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VARIETIES OF POLYADIC GROUPS

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ABSTRACT. In this note the class of all n-ary groups is considered as the class of some universal algebras with different systems of fundamental operations. In any such case we give the minimal systems of identities defining this class.

1. Introduction

Wilhelm Dörnte, inspired by E. Noether, introduced in 1928 (see [1]) the notion of n-group (called also n-ary group or polyadic group), which is a natural generalization of the notion of group. The idea of such investigations seems to be going back to E. Kasner's lecture at the fifty-third annual meeting of the American Association for the Advancement of Science, reported (by L. G. Weld) in The Bulletin of the American Mathematical Society in 1904 (see [2]). The second paper which plays a very important role in the theory of n-ary groups is the large paper (143 pages) of E. L. Post [3].

We shall use the following abreviated notation: the sequence $x_i, x_{i+1}, ..., x_j$ will be denoted by x_i^j. For $j < i$ x_i^j is the empty symbol. In this convention $f(x_1^n)$ denotes $f(x_1, x_2, ..., x_n)$. The word

$$f(x_1, x_2, ..., x_k, x, ..., x, x_{k+t+1}, ..., x_n),$$

where x appears t times, will be denoted by $f(x_1^k, \overset{(t)}{x}, x_{k+t+1}^n)$. For $t \leq 0$ the symbol $\overset{(t)}{x}$ will be empty.

If $m = k(n-1)+1$, then the m-ary operation g given by

$$g(x_1^{k(n-1)+1}) = \underbrace{f(f(..., f(f}_{k}(x_1^n), x_{n+1}^{2n-1}), ...), x_{(k-1)(n-1)+2}^{k(n-1)+1})$$

will be denoted by $f_{(k)}$. In certain situations, when the arity of g does not play a crucial role, or when it will differ depending on additional assumptions, we write $f_{(.)}$, to mean $f_{(k)}$ for some $k = 1, 2, ...$

A non-empty set G with an n-ary operation $f : G^n \longrightarrow G$ will be called *an n-groupoid* or *an n-ary groupoid* and will be denoted by $(G; f)$. An n-groupoid $(G; f)$ will be called *an n-group* or *an n-ary group* if and only if

1^0 for all $x_1, x_2, ..., x_{2n-1} \in G$ *the* (i, j)*-associative law*

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}) \tag{1}$$

hold for every $i, j \in \{1, 2, ..., n\}$,

2^0 for all $x_0, x_1, ..., x_{k-1}, x_{k+1}, ..., x_n \in G$ $(k = 1, 2, ..., n)$ there exist a unique $z \in G$ such that

$$f(x_1^{k-1}, z, x_{k+1}^n) = x_0. \tag{2}$$

Condition 1^0 is called *associativity*, and algebras (G, f) fulfilling 1^0 are called *n-semigroups*. Algebras fulfilling only 2^0 are called *n-quasigroups*.

The above definition is a generalization of H. Weber's formulation of axioms of a group (from 1896). Similar generalization of L. E. Dickson's (with the neutral element) one leads to some narrower class of n-groups derived from 2-groups (i.e. classical groups).

It is interesting that there exists no nontrivial (on a non one-element set) theory of infinitary groups, i.e. ω-groups for countable infinite ordinal ω, but there exist infinitary quasigroups of any (finite and infinite) order [19]. Therefore we shall consider n-ary groups (n-ary groupoids) only in the case when $n \geq 2$ is a fixed (but arbitrary) natural number.

It is worthwhile to note that, under the assumption 1^0, it suffices only to postulate the existence of a solution of (2) at the places $k = 1$ and $k = n$ or at one place k other than 1 and n. Then one can prove uniqueness of the solution of (2) for all $k = 1, ..., n$ (see [3], p. 213[17]). Also the following Proposition is true (see [4]).

Proposition 1.1. *An n-groupoid* $(G; f)$ *is an n-group if and only if (at least) one of the following conditions is satisfied:*

(a) *the* $(1, 2)$*-associative law holds and the equation* (2) *is solvable for* $k = n$ *and uniquely solvable for* $k = 1$,

(b) *the* $(n-1, n)$*-associative law holds and the equation* (2) *is solvable for* $k = 1$ *and uniquely solvable for* $k = n$,

(c) *the* $(i, i+1)$*-associative law holds for some* $i \in \{2, ..., n-2\}$ *and the equation* (2) *is uniquely solvable for* i *and some* $k > i$.

2. Varieties of n-ary groups

In an n-quasigroup $(G; f)$ for every $s \in \{1,2,...,n\}$ one can define *the s-th inverse n-ary operation* $f^{(s)}$ putting

$$f^{(s)}(x_1^n) = y \quad \textit{if and only if} \quad f(x_1^{s-1}, y, x_{s+1}^n) = x_s .$$

Obviously, the operation $f^{(s)}$ is the s-th inverse operation for f if and only if

$$(3) \qquad f^{(s)}(x_1^{s-1}, f(x_1^n), x_{s+1}^n) = x_s ,$$

for all $x_1, ..., x_n \in G$. Therefore (as in the binary case) the class of all n-quasigroups (and in the consequence the class of all n-groups) may be treated as the variety of equationally definable algebras with $n+1$ fundamental n-ary operations $f, f^{(1)}, f^{(2)}, ..., f^{(n)}$. Such variety is defined by (1) and (3). Obviously (1) and (3) must hold for all $i, j, s \in \{1, 2, ..., n\}$.

An n-group ($n > 2$) may be considered also as an algebra with three n-ary operations. Namely, as a consequence of Proposition 1.1 we obtain the following characterization.

Corollary 2.1. *Every n-ary group ($n > 2$) may be considered as an algebra $(G; f, f^{(j)}, f^{(k)})$ of the type (n, n, n) with the $(i, i+1)$-associative operation f where*

(a) $i = j = 1$ *and* $k = n$, *or*
(b) $i = n-1$, $j = 1$ *and* $k = n$, *or*
(c) $i \in \{2, ..., n-2\}$ *is fixed and* $k > j = i$.

Corollary 2.2. *The class of algebras with three n-ary ($n > 2$) operations f, g, h is the variety of all n-ary groups $(G; f)$ if and only if (at least) one of the following axiom systems is satisfied:*

$$(a) \quad \begin{cases} f(f(x_1^n), x_{n+1}^{2n-1}) = f(x_1, f(x_2^{n+1}), x_{n+2}^{2n-1}), \\ g(f(y, x_2^n), x_2^n) = y, \\ h(x_1^{n-1}, f(x_1^{n-1}, y)) = y, \end{cases}$$

$$(b) \quad \begin{cases} f(x_1^{n-2}, f(x_{n-1}^{2n-2}), x_{2n-1}) = f(x_1^{n-1}, f(x_n^{2n-1})), \\ g(f(y, x_2^n), x_2^n) = y, \\ h(x_1^{n-1}, f(x_1^{n-1}, y)) = y, \end{cases}$$

$$(c) \quad \begin{cases} f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^i, f(x_{i+1}^{n+i}), x_{n+i+1}^{2n-1}), \\ g(x_1^{i-1}, f(x_1^{i-1}, y, x_{i+1}^n), x_{i+1}^n) = y, \\ h(x_1^{s-1}, f(x_1^{s-1}, y, x_{s+1}^n), x_{s+1}^n) = y, \quad \textit{where } 1 < i < s < n \textit{ are fixed.} \end{cases}$$

Note that axiom systems given by (a) and (b) (also in Corollary 2.1) are valid for $n = 2$, too. But the greater part of characterizations of n-ary groups obtained by several authors are valid only for $n > 2$. Characterizations which are valid also for $n = 2$ are given for example in [7], [9] and [8]. Since in all these characterizations f is an associative operation, then founded systems of defining identities are not minimal.

We give such minimal system basing on result obtained in [8].

Corollary 2.3. *The class of all n-ary groups* $(n \geq 2)$ *may be considered as the variety of algebras with one* $(1,2)$*-associative (or* $(n-1,n)$*-associative) n-ary operation* f *and one* $(n-1)$*-ary operation* h *satisfying the following two axioms:*

(a) $f(h(x_1^{n-2}, z), x_1^{n-2}, f(z, x_1^{n-2}, y)) = y\,,$

(b) $f(f(y, x_1^{n-2}, z), x_1^{n-2}, h(x_1^{n-2}, z)) = y\,.$

Proof. If an algebra $(G; f, h)$ satisfies the above conditions, then as in [8] one can prove that (2) has a unique solution at the place $k = 1$ and $k = n$, which together with our Proposition 1.1 proves that $(G; f)$ is an n-group.

Conversely, if $(G; f)$ is an n-group then for every $x_1, ..., x_{n-2} \in G$ there exists a unique element $v \in G$ such that

$$y = f(y, x_1^{n-2}, v) = f(x_1^{n-2}, v, y) = f(y, v, x_1^{n-2}) = f(v, x_1^{n-2}, y)$$

for each $y \in G$ (see [3], 214-215). Hence for every $z, x_1, ..., x_{n-2} \in G$ there exists only one $u \in G$ such that

$$f(u, x_1^{n-3}, f(x_{n-2}, z, x_1^{n-2}), y) = y$$

holds for each $y \in G$. Since u depends on $n-1$ elements $z, x_1, ..., x_{n-2}$, it may be treated as the value of an $(n-1)$-ary operation h. Obviously h satisfies (a) and (b). This completes the proof.

As it is well known in an n-group $(G; f)$ the equation

(4) $$f(\overset{(n-1)}{x}, z) = x$$

has a unique solution $z \in G$, which is called *the skew element* to x and is denoted by $\overline{x}$. Since for every $x \in G$ there exists only one skew element, then the solution of (4) induces on G a new unary operation $x \to \overline{x}$. Thus an n-group $(G; f)$ may be considered as an algebra $(G, f; ^{-})$ with two fundamental opertations: an n-ary one f and an a unary one $x \to \overline{x}$. The variety of such n-groups is defined (see [6]) by three identities: one of the type (1) and two so-called *Dörnte's identities*

(5) $$f(\overset{(i-1)}{x}, \overline{x}, \overset{(n-i-1)}{x}, y) = y\,,$$

(6) $$f(y,\overset{(n-j-1)}{x},\overline{x},\overset{(j-1)}{x}) = y\,.$$

In an n-group the last two identities hold for all $i,j \in \{1,2,...,n-1\}$, but one can prove (see for example [6],[4]) that (5) and (6) determine (together with (1)) an n-group if it hold for some fixed i,j. The minimal base of such variety is given by the following theorem (proved in [7]).

Theorem 2.4. *Let $(G;f,{}^{-})$ be an n-ary groupoid $(n>2)$ with a unary operation $x \rightarrow \overline{x}$. Then $(G;f,{}^{-})$ is an n-group if and only if f is $(1,2)$ or $(n-1,n)$-associative and Dörnte's identities hold for some fixed $i,j \in \{1,2,...,n-1\}$.*

As a consequence we obtain

Corollary 2.5. *The class of all n-ary groups $(n \geq 2)$ may be considered as the variety of algebras with one $(1,2)$-associative (or $(n-1,n)$-associative) n-ary operation f and one unary operation $x \rightarrow \widehat{x}$ satisfying the following two axioms:*

(a) $f(\widehat{x},\overset{(n-2)}{x},f(\overset{(n-1)}{x},y)) = y\,,$

(b) $f(f(y,\overset{(n-1)}{x}),\overset{(n-2)}{x},\widehat{x}) = y\,.$

Theorem 2.6. *The class of algebras $(G;f,g,h)$ with one $(1,2)$-associative (or $(n-1,n)$-associative) n-ary $(n>2)$ operation f and two $(n-2)$-ary operations g and h is the variety of n-ary groups if and only if the following two identities*

(7) $$f(x_1^{i-1},g(x_1^{n-2}),x_i^{n-2},y) = y\,,$$

(8) $$f(y,x_1^{j-1},h(x_1^{n-2}),x_j^{n-2}) = y$$

hold for some fixed $i,j \in \{1,2,...,n-1\}$.

Proof. From [3] (p.215) follows that in every n-group $(G;f)$ there exists an $(n-2)$-ary operation g satisfying (7). Similarly there exists an $(n-2)$-ary operation h satisfying (8). Thus (7) and (8) hold in every n-group.

To prove the converse observe first that putting in (7) $x = x_1 = ... = x_{n-2}$ and $g(\overset{(n-2)}{x}) = \overline{x}$ we obtain the identity

(9) $$f(\overset{(i-1)}{x},\overline{x},\overset{(n-i-1)}{x},y) = y\,.$$

Similarly, for $h(\overset{(n-2)}{x}) = \widehat{x}$ from (8) follows

(10) $$f(y,\overset{(j-1)}{x},\widehat{x},\overset{(n-j-1)}{x}) = y\,.$$

If f is $(1,2)$-associative, then (10) implies

$$f(x_1, f(x_2^{n+1}), x_{n+2}^{2n-1}) = f(f(x_1, f(x_2^{n+1}), x_{n+2}^{2n-1}), \overset{(j-1)}{x}, \widehat{x}, \overset{(n-j-1)}{x}) =$$
$$f(x_1, f(f(x_2^{n+1}), x_{n+2}^{2n-1}, x), \overset{(j-2)}{x}, \widehat{x}, \overset{(n-j-1)}{x}) =$$
$$f(x_1, f(x_2, f(x_3^{n+2}), x_{n+3}^{2n-1}, x), \overset{(j-2)}{x}, \widehat{x}, \overset{(n-j-1)}{x}) =$$
$$f(f(x_1^2, f(x_3^{n+2}), x_{n+3}^{2n-1}), \overset{(j-1)}{x}, \widehat{x}, \overset{(n-j-1)}{x}) = f(x_1^2, f(x_3^{n+2}), x_{n+3}^{2n-1}).$$

This proves $(1,3)$-associativity of f. Now, using $(1,2)$ and $(1,3)$-associativity we prove $(1,4)$-associativity. Similarly we can prove $(1,k)$-associativity for all $k = 5, 6, ..., n$. Thus $(G; f)$ is an n-semigroup.

In the case of $(n-1, n)$-associativity the proof is analogous.

To prove that $(G; f)$ is an n-group it is sufficient to solve (2) for $k = 1$ and $k = n$. In the same manner as in the proof of Theorem 2 in [4] one can verify that if (9) holds for $2 \leq i \leq n-1$ then the element

$$z = f_{(.)}(\overset{(i-2)}{x_{n-1}}, \overline{x}_{n-1}, \overset{(n-i-1)}{x_{n-1}}, \overset{(i-2)}{x_{n-2}}, \overline{x}_{n-2}, \overset{(n-i-1)}{x_{n-2}}, ..., \overset{(i-2)}{x_1}, \overline{x}_1, \overset{(n-i-1)}{x_1}, x_0)$$

is a solution of the equation $f(x_1^{n-1}, z) = x_0$.

Similarly, under the assumption $1 \leq j \leq n-2$ in (10), the element

$$z = f_{(.)}(x_0, \overset{(j-1)}{x_n}, \widehat{x}_n, \overset{(n-j-2)}{x_n}, \overset{(j-1)}{x_{n-1}}, \widehat{x}_{n-1}, \overset{(n-j-2)}{x_{n-1}}, ..., \overset{(j-1)}{x_2}, \widehat{x}_2, \overset{(n-j-2)}{x_2})$$

is a solution of the equation $f(z, x_2^n) = x_0$.

Thus $(G; f)$ is an n-group if (9) and (10) hold with the restriction:

(11) $\quad 2 \leq i \leq n-1 \quad$ *and* $\quad 1 \leq j \leq n-2$.

We have still to consider the following cases:

(12) $\quad i = 1, \qquad j = n-1,$

(13) $\quad i = 1, \qquad 2 \leq j \leq n-2,$

(14) $\quad j = n-1, \quad 2 \leq i \leq n-2,$

(15) $\quad i = n-1, \quad j = n-1,$

(16) $\quad i = 1, \qquad j = 1.$

Let (9) and (10) hold for $i = 1$ and $j = n-1$. Then

$$f(\overline{x}, \overset{(n-2)}{x}, y) = f(y, \overset{(n-2)}{x}, \widehat{x}) = y,$$

which gives

$$f(\overline{x}, \overset{(n-1)}{x}) = f(\overset{(n-2)}{x}, \widehat{x}) = x \quad and \quad \overline{x} = f(\overline{x}, \overset{(n-2)}{x}, \widehat{x}) = \widehat{x}\,.$$

As a consequence we obtain

$$y = f(\bar{x}, \overset{(n-2)}{x}, y) = f(\bar{x}, f(\overset{(n-1)}{x}, \bar{x}), \overset{(n-3)}{x}, y) =$$

$$f(f(\bar{x}, \overset{(n-1)}{x}), \bar{x}, \overset{(n-3)}{x}, y) = f(x, \bar{x}, \overset{(n-3)}{x}).$$

By a similar calculation we get

$$y = f(y, \overset{(n-3)}{x}, \bar{x}, x) = f(y, \overset{(n-3)}{x}, \hat{x}, x).$$

Thus the case (12) is reduced to (11) and $(G; f)$ is an n-group.

If (9) and (10) hold with the restriction (13), then

$$x = f(\bar{x}, \overset{(n-1)}{x}) = f(\overset{(j)}{x}, \hat{x}, \overset{(n-j-1)}{x}),$$

which implies

$$y = f(\bar{x}, \overset{(n-2)}{x}, y) = f(\bar{x}, \overset{(n-j-1)}{x}, f(\overset{(j)}{x}, \hat{x}, \overset{(n-j-1)}{x}), \overset{(j-2)}{x}, y) =$$

$$f(f(\bar{x}, \overset{(n-1)}{x}), \hat{x}, \overset{(n-3)}{x}, y) = f(x, \hat{x}, \overset{(n-3)}{x}, y).$$

Hence

$$f(x, \bar{x}, \overset{(n-3)}{x}, y) = f(x, \bar{x}, \overset{(n-3)}{x}, f(x, \hat{x}, \overset{(n-3)}{x}, y)) =$$

$$f(x, f(\bar{x}, \overset{(n-2)}{x}, \hat{x}), \overset{(n-3)}{x}, y) = f(x, \hat{x}, \overset{(n-3)}{x}, y) = y.$$

This proves that (9) holds also for $i = 2$. Therefore (13) may be reduced to (11) and $(G; f)$ is an n-group. By a similar argumentation the case (14) may be reduced to (11).

Now we consider the case (15). In this case the identity (9) has the form $f(\overset{(n-2)}{x}, \bar{x}, y) = y$, which in particular implies

$$f(\overset{(n-2)}{x}, \bar{x}, x) = f(\overset{(n-2)}{\bar{x}}, \bar{\bar{x}}, x) = x,$$

where $\bar{\bar{x}} = g(\overset{(n-2)}{\bar{x}})$. Using these identities it is not difficult to verify that the solution z of the equation

$$f(\overset{(n-3)}{x}, \bar{x}, x, z) = y$$

has the form

$$z = f_{(n-2)}(\overset{(n-3)}{x}, \overline{x}, \overset{(n-3)}{\overline{\overline{x}}}, \overline{\overline{x}}, \underbrace{\underbrace{\overset{(n-3)}{x}, \overline{x}}, \dots, \underbrace{\overset{(n-3)}{x}, \overline{x}}}_{(n-3)\ times}, y) =$$

$$f_{(n-3)}(\overset{(n-3)}{x}, f(\overset{(n-2)}{\overline{x}}, \overline{\overline{x}}, x), \overset{(n-4)}{x}, \overline{x}, \overset{(n-3)}{x}, \overline{x}, \dots, \overset{(n-3)}{x}, \overline{x}, y) =$$

$$f_{(n-3)}(\overset{(n-3)}{x}, \overset{(n-3)}{x}, \overline{x}, \dots, \overset{(n-3)}{x}, \overline{x}, y) =$$

$$f_{(n-4)}(\overset{(n-4)}{x}, f(\overset{(n-2)}{x}, \overline{x}, x), \overset{(n-4)}{x}, \overline{x}, \dots, \overset{(n-3)}{x}, \overline{x}, y) =$$

$$f_{(n-4)}(\overset{(n-4)}{x}, \overset{(n-3)}{x}, \overline{x}, \dots, \overset{(n-3)}{x}, \overline{x}, y) = \dots =$$

$$f(x, f(\overset{(n-2)}{x}, \overline{x}, x), \overset{(n-4)}{x}, \overline{x}, y) = f(\overset{(n-2)}{x}, \overline{x}, y) = y.$$

Hence in this case holds also $f(\overset{(n-3)}{x}, \overline{x}, x, y) = y$, which reduces (15) to (14). Analogously (16) may be reduced to (13). This completes our proof.

Note that in general $g(x_1^{n-2}) \neq h(x_1^{n-2})$, but as it easy to show $g(x, \dots, x) = h(x, \dots, x)$ for all $x \in G$. Moreover, using the Post's Coset Theorem (see [3]), one can prove that in the case $i = j$ we have $g(x_1^{n-2}) = h(x_1^{n-2})$. Hence as a simple consequence of Theorem 2.6 we obtain

Corollary 2.7. *The class of algebras $(G; f, g)$ with one $(1,2)$-associative (or $(n-1, n)$-associative) n-ary $(n > 2)$ operation f and one $(n-2)$-ary operation g is the variety of n-ary groups if and only if the following two identities*

(a) $f(x_1^{i-1}, g(x_1^{n-2}), x_i^{n-2}, y) = y$,
(b) $f(y, x_1^{i-1}, g(x_1^{n-2}), x_i^{n-2}) = y$

hold for some fixed $i = 1, 2, \dots, n-1$.

Corollary 2.8. *The variety of n-ary groups $(n > 2)$ is the class of algebras $(G; f, g, h)$ with one associative n-ary operation f and two $(n-2)$-ary operations g and h satisfying for some fixed $i, j \in \{1, 2, \dots, n-1\}$ the identity*

(17) $$f_{(2)}(x_1^{i-1}, g(x_1^{n-2}), x_i^{n-2}, y, x_1^{j-1}, h(x_1^{n-2}), x_j^{n-2}) = y.$$

Proof. In every n-group $(G; f)$ there exist (by Theorem 2.6) two $(n-2)$-ary operations g and h satisfying (7) and (8). Hence (17) is satisfied, too.

Conversely, if the identity (17) holds in an n-semigroup $(G; f)$, then putting $x = x_1 = \dots = x_{n-2}$, $g(\overset{(n-2)}{x}) = \tilde{x}$ and $h(\overset{(n-2)}{x}) = \hat{x}$ in (17) we obtain

$$f_{(2)}(\overset{(i-1)}{x}, \tilde{x}, \overset{(n-i-1)}{x}, y, \overset{(j-1)}{x}, \hat{x}, \overset{(n-j-1)}{x}) = y .$$

Using the same method as in the proof of Theorem 4 from [7] one can prove that $(G; f)$ is an n-group, which completes the proof.

Analogously as in Theorem 2.6, using the Post's Coset Theorem, one can prove that $\tilde{x} = \hat{x}$ for every $x \in G$. Thus as a simple consequence we obtain

Corollary 2.9. *The variety of n-ary groups $(n > 2)$ may be considered as the class of n-ary semigroups $(G; f)$ with one unary operation $x \to \hat{x}$ satisfying for some fixed $i, j \in \{2, 3, ..., n\}$ the identity*

$$f_{(2)}(\overset{(i-2)}{x}, \hat{x}, \overset{(n-i)}{x}, y, \overset{(n-j)}{x}, \hat{x}, \overset{(j-2)}{x}) = y .$$

Observe that from Corollary 2.1 (a) follows that the class of n-ary groups $(n \geq 2)$ may be considered as the subvariety of the variety of n-ary quasigroups. For $n \geq 3$ this class may be considered also as the subvariety of the class of inversive n-ary semigroups described in [20] and may be defined by a system of identities containing some identities which are characteristic for inversive n-semigroups.

Proposition 2.10. *The class of all n-ary groups $(n > 2)$ may be considered as the variety of algebras $(G; f, g, h)$ of the type $(n, n-2, 3)$ defined by*

(a) $f(x_1^n) = h(x_1, g(x_2^{n-1}), x_n)$,
(b) $h(y, x, x) = h(x, x, y) = y$,
(c) $h(h(x_1^3), x_4^5) = h(x_1, h(x_4, x_3, x_2), x_5) = h(x_1^2, h(x_3^5))$,
(d) $g(x_1^{n-3}, g(x_1^{n-2})) = x_{n-2}$,

where the operation f is $(1,2)$ or $(n-1, n)$-associative.

Proof. Any n-group $(n \geq 3)$ is an inversive n-semigroup in which there exist two operations g and h satisfying the above identities (see [20]).

Conversely, if an algebra $(G; f, g, h)$ satisfies the above conditions, then for all $x, y \in G$ and $\hat{x} = g(\overset{(n-2)}{x})$, we obtain

$$f(y, \overset{(n-2)}{x}, \hat{x}) = h(y, g(\overset{(n-2)}{x}), \hat{x}) = h(y, \hat{x}, \hat{x}) = y$$

and

$$f(\hat{x}, \overset{(n-2)}{x}, y) = h(\hat{x}, \hat{x}, y) = y,$$

which together with the $(1,2)$-associativity of f implies the $(1,3)$-associativity. Indeed,

$$f(x_1, f(x_2^{n+1}), x_{n+2}^{2n-1}) = f(f(x_1, f(x_2^{n+1}), x_{n+2}^{2n-1}), \overset{(n-2)}{x}, \hat{x}) =$$

$$f(x_1, f(f(x_2^{n+1}), x_{n+2}^{2n-1}, x), \overset{(n-3)}{x}, \hat{x}) = f(x_1, f(x_2, f(x_3^{n+2}), x_{n+3}^{2n-1}, x), \overset{(n-3)}{x}, \hat{x}) =$$

$$f(f(x_1^2, f(x_3^{n+2}), x_{n+3}^{2n-1}), \overset{(n-2)}{x}, \hat{x}) = f(x_1^2, f(x_3^{n+2}), x_{n+3}^{2n-1}).$$

Now, using $(1,2)$ and $(1,3)$-associativity we prove the $(1,4)$-associativity. Similarly we can prove $(1,j)$-associativity for $j = 5, 6, ..., n$. Thus $(G; f)$ is an n-semigroup. By Theorem 13 from [20] it is an n-group.

In the case of the $(n-1, n)$-associativity the proof is analogous.

Moreover, the above proof suggest the following characterization of n-groups.

Corollary 2.11. *The class of all n-ary groups* $(n > 2)$ *may be considered as the variety of algebras* $(G; f, g, h)$ *of the type* $(n, n-2, 3)$ *defined by*

(a) $f(x_1^n) = h(x_1, g(x_2^{n-1}), x_n)$,

(b) $h(y, x, x) = h(x, x, y) = y$,

(c) $f(f(x_1^n), x_{n+1}^{2n-1}) = f(x_1, f(x_2^{n+1}), x_{n+2}^{2n-1})$.

Proof. As in the previous proof we can prove that $(G; f)$ is an n-semigroup with a unary operation $x \to \overline{x} = g(\overset{(n-2)}{x})$ and satisfies the assumption of Theorem 2.4. Hence it is an n-group.

Conversely, if $(G; f)$ is an n-group, then by Post's Coset Theorem (see [3]) there exists a binary group $(G^*, \cdot)$ such that $f(x_1^n) = x_1 \cdot x_2 \cdot ... \cdot x_n$ for all $x_1, x_2, ..., x_n \in G$. Hence $g(x_1^{n-2}) = (x_1 \cdot x_2 \cdot ... \cdot x_{n-2})^{-1}$ and $h(x, y, z) = x \cdot y^{-1} \cdot z$ are operations fulfilling (a) and (b), which completes the proof.

Remark that in general the operations g from Proposition 2.10 and Corollary 2.11 are not identical because the second not satisfies (d), in general.

It is worth remaining that the operation h satisfying (b) is so-called *Mal'cev operation.* The existence of such operation in the set of all polynomials of some variety of general algebras is equivalent to the commutativity of congruence on each algebra from this variety. Moreover, the lattice of all congruences of a fixed algebra from such variety is modular (see for example [21]). Thus for every fixed $n \geq 2$ the class of all n-groups is a Mal'cev variety and the lattice of all congruences of a fixed n-groups is modular. For $n = 2$ this fact is known, for $n \geq 2$ it was proved in [22].

Theorem 2.12. *The class of algebras $(G; f, g, h)$ with one associative n-ary $(n \geq 2)$ operation f and two binary operations g and h is the variety of all n-ary groups if and only if for some fixed $i, j \in \{1, 2, ..., n-1\}$ the following two identities hold:*

$$f(\overset{(i)}{x}, \overset{(n-1-i)}{y}, g(x, y)) = y\,, \tag{18}$$

$$f(h(x, y), \overset{(n-1-j)}{y}, \overset{(j)}{x}) = y\,. \tag{19}$$

Proof. It is well known that in every n-group $(n \geq 2)$ the solution z of the equation $f(\overset{(i)}{x}, \overset{(n-1-i)}{y}, z) = y$ there exists and depends only on x and y. Thus z may be treated as the value of a binary operation g satisfying (18). The similar argumentation shows that there exists a binary operation h satisfying (19). (In general $g(x, y) \neq h(x, y)$, but $g(x, x) = h(x, x)$ for all $x \in G$.)

Conversely, let $(G; f)$ be an n-semigroup with two binary operation satisfying (18) and (19). Then in a similar way as in the proof of Theorem 2 in [4] one can verify that for $2 \leq i \leq n-1$ the element

$$z = f_{(\cdot)}(\overset{(i-1)}{x_{n-1}}, \overset{(n-1-i)}{x_{n-2}}, g(x_{n-1}, x_{n-2}), \overset{(i-2)}{x_{n-2}}, \overset{(n-1-i)}{x_{n-3}}, g(x_{n-2}, x_{n-3}), \ldots$$

$$\ldots, \overset{(i-2)}{x_2}, \overset{(n-1-i)}{x_1}, g(x_2, x_1), \overset{(i-2)}{x_1}, \overset{(n-1-i)}{x_0}, g(x_1, x_0))$$

is a solution of the equation $f(x_1^{n-1}, z) = x_0$.

For $i = 1$ this solution has the form

$$z = f_{(\cdot)}(\overset{(n-2)}{x_0}, g(x_{n-1}, x_0), \overset{(n-3)}{x_0}, g(x_2, x_0), \ldots, \overset{(n-3)}{x_0}, g(x_1, x_0)).$$

Similarly, the solution of $f(z, x_2^n) = x_0$ has the form

$$z = f(h(x_n, x_0), \overset{(n-1-j)}{x_0}, \overset{(j-2)}{x_n}, h(x_n, x_{n-1}), \overset{(n-1-j)}{x_n}, \overset{(j-1)}{x_{n-1}}, \ldots$$

$$\ldots, h(x_4, x_3), \overset{(n-1-j)}{x_4}, \overset{(j-2)}{x_3}, h(x_3, x_2), \overset{(n-1-j)}{x_3}, \overset{(j-1)}{x_2})$$

for $j \in \{2, ..., n-1\}$, and

$$z = f(h(x_n, x_0), \overset{(n-3)}{x_0}, \ldots, h(x_3, x_2), \overset{(n-3)}{x_0}, h(x_2, x_0), \overset{(n-2)}{x_0})$$

for $j = 1$.

This proves (by Proposition 1.1) that $(G; f)$ is an n-group.

As a simple consequence of Theorem 2.12 we obtain

Corollary 2.13. *The class of algebras* $(G; f, g, h)$ *with one associative n-ary* $(n \geq 2)$ *operation* f *and two binary operations* g *and* h *is the variety of all n-ary groups if and only if the following two identities hold:*

(i) $f(\overset{(n-1)}{x}, g(x, y)) = y$,

(ii) $f(h(x, y), \overset{(n-1)}{x}) = y$.

Corollary 2.14. *An n-semigroup* $(G; f)$ *is an n-group* $(n \geq 2)$ *if and only if for every* $x, y \in G$ *and some fixed* $i, j \in \{1, 2, ..., n-1\}$ *there exists* $z \in G$ *such that*

(i) $f(\overset{(i)}{x}, \overset{(n-1-i)}{y}, z) = y$,

(ii) $f(z, \overset{(n-1-j)}{y}, \overset{(j)}{x}) = y$.

In particular, for $i = j = n - 1$ we obtain the following result proved in [10].

Corollary 2.15. *An n-semigroup* $(G; f)$ *is an n-group* $(n \geq 2)$ *if and only if for every* $x, y \in G$ *there exists* $z \in G$ *such that*

(i) $f(\overset{(n-1)}{x}, z) = y$,

(ii) $f(z, \overset{(n-1)}{x}) = y$.

3. Subvarieties

In this part basing on the results of previous section we describe some subvarieties of the variety of all n-groups.

In the first place we consider the class of idempotent n-groups. This class is the variety selected from the variety of n-groups by the identity $f(x, ..., x) = x$. Since in idempotent n-groups $(G; f)$ the operation $x \longrightarrow \overline{x}$ is the identity mapping, i.e. $x = \overline{x}$ for all $x \in G$, then by Theorem 2.4 this class has the following description, which for $n = 2$ trivially yields one-element groups.

Proposition 3.1. *The class of all idempotent n-ary groups* $(n \geq 2)$ *is the variety of algebras* $(G; f)$ *with one* $(1, 2)$ *or* $(n-1, n)$*-associative n-ary operation* f *such that the equalities*

$$f(\overset{(n-1)}{x}, y) = f(y, \overset{(n-1)}{x}) = y$$

holds for every $x, y \in G$.

As a consequence of Theorem 2.6 we obtain

Corollary 3.2. *The class of algebras* $(G; f, g)$ *with one* $(1,2)$*-associative (or* $(n-1,n)$*-associative)* n*-ary* $(n > 2)$ *operation* f *and two idempotent* $(n-2)$*-ary operations* g *and* h *is the variety of idempotent* n*-ary groups if and only if for some fixed* $i, j \in \{1, 2, ..., n-1\}$ *the following two identities hold*

(a) $f(x_1^{i-1}, g(x_1^{n-2}), x_i^{n-2}, y) = y$,
(b) $f(y, x_1^{j-1}, h(x_1^{n-2}), x_j^{n-2}) = y$.

In a similar way as Theorem 2.12 we can prove

Proposition 3.3. *The class of algebras* $(G; f, g, h)$ *with one associative* n*-ary* $(n \geq 2)$ *operation* f *and two idempotent binary operations* g *and* h *is the variety of all idempotent* n*-ary groups if and only if for some fixed* $i, j \in \{1, 2, ..., n-1\}$ *the following two identities hold:*

(i) $f(\overset{(i)}{x}, \overset{(n-1-i)}{y}, g(x, y)) = y$,

(ii) $f(h(x, y), \overset{(n-1-j)}{y}, \overset{(j)}{x}) = y$.

Corollary 3.4. *The class of algebras* $(G; f, g, h)$ *with one associative* n*-ary* $(n \geq 2)$ *operation* f *and two idempotent binary operations* g *and* h *is the variety of all idempotent* n*-ary groups if and only if the following identities hold:*

(i) $f(\overset{(n-1)}{x}, g(x, y)) = y$,

(ii) $f(h(x, y), \overset{(n-1)}{x}) = y$.

The variety of idempotent n-ary groups may be considered also as the variety of n-groups in which all inverse operations are idempotent. The minimal system of identities defining such variety is given (for example) by Corollary 2.1 and Corollary 2.2, where all operations $f^{(i)}, g, h$ are idempotent.

On the other hand, it is easy to see that if in Corollary 2.3 an operation f is idempotent, then also g and h are idempotent. The converse is not true. For example, in an algebra $(Z_4; f, g, h)$, where $f(x, y, z) = (x+y+z)(mod\,4)$ and $g(x, y) = h(x, y) = (2x + 3y)(mod\,4)$, the conditions (a) and (b) are satisfied. Moreover, g and h are idempotent, but a 3-group (Z_4, f) is not idempotent.

We say that an n-group $(G;f)$ is *σ-commutative* if $f(x_{\sigma(1)},...,x_{\sigma(n)})$ is invariant under a permutation $\sigma\in S_n$. An n-group which is σ-commutative for every $\sigma\in S_n$ is called *commutative.* It is not difficult to prove (see [7]) that an n-group is commutative iff it is σ-commutative for some fixed $\sigma=(i,i+1)$. Moreover, this fact together with Hosszú Theorem [11] gives

Lemma 3.5. *An n-group $(G;f)$ is commutative if and only if there exists an element $a\in G$ such that for all $x,y\in G$ and some $2\le i\le n$ holds*

$$f(\overset{(i-2)}{a},x,y,\overset{(n-i)}{a})=f(\overset{(i-2)}{a},y,x,\overset{(n-i)}{a}).$$

Theorem 3.6. *The class of all n-ary commutative groups $(n>2)$ may be considered as the variety of algebras with one $(1,2)$-associative n-ary operation f and one unary operation $x\to\hat{x}$ satisfying for some fixed $2\le i\le n$ and $3\le j\le n$ the following two identities:*

(a) $f(y,\overset{(i-2)}{x},\hat{x},\overset{(n-i)}{x})=y$,

(b) $f(x,y,\overset{(j-3)}{x},\hat{x},\overset{(n-j)}{x})=y$.

Proof. Since every commutative n-group satisfies these conditions we prove the converse. Let $(G;f)$ be an $(1,2)$-associative n-groupoid satisfying (a) and (b). Since (a) implies $f(\overset{(i-1)}{x},\hat{x},\overset{(n-i)}{x})=x$, then (b) together with the $(1,2)$-associativity gives

$$y=f(x,y,\overset{(j-3)}{x},\hat{x},\overset{(n-j)}{x})=f(f(\overset{(i-1)}{x},\hat{x},\overset{(n-i)}{x}),y,\overset{(j-3)}{x},\hat{x},\overset{(n-j)}{x})=$$

$$f(x,f(\overset{(i-2)}{x},\hat{x},\overset{(n-i)}{x},y),\overset{(j-3)}{x},\hat{x},\overset{(n-j)}{x})=f(\overset{(i-2)}{x},\hat{x},\overset{(n-i)}{x},y).$$

Thus by Theorem 2.4 an algebra $(G;f,\hat{}\ )$ is an n-group and $\hat{x}$ is the skew element. Therefore (a) and (b) are valid for all $2\le i\le n$ and $3\le j\le n$. Moreover,

$$f(x,y,\overset{(n-2)}{a})=f(f(y,x,\overset{(j-3)}{y},\hat{y},\overset{(n-j)}{y}),y,\overset{(n-2)}{a})=$$

$$f(y,f(x,\overset{(j-3)}{y},\hat{y},\overset{(n-j+1)}{y}),\overset{(n-2)}{a})=f(y,x,\overset{(n-2)}{a})$$

for all $a,x,y\in G$, which by Lemma 3.5 completes the proof.

As a consequence of the above Theorem and Theorem 2.6 we obtain the following characterization of commutative n-groups.

Corollary 3.7. *The class of algebras* $(G; f, g)$ *with one* $(1,2)$*-associative n-ary* $(n > 2)$ *operation* f *and one* $(n-2)$*-ary operation* g *is the variety of commutative n-ary groups if and only if for some fixed* $j \in \{1, 2, ..., n-1\}$ *the following two identities hold*

(a) $f(y, x_1^{j-1}, g(x_1^{n-2}), x_j^{n-2}) = y$,

(b) $f(, x_1, y, x_2^{j-1}, g(x_1^{n-2}), x_j^{n-2}) = y$.

In the theory of n-semigroups the following identities

$$f(x_1, x_2^{n-1}, x_n) = f(x_n, x_2^{n-1}, x_1)$$

and

$$f(f(x_{11}^{1n}), f(x_{21}^{2n}), ..., f(x_{n1}^{nn})) = f(f(x_{11}^{n1}), f(x_{12}^{n2}), ..., f(x_{1n}^{nn}))$$

play a very important role.

The first of them is called *semi-commutativity* (an n-group with this identity is called, by Dörnte [1], *semiabelian.*) The second of them is a natural generalization of the medial (entropic) law for groupoids. An n-semigroup satisfying this identity is called *medial* or *Abelian* (see [12]) since an n-semigroup $(G; f)$ treated as an algebra $(G; f, f)$ of the type (n, n) is Abelian in the sense of [13] (p. 87).

Each semi-commutative n-semigroup is medial [12], but for every $n \geq 2$ there exist medial n-semigroups which are not semi-commutative [5]. An n-ary group is medial iff it is semi-commutative [12], or equivalently (see [5] and [14]), iff for some fixed $a \in G$ the identity $f(x, \overset{(n-2)}{a}, y) = f(y, \overset{(n-2)}{a}, x)$ is true. Hence the class of all medial n-groups is the variety defined by the last identity, the $(1,2)$-associativity and (6) (or by the $(n-1, n)$-associativity and (5)).

4. Open problems

From the proof of Theorem 3 in [12] follows that any medial n-group satisfies the identity

(20) $\overline{f(x_1^n)} = f(\overline{x}_1, \overline{x}_2, ..., \overline{x}_n)$.

Hence an n-group $(G; f)$ is Abelian as an algebra $(G; f, ^-)$. Note that (20) holds also in some non-medial n-groups. It holds for example in all idempotent n-groups. Therefore the following problems (announced in [5]) seems to be interesting:

Problem 1. *Describe the variety of all n-groups satisfying* (20).

Let $(G; f, ^{-})$ be an n-group and let $\overline{x}$ be the skew element to x. Moreover, let $\overline{x}^{(0)} = x$ and let $\overline{x}^{(s+1)}$ be the skew element to $\overline{x}^{(s)}$ for $s \geq 0$. In the other words: $\overline{x}^{(1)} = \overline{x}$, $\overline{x}^{(2)} = \overline{\overline{x}}$, $\overline{x}^{(3)} = \overline{\overline{\overline{x}}}$, etc.

Problem 2. *Describe the class of n-groups in which there exists s such that $\overline{x}^{(s)} = \overline{x}^{(t)}$ for all elements and all $t \geq s$.*

Some results connected with this problem are obtained in [15] and [16].

Problem 3. *Describe the class of n-groups in which $\overline{x}^{(s)} \neq \overline{x}^{(t)}$ for all $s \neq t$ and $x \in G$.*

Problem 4. *Describe the variety $\mathbf{V}_s$ of n-groups in which $\overline{x}^{(s)} = x$ for all $x \in G$.*

The class $\mathbf{V}_1$ is the variety of idempotent n-groups. Obviously $\mathbf{V}_1 \subset V_s$ for every natural s. Moreover, $\mathbf{V}_s \cap V_{s+1} = V_1$ and $\mathbf{V}_s \subset V_{st}$ for any natural s, t. Any $\mathbf{V}_s$ contains the variety of medial n-groups (and in the consequence - the varierty of commutative n-groups). Since $\overline{\overline{x}} = x$ for all 3-groups [1], the variety $\mathbf{V}_{2s}$ contains the variety of all 3-groups.

As it is known (see [18]) in some n-ary algebras there exist so-called *splitting automorphisms*, i.e. automorphism ψ satisfying for every $i = 1, 2, ... n$ the condition $\psi(f(x_1^n)) = f(x_1^{i-1}, \psi(x_i), x_{i+1}^n)$. Such automorphisms there exist also in some n-ary groups $(n > 2)$. For example, it is easy to see that $\psi_a(x) = (x + a)(mod\, n)$ is a splitting automorphism of an (n+1)-group $(Z_n; f)$ defined by $f(x_1^{n+1}) = (x_1 + ... + x_{n+1} + b)(mod\, n)$. Moreover, in some n-groups the unary operation $x \longrightarrow \overline{x}$ is a splitting automorphism. Such n-groups are called *distributive*. The class of distributive n-groups forms a variety selected from the variety of all n-groups by the identity

$$\overline{f(x_1^n)} = f(x_1^{i-1}, \overline{x}_i, x_{i+1}^n), \tag{21}$$

where $i = 1, 2, ..., n$.

Every distributive n-group satisfies (20) and it is a set-theoretic union of disjoint and isomorphic subgroups of the form $\{x, \overline{x}, ..., \overline{x}^{(t-1)}\}$, where t is fixed. Hence a distributive n-group is idempotent or has no any idempotents [17]. Moreover, $\{\phi, \phi^2, \phi^3, ..., \phi^t\}$, where $\phi(x) = \overline{x}$ is an invariant subgroup of the group of all splitting automorphisms.

In every medial distributive n-group $(G; f)$ an operation f is distributive with respect to itself, i.e. the identity

$$f(x_1^{i-1}, f(y_1^n), x_{i+1}^n) = f(f(x_1^{i-1}, y_1, x_{i+1}^n), ..., f(x_1^{i-1}, y_n, x_{i+1}^n)),$$

holds for all $i = 1, 2, ..., n$. Such n-groups, called *autodistributive*, are described in [16] and [5]. The class of autodistributive n-groups ($n > 3$) is a proper subvariety of the variety of distributive n-groups. For $n = 3$ these varieties are equal; for $n = 2$ are trivial.

Problem 5. *Describe the variety of all n-groups satisfying* (20) *and* (21).

Problem 6. *Describe the class of all n-groups in which there exists at least one non-trivial splitting automorphism.*

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