued fields* (to appear).

Department of Mathematics, Technical University of Civil Engineering, Lacul Tei 124, Sec. 2, R-72302, Bucharest, Romania